Initial intervals of partial orders and reverse mathematics

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Initial intervals and ideals

Definition

Let P be a partial order.

- $I \subseteq P$ is an initial interval of P if $\forall x, y \in P(x \leq_P y \wedge y \in I \implies x \in I)$;
- An initial interval I of P is an ideal if $\forall x, y \in I \, \exists z \in I(x \leq_P z \wedge y \leq_P z)$ (compatibility).
- $A \subseteq P$ is an antichain if $\forall x, y \in A\,(x = y \vee x \perp y)$;
- $S \subseteq P$ is a strong antichain in P if $\forall x, y \in S(x = y \vee \neg \exists z \in P\, x, y \leq_P z)$;

The theorems

Theorem (Bonnet 1975)

A partial order P is FAC (no infinite antichains) if and only if every initial interval of P is a finite union of ideals.

Theorem (Erdös-Tarski 1943)

If a partial order P has no infinite strong antichains, then there are no arbitrarily large strong antichains in P.

Theorem (Bonnet 1973)

A countable partial order P has countably many initial intervals iff it is scattered (no copy of $\mathbb{Q}$ in P) and FAC.

Results

Theorem

Over RCA_0*, the following are pairwise equivalent:*

1. ACA_0*;*
2. *if a partial order is FAC, then every initial interval is a finite union of ideals (Bonnet 1975);*
3. *every partial order with no infinite strong antichains has a finite bound on the size of strong antichains (Erdös-Tarski 1943).*

Results

Theorem

Over ACA_0*, the following are equivalent:*

1. ATR_0*;*
2. *every scattered FAC partial order has countably many initial intervals (Bonnet 1973).*

Results

Theorem

Provably in WKL_0 *but not in* RCA_0,

1. *every partial order which is not FAC contains an initial interval which is not a finite union of ideals.*
2. *every partial order which is not FAC has uncountably many initial intervals.*

Theorem (Patey)

Neither 1. nor 2. is provable in WWKL_0.

Reversal of Bonnet 1975

Theorem

Over RCA_0, the statement "if a partial order is FAC, then every initial interval is a finite union of ideals" implies ACA_0.

Proof.

Fix a one-to-one function $f\colon \mathbb{N} \to \mathbb{N}$. We build a partial order P which encodes the range of f. We use false and true stages (n is false if $f(i) < f(n)$ for some $i > n$). We can arrange false and true stages into an $\omega + \omega^*$-chain. We then add an ω-chain "above" the false stages. Antichains have at most two elements. Apply the statement. The ideal which contains the ω-chain separates false from true stages. □

Initial intervals and trees

Let P be a partial order and $\mathcal{I}(P)$ be the class of initial intervals.

- P has countably many initial intervals if there exists $\{I_n : n \in \mathbb{N}\}$ such that $\forall I \in \mathcal{I}(P) \, \exists n \in \mathbb{N} \, I = I_n$.
- P has perfectly many initial intervals if there exists a nonempty perfect tree $T \subseteq 2^{<\mathbb{N}}$ such that $[T] \subseteq \mathcal{I}(P)$.

The tree of initial intervals of P is $T(P) \subseteq 2^{<\mathbb{N}}$:
$\sigma \in T(P)$ iff for all $x, y < |\sigma|$:

- $\sigma(x) = 1$ implies $x \in P$;
- $\sigma(y) = 1$ and $x \leq_P y$ imply $\sigma(x) = 1$.

Initial intervals and trees

- P has countably many initial intervals iff $T(P)$ has countably many paths (within RCA_0).
- P has perfectly many initial intervals iff $T(P)$ contains a perfect subtree (within RCA_0).
- "P has uncountably many initial intervals" is $\boldsymbol{\Sigma}^1_1$ (within ATR_0 via the Perfect Tree Theorem).

A proof of Bonnet 1973

Theorem

ATR_0 *proves that every scattered FAC partial order has countably many initial intervals.*

We want to iterate the following lemma:

Lemma (one-step lemma)

ACA_0 *proves that if P has perfectly many initial intervals, then there exists $x \in P$ such that either*

(1) *both $\downarrow x$ and $\uparrow x$ have uncountably many initial intervals, or*

(2) *$\perp x$ has uncountably many initial intervals.*

The proof in ATR_0

We aim to define by recursion a pruned tree $T \subseteq \mathbb{N}^{<3}$ labeled with pairs (x_σ, P_σ) such that

- $P_\sigma \subseteq P$ has uncountably many initial intervals;
- if $\sigma \in T$ splits then
  (1) $P_{\sigma 0} = P_\sigma \cap \downarrow x_\sigma$ and $P_{\sigma 1} = P_\sigma \cap \uparrow x_\sigma$
- if $\sigma \in T$ does not split then
  (2) $P_{\sigma 3} = P_\sigma \cap \perp x$.

This recursion is $\mathbf{\Sigma}^1_1$ within ATR_0.
We overcome this and make it arithmetical by using countable coded ω-models. The key point is that $T(P_\sigma)$ is indeed computable in P.

The proof in ATR_0

Theorem

ATR_0 *proves that for all* X *there exists a countable coded* ω*-model* M *of* $\mathbf{\Sigma}^1_1$-DC_0 *such that and* $X \in M$ *and* M *satisfies the Perfect Tree Theorem for all trees computable in* X*.*

Proof of Theorem

- Assume P has uncountably many initial intervals and let U be a perfect subtree of $T(P)$;
- Let M be an ω-model as above;
- Define T by recursion using M as a parameter.
- At each stage, apply the one-step lemma.

A proof of Bonnet 1975

Theorem

WKL_0 *proves that every partial order which is not FAC contains an initial interval which is not a finite union of ideals.*

Proof.

Let P be a partial order with an infinite antichain A. We define an infinite binary tree such that any path is an initial interval which contains A and no elements above A. By WKL_0, there is a path and so an initial interval I with this property. I cannot be a finite union of ideals because A is a strong antichain in I. By the finite pigeonhole principle, two elements of A would be in the same ideal and so compatible. □

Unprovability in RCA_0

Let NCF be the statement "every partial order which is not FAC contains an initial intervals which is not the downward closure of a finite set".

Theorem

NCF *is false in* **REC**.

Proof.

Diagonalize against any Φ_e computing an initial interval.
To defeat Φ_e, fix x and wait until you see $x \in \Phi_e$ or $x \notin \Phi_e$.
In the first case, build below x. In the second case, above x. □

Nonimplications by conservativity

NCF is a restricted $\mathbf{\Pi}^1_2$ statement and so is not implied by $\mathbf{\Pi}^0_1\mathsf{G}$.

Unprovability in WWKL_0

Theorem (Patey)

There exists a computable partial order with a computable antichain such that the class of oracles computing initial intervals which are not the closure of a finite set is null.

Proof.

Diagonalize against any Φ_e^X computing an initial interval with enough measure.
To defeat Φ_e^X, fix x and wait until you see $\mu(X : x \in \Phi_e^X) \geq 0.49$ or $\mu(X : x \notin \Phi_e^X) \geq 0.49$.
In the first case, build below x. In the second, above x. $\square$

Corollary

WWKL_0 *does not prove* NCF.

Proof.

Given P as above, let X be a ML random real not in the class. Thus X computes only "trivial" initial intervals of P. Build an ω-model of WWKL_0 below X as usual. Inside the model, every initial interval of P is the closure of a finite set. In particular, initial intervals are uniformly computable, and hence P has countably many initial intervals. □

Corollary

WWKL_0 *does not prove the "easy" directions of Bonnet 75 and Bonnet 73.*

Open questions

Is "every partial order which is not FAC has an initial interval which is not a finite union of ideals" equivalent to WKL_0?

Gregory Igusa showed that any computable instance has a solution which is not a PA degree. On the other hand, WKL_0 is known to be very robust.

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Is “every partial order which is not FAC has uncountably many initial intervals” is equivalent to WKL_0?

Thanks