BROWN'S LEMMA IN SECOND-ORDER ARITHMETIC

EMANUELE FRITTAION

Abstract. Brown's lemma states that in every finite coloring of the natural numbers there is a homogeneous piecewise syndetic set. We show that Brown's lemma is equivalent to $\mathsf{I}\Sigma^0_2$ over RCA^*_0. We show in contrast that (the infinite) van der Waerden's theorem is equivalent to $\mathsf{B}\Sigma^0_2$ over RCA^*_0. We finally consider the finite version of Brown's lemma and show that it is provable in RCA_0 but not in RCA^*_0.

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1. Introduction

In the present paper we study Brown's lemma in the context of second-order arithmetic. Brown's lemma asserts that piecewise syndetic sets are large, in the sense that whenever we partition the natural numbers into finitely many sets, at least one set must be piecewise syndetic.

Definition 1.1 (Piecewise syndetic)**.** A set $X \subseteq \mathbb{N}$ has gaps bounded by $d \in \mathbb{N}$ if the difference between any two consecutive elements of X is $\leq d$. A set $X \subseteq \mathbb{N}$ is *piecewise d-syndetic* if it contains arbitrarily

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large finite sets with gaps bounded by d. We say that X is *piecewise syndetic* if it is piecewise d-syndetic for some d.

Let $r \in \mathbb{N}$ and $C\colon \mathbb{N} \to r$ be a coloring of $\mathbb{N}$ in r colors. As usual, we identify r with the finite set $\{0, \ldots, r-1\}$. We say that $H \subseteq \mathbb{N}$ is C-homogeneous if there exists $i < r$ such that $C(x) = i$ for all $x \in H$.

Theorem 1.2 (Brown's lemma)**.** *Every finite coloring $C\colon \mathbb{N} \to r$ has a C-homogeneous piecewise syndetic set.*

Note that piecewise syndetic sets are closed under supersets and so we can recast Brown's lemma as follows.

Theorem 1.3 (Brown's lemma, restated)**.** *Let $\mathbb{N} = X_0 \cup \ldots X_{r-1}$ be a finite partition of the natural numbers. Then X_i is piecewise syndetic for some $i < r$.*

In fact, Brown's lemma generalizes to finite partitions of piecewise syndetic sets. Recall that a class of sets $\mathbb{S} \subseteq \mathcal{P}(\mathbb{N})$ is called *partition regular* if for every $X \in \mathbb{S}$ and for every finite partition $X = X_0 \cup \ldots X_{r-1}$ there exists $i < r$ such that $X_i \in \mathbb{S}$.

Theorem 1.4 (Partition regularity of piecewise syndetic sets)**.** *Let $X \subseteq \mathbb{N}$ be piecewise syndetic and $X = X_0 \cup \ldots X_{r-1}$ be a finite partition. Then X_i is piecewise syndetic for some $i < r$.*

Theorem 1.2 was originally proved by Brown [Bro68] (see also [Bro69, Bro71]) in the context of locally finite semigroups. An ergodic-theoretic proof of Theorem 1.2 as well as Theorem 1.4 can be found in Furstenberg [Fur81, Theorem 1.23, Theorem 1.24]. For an algebraic-theoretic proof of Theorem 1.4 based on the characterization of piecewise syndetic sets in terms of $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$, see Hindman [HS12].

As piecewise syndetic sets are arithmetically (Σ^0_3-)definable and closed under supersets, Brown's lemma does not actually assert the existence of a set. In other words, this is a Π^1_1-statement. Notice that a true statement of the form

† Every finite coloring $C\colon \mathbb{N} \to r$ has a C-homogeneous *large* set,

where *large* is a property about sets closed under supersets, is computably true, that is, it is true in the model of recursive sets.

A statement like † has no computational power but only inductive strength. In fact, † is likely to be provable in $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_n$ if *large* is Σ^0_{n+1}-definable, and in $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_n$ if *large* is Π^0_n-definable. For instance the infinite pigeonhole principle

$\mathsf{RT}^1_{<\infty}$ Every finite coloring $C\colon \mathbb{N} \to r$ has an infinite C-homogeneous set,

is provable and actually equivalent to $\mathsf{B}\Sigma^0_2$ over RCA_0. Therefore $\mathsf{RT}^1_{<\infty}$ and Brown's lemma serve as an example for $n = 2$.

Brown's lemma is related to the well-known van der Waerden's theorem.

Theorem 1.5 (van der Waerden's theorem). *Every finite coloring $C\colon \mathbb{N} \to r$ has a C-homogeneous set with arbitrarily long arithmetic progressions.*

Say that $X \subseteq \mathbb{N}$ is AP if it contains arbitrarily long arithmetic progressions. Note that AP sets are also closed under supersets and so we can restate van der Waerden's theorem as follows.

Theorem 1.6 (van der Waerden's theorem, restated). *Let $\mathbb{N} = X_0 \cup \ldots X_{r-1}$ be a finite partition of the natural numbers. Then X_i is AP for some $i < r$.*

Also van der Waerden's theorem generalizes to finite partitions of AP sets.

Theorem 1.7 (Partition regularity of AP sets). *Let $X \subseteq \mathbb{N}$ be AP and $X = X_0 \cup \ldots X_{r-1}$ be a finite partition. Then X_i is AP for some $i < r$.*

It is known that every piecewise syndetic set contains arbitrarily long arithmetic progressions and so Brown's lemma is a generalization of van der Waerden's theorem. On the other hand, AP sets need not be piecewise syndetic and so partition regularity for piecewise syndetic sets does not imply partition regularity for AP sets. The proof that piecewise syndetic sets are AP uses the finite version of van der Waerden's theorem (see Rabung [Rab75]).

Theorem 1.8 (van der Waerden's theorem, finite). *For all r, l there exists n such that every coloring $C\colon n \to r$ has a C-homogeneous arithmetic progression of length l.*

Let $W(r, l)$ be the least number n such that every r-coloring of n has a homogeneous arithmetic progression of length l. Shelah [She88] proved that the function $W(l, r)$ has primitive recursive upper bounds by giving a new proof of Hales-Jewett theorem which avoids the use of double induction ([1]) and can be formalized in RCA_0 ([2]).

Theorem 1.9 (Folklore, essentially Shelah [She88])**.** *The finite version of van der Waerden's theorem is provable in* RCA_0*.*

Working in the framework of reverse mathematics we prove that Brown's lemma is equivalent to $\mathsf{I}\Sigma^0_2$ over RCA^*_0 and so is the statement that piecewise syndetic sets are partition regular (Theorem 5.5). Note that Brown's lemma trivially implies the infinite pigeonhole principle $\mathsf{RT}^1_{<\infty}$ and therefore $\mathsf{B}\Sigma^0_2$. In contrast, we show that van der Waerden's theorem and the statement that AP sets are partition regular are both equivalent to $\mathsf{B}\Sigma^0_2$ over RCA^*_0 (Theorem 4.1).

We finally consider the finite version of Brown's lemma.

Definition 1.10. Let $H \subseteq \mathbb{N}$ be finite. Define the *gap size* of H, denoted $gs(H)$, as the largest difference between two consecutive elements of H. In other words, the gap size of H is the least d such that H has gaps bounded by d. If $|H| \leq 1$ let $gs(H) = 1$.

Theorem 1.11 (Brown's Lemma, finite)**.** *Let* $f\colon \mathbb{N} \to \mathbb{N}$*. Then for all* $r > 0$ *there exists* n *such that every coloring* $C\colon n \to r$ *has a* C*-homogeneous set* H *with* $|H| > f(gs(H))$*.*

In Brown [Bro81] Theorem 1.11 is attributed to Justin [Jus71]. The finite version of Brown's lemma is reminiscent of the (relativized) Paris-Harrington principle. We can think of $|H| > f(gs(H))$ as a largeness condition on H. Note on the other hand that a natural Paris-Harrington miniaturization of van der Waerden's theorem, that is, the

[1]All previous proofs of Hales-Jewett theorem proceeded by Π_2-induction on n in order to show

$$(\forall c)(\exists k)HJ(n, c) = k,$$

where $HJ(n, c) = k$ says that k is the Hales-Jewett number for n and c, and resulted in Ackermannian upper bounds. Shelah's famous proof shows

$$(\exists k)HJ(n, c) = k$$

by Σ_1-induction on n, where c is treated as a fixed parameter.

[2]By the celebrated work of Gowers [Gow98, Gow01] on Szemerédi's theorem it follow that we actually have elementary recursive upper bounds for van der Waerden numbers. Gowers' bounds are the following:

$$W(r, l) \leq 2^{2^{f(r,l)}}, \text{ where } f(r, l) = r^{2^{2^{l+9}}}.$$

However, the proof is far from elementary and so we can only conjecture that van der Waerden's theorem is provable in EFA. On the other hand, the lower bounds for van der Waerden numbers are exponential (see [GRS90]), and hence van der Waerden's theorem is not provable in bounded arithmetic.

statement "For all $f\colon \mathbb{N} \to \mathbb{N}$ and for all $r > 0$ there exists n such that every coloring $C\colon n \to r$ has a C-homogeneous arithmetic progression $H = \{x, x+d, x+2d, \ldots\}$ such that $|H| > f(x)$ or $|H| > f(d)$", does not hold (see Brown [Bro81, Fact 1]).

Definition 1.12. For $f\colon \mathbb{N} \to \mathbb{N}$, let BL_f be the statement "For all $r > 0$ there exists n such that every $C\colon n \to r$ has a C-homogeneous set H such that $|H| > f(gs(H))$.

We show that the finite version of Brown's lemma is provable in RCA_0 (Theorem 6.2) but not in RCA_0^*. We obtain our unprovability result by showing that BL_f is not provable in EFA for $f(d) = 2^d$ (Theorem 6.7). On the other hand, BL_f is provable in EFA for $f(d) = d$ (Theorem 6.5). We can therefore see the finite version of Brown's lemma as a natural example of phase transition with respect to EFA.

In the appendix we discuss the original proof of Brown's lemma and point out a disguised use of ACA_0.

2. Preliminaries

We assume familiarity with the basic systems of reverse mathematics. The standard reference is [Sim09]. In this paper we are mainly concerned with the base systems RCA_0^*, RCA_0 and the induction schemes $\mathsf{I}\Sigma^0_2$ and $\mathsf{B}\Sigma^0_2$. Recall that RCA_0^* consists of the usual axioms for addition, multiplication, exponentiation plus Σ^0_0-induction and Δ^0_1-comprehension. The system RCA_0 consists of the usual axioms for addition and multiplication plus Σ^0_1-induction and Δ^0_1-comprehension. We have $\mathsf{RCA}_0 = \mathsf{RCA}_0^* + \mathsf{I}\Sigma^0_1$. We use the following fact repeatedly (see Hirst [Hir87] and Yokoyama [Yok13]).

Theorem 2.1 (RCA_0^*)**.** $\mathsf{RT}^1_{<\infty}$ *is equivalent to* $\mathsf{B}\Sigma^0_2$.

By Simpson and Smith [SS86], the first order part of RCA_0^* and WKL_0^*, that is, RCA_0^* plus weak König's lemma, "Every infinite binary tree has an infinite path", is $\mathsf{B}\Sigma_1 + \exp$, and WKL_0^* is Π_2-conservative over $\mathsf{I}\Sigma_0 + \exp$, also known as EFA (Elementary Function Arithmetic).

Recall that the provably recursive functions of EFA are exactly the elementary recursive functions and that every elementary recursive function $f(n)$ is dominated by $2_k(n)$ for some number k, where $2_k(n)$ is the *superexponential function* defined by $2_0(n) = n$ and $2_{k+1}(n) = 2^{2_k(n)}$ (see for instance [SW12]).

3. Piecewise syndetic sets

In the present paper we do not consider the characterization of piecewise syndetic sets in terms of $\beta\mathbb{N}$ (cf. Hindman [HS12, Theorem 4.40]).

The purpose of this section is to show that, with few exceptions, most of the elementary characterizations for piecewise syndetic sets can be proved within RCA_0^*.

Definition 3.1. An infinite set $X \subseteq \mathbb{N}$ is *syndetic* if there exists $d \in \mathbb{N}$ such that X has gaps bounded by d. We say that X is d-syndetic if d is such a witness. A set $X \subseteq \mathbb{N}$ is *thick* if it contains arbitrarily large intervals of natural numbers.

Proposition 3.2 (RCA_0^*)**.** *Let $X \subseteq \mathbb{N}$. The following are equivalent:*
(1) *X is piecewise syndetic;*
(2) *X is the intersection of a syndetic set and a thick set.*

Proof. We argue in RCA_0^*. Implication (2) $\to$ (1) is straightforward. For (1) $\to$ (2) suppose X is piecewise d-syndetic. Let $Z = \bigcup_{s<d}(X + s) = \bigcup_{s<d}\{x + s \colon x \in X\}$. We claim that Z is thick. Fix n and let H be an n-element subset of X such that $gs(H) \leq d$. Then $I = \bigcup_{s<d}(H+s) \subseteq Z$ is an interval of size at least n. Define $Y = X \cup (\mathbb{N} \setminus Z)$. Clearly $X = Y \cap Z$. It suffices to show that Y is syndetic. Suppose not. Then $A = \mathbb{N} \setminus Y$ is thick and $A \subseteq Z$. In particular $X \cap A = \emptyset$. Let $J \subseteq A$ be an interval of size at least d. Every element of J is of the form $x + s$ with $x \in X$ and $s < d$. It follows that $X \cap J \neq \emptyset$ since otherwise $|J| < d$, and so $X \cap A \neq \emptyset$, a contradiction. □

Definition 3.3. We say that $I \subseteq X$ is an *interval* of X if for all $x < y$ in I we have $\{z \in X \colon x < z < y\} \subseteq I$.

Proposition 3.4. *Let $X \subseteq \mathbb{N}$. The following are equivalent:*
(1) *there is d such that $gs(H) \leq d$ for arbitrarily large finite sets $H \subseteq X$, that is, X is piecewise syndetic;*
(2) *there is d such that $gs(H) = d$ for arbitrarily large finite sets $H \subseteq X$;*
(3) *there is d such that $gs(I) \leq d$ for arbitrarily large intervals I of X;*
(4) *there is d such that $gs(I) = d$ for arbitrarily large intervals I of X.*

Most implications are trivial. The strongest is (1) $\to$ (4) and is provable in $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. We actually have a reversal.

Proposition 3.5 (RCA_0^*)**.** *The following are equivalent:*
(1) $\mathsf{B}\Sigma^0_2$*;*
(2) *if X is piecewise syndetic then there exists d such that $gs(I) = d$ for arbitrarily large intervals I of X.*

Proof. We work in $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. Suppose that X is piecewise e-syndetic. For all n, search for an n-element interval I_n of X such that $gs(I_n) \leq e$.

Define a coloring $C\colon \mathbb{N} \to e+1$ by letting $C(n) = gs(I_n)$. By $\mathsf{RT}^1_{<\infty}$ there exists $d \leq e$ such that $\{n\colon C(n) = e\}$ is infinite. Then d is as desired.

For the other direction we work in RCA_0^* assume (2). We aim to show $\mathsf{RT}^1_{<\infty}$. Let $C\colon \mathbb{N} \to r$. For convenience we assume $C(n) > 0$ for all n. We define a piecewise r-syndetic set $X = \{x_0 < x_1 < \ldots\} \subseteq \mathbb{N}$ such that for every $n > 0$ there is an interval I_n of X of size n with $gs(I_n) = C(n) < r$ and $\min(I_{n+1}) - \max(I_n) = n$.

$$1\,\underbrace{1\overbrace{0\ldots\ldots\ldots0}^{C(2)-1\text{ times}}1}_{|I_2|=2}\,0\,\underbrace{1\overbrace{0\ldots\ldots\ldots0}^{C(3)-1\text{ times}}1\overbrace{0\ldots\ldots\ldots0}^{C(3)-1\text{ times}}1}_{|I_3|=3}\,001\ldots$$

Let $x_0 = 0$, $x_1 = x_0+1$ and $x_2 = x_1+C(2)$, $x_3 = x_2+2$, $x_4 = x_3+C(3)$ and $x_5 = x_4 + C(3)$. In general, suppose we have defined x_k and $k+1 = n(n+1)/2 = 1+2+3+\ldots+n$. Then

$$\begin{aligned} x_{k+1} &= x_k + n; \\ x_{k+i+1} &= x_{k+i} + C(n+1) \text{ for } 0 < i < n+1. \end{aligned}$$

Since $x_k \leq rk(k+1)/2$, we can define x_k by bounded primitive recursion and X by Δ^0_1-comprehension. By construction, $I_n = \{x_k, x_{k+1}, \ldots, x_{k+n-1}\}$, where $k = (n-1)n/2$, is an interval of X of size n and gaps bounded by r, and hence X is piecewise r-syndetic. By (2), there exists d such that $gs(I) = d$ for arbitrarily long intervals I of X. We claim that $d < r$ and $\{n \in \mathbb{N}\colon C(n) = d\}$ is infinite. Suppose for a contradiction that $d \geq r$ or there exists n such that $C(m) \neq d$ for all $m > n$. In both cases there exists $n > d$ such that $C(m) \neq d$ for all $m > n$. Let $k+1 = 2+3+\ldots+n$. It follows that if I is a 2-element interval of X such that $gs(I) = d$ then $I \subseteq x_k+1$ and hence $|I| \leq x_k+2$, against our assumption on d. □

One might ask what is the reverse mathematics of all other nontrivial implications. By Proposition 3.5, we know that they are provable in $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. In particular, the statement "for every piecewise syndetic set X there exists d such that $gs(H) = d$ for arbitrarily large finite sets $H \subseteq X$" seems to require $\mathsf{B}\Sigma^0_2$. Note however that the construction in the reversal of Proposition 3.5 does not work for such a statement. In fact, X contains arbitrarily large subsets H with $gs(H) = d$ whenever d is a multiple of an infinite color of C.

4. Brown's lemma vs van der Waerden's theorem

The proof that every piecewise syndetic set contains arbitrarily long arithmetic progressions is based on the finite version of van der Waerden's theorem and can be formalized in RCA_0^*. Therefore it is not surprising that within RCA_0 Brown's lemma implies van der Waerden's theorem. Here we establish this fact by showing that van der Waerden's theorem is equivalent to $\mathsf{B}\Sigma^0_2$ over RCA_0^*.

Theorem 4.1 (RCA_0^*). *The following are equivalent:*
(1) $\mathsf{B}\Sigma^0_2$*;*
(2) *van der Waerden's theorem;*
(3) *partition regularity of AP sets.*

Proof. We argue in RCA_0^*. Clearly (3) $\rightarrow$ (2). As van der Waerden's theorem implies $\mathsf{RT}^1_{<\infty}$, which is equivalent to $\mathsf{B}\Sigma^0_2$, we have (2) $\rightarrow$ (1). It remains to show (1) $\rightarrow$ (3). Let $X = X_0 \cup \ldots X_{r-1}$ be a finite partition of an AP set. Suppose for a contradiction that none of the X_i's are AP. Then for all $i < r$ there exists l such that no arithmetic progression of length l lies within X_i. By $\mathsf{B}\Sigma^0_2$ there exists l large enough such that for all $i < r$ no arithmetic progression of length l lies within X_i. By the finite van der Waerden's theorem, let n be such that every coloring $C \colon n \to r$ has a C-homogeneous arithmetic progression of length l. Let $\{x_0 < x_1 < \ldots < x_{n-1}\} \subseteq X$ be an arithmetic progression of length n. Define a coloring $C \colon n \to r$ be letting $C(m) = i$ iff $x_m \in X_i$. Then there exists a C-homogeneous arithmetic progression $m_0 < m_1 < \ldots m_{l-1}$ of length l. It follows that $x_{m_0} < x_{m_1} < \ldots x_{m_{l-1}}$ is an arithmetic progression of length l which lies entirely within X_i for some $i < r$, for the desired contradiction. □

Notice that we use the finite van der Waerden's theorem in the course of the proof.

Corollary 4.2 (RCA_0^*). *Brown's lemma implies van der Waerden's theorem.*

Proof. Over RCA_0^* Brown's lemma implies $\mathsf{RT}^1_{<\infty}$, which is equivalent to $\mathsf{B}\Sigma^0_2$. □

5. Brown's lemma

We first show that Brown's lemma is provable in $\mathsf{I}\Sigma^0_2$. The argument is due to Kreuzer [Kre12].

Lemma 5.1 (RCA_0^*). *If $X \subseteq \mathbb{N}$ is d-syndetic and $X = X_0 \cup X_1$ is a 2-partition, then either each X_i is piecewise d-syndetic or one of them is syndetic.*

Proof. Suppose X_0 is not syndetic. This means that for arbitrarily long intervals I of $\mathbb{N}$ we have $X_0 \cap I = \emptyset$ and so $X \cap I = X_1 \cap I$. Therefore X_1 is piecewise d-syndetic. □

Theorem 5.2 (RCA_0). $\mathsf{I}\Sigma^0_2$ *implies Brown's Lemma.*

Proof. Let $C\colon \mathbb{N} \to r$ be a finite coloring. By bounded Σ^0_2-comprehension let

$$I = \{A \subseteq r\colon \bigcup_{i \in A} \{x \in \mathbb{N}\colon C(x) = i\} \text{ is syndetic}\}.$$

I is nonempty as $r \in I$. Let $A \in I$ be minimal (w.r.t. $\subseteq$). Notice that $A \neq \emptyset$. Suppose that the union is d-syndetic. By Lemma 5.1 and by the minimality of I, for every $i \in A$ the set $\{x \in \mathbb{N}\colon C(x) = i\}$ must be piecewise d-syndetic. □

Note that the proof of Theorem 5.2 does not generalize to partitions of piecewise syndetic sets. However we have the following:

Theorem 5.3. *Over* RCA^*_0, *the following are equivalent:*

(1) *Brown's lemma;*
(2) *partition regularity of piecewise syndetic sets.*

Proof. We argue in RCA^*_0 and assume Brown's lemma. In particular we have Σ^0_1-induction. Let $X = X_0 \cup X_1 \ldots \cup X_{r-1}$ be a finite partition of X and suppose that X is piecewise d-syndetic. By primitive recursion we can define a piecewise d-syndetic subset $\{x_0 < x_1 < x_2 < \ldots\}$ of X such that for every $n > 0$ the n-element subset $I_n = \{x_k, x_{k+1}, x_{k+2}, \ldots, x_{k+n-1}\} \subseteq X$ has gaps bounded by d, where $k = 1 + 2 + \ldots + (n-1)$. Define $C\colon \mathbb{N} \to r$ by letting $C(n) = i$ iff $x_n \in X_i$. By Brown's Lemma, there exists a C-homogeneous piecewise syndetic set Y. Say Y is piecewise e-syndetic and homogeneous for color i.

We claim that $Z = \{x_n\colon n \in Y\} \subseteq X_i$ is piecewise $e \cdot d$-syndetic. To this end it is enough to show that for every n there exists an n-element subset A of Y such that $gs(A) \leq e$ and $\{x_n\colon n \in A\} \subseteq I_m$ for some m. Let $n > 0$ be given. Set $k = 1 + 2 + \ldots + (2ne - 1)$ so that $x_k \in I_{2ne}$. Now consider $p = k + 2n$. As Y is piecewise e-syndetic, there exists a p-element subset $\{i_0 < \ldots < i_{p-1}\}$ of Y with gaps bounded by e. Let $A = \{i_k < i_{k+1} < \ldots < i_{p-1}\}$. Notice that $|A| = 2n$ and $gs(A) \leq e$. Since $i_k \geq k$ we have that $x_{i_k} \in I_m$ with $|I_m| \geq 2ne$. As $gs(A) \leq e$ we have that $i_{p-1} - i_k \leq 2ne$ and so $|\{x_i\colon i_k \leq i \leq i_p\}| \leq 2ne + 1$. It follows that $\{x_i\colon i_k \leq i \leq i_p\} \subseteq I_m \cup I_{m+1}$. This ensures that $\{x_i\colon i \in A\} \subseteq I_m \cup I_{m+1}$. Then either the first n elements are in I_m or the last n elements are in I_{m+1}. In both cases we are done. □

We now turn to the reversal. Actually we show that the following weak version of Brown's lemma already implies $\mathsf{I}\Sigma^0_2$ over RCA^*_0.

Theorem 5.4 (weak Brown's lemma)**.** *For every coloring $C\colon \mathbb{N} \to r$ there exists $d \in \mathbb{N}$ such that $gs(H) \leq d$ for arbitrarily large C-homogeneous sets H.*

Theorem 5.5. *Over RCA^*_0, the following are equivalent:*
(1) $\mathsf{I}\Sigma^0_2$*;*
(2) *Brown's lemma;*
(3) *partition regularity of piecewise syndetic sets;*
(4) *weak Brown's lemma.*

We use the following diagonalization lemma.

Lemma 5.6 (RCA^*_0)**.** *There exists a function $D\colon \mathbb{N}\times\mathbb{N} \to 2$ such that for all d the 2-coloring $D(d,\cdot)$ has no arbitrarily large C-homogeneous sets H with $gs(H) \leq d$.*

Proof. For all $d > 0$ let

$$D(d,\cdot) = \overbrace{0\dots0}^{d\text{ times}}\overbrace{1\dots1}^{d\text{ times}}\overbrace{0\dots0}^{d\text{ times}}\overbrace{1\dots1}^{d\text{ times}}\dots$$

We define D by $D(d,x) = \lfloor \frac{x}{d} \rfloor \pmod 2$. Fix $d > 0$ and let $C(x) = D(d,x)$. Suppose H is C-homogeneous with gaps bounded by d. We claim that $|H| \leq d$. Let $H = \{x_0 < x_1 < \ldots < x_l\}$ and $m = \lfloor \frac{x_0}{d} \rfloor$ so that $md \leq x_0 < (m+1)d$. We show by induction that $x_l < (m+1)d$ for all l. The case $l = 0$ is true. Suppose $x_l < (m+1)d$. Since $x_{l+1} - x_l \leq d$ we have that $x_{l+1} < (m+2)d$. If $(m+1)d \leq x_{l+1}$ then $\lfloor \frac{x_{l+1}}{d} \rfloor = m+1$ and $C(x_{l+1}) = m+1 \pmod 2 \neq m \pmod 2 = C(x_l)$, against H being homogeneous. Therefore $H \subseteq [md,(m+1)d)$ and hence $|H| \leq d$. □

Proof of Theorem 5.5. Implication (1) → (2) is Theorem 5.2 and (2) ↔ (3) is Theorem 5.3. Clearly (2) → (4). It remains to show (4) → (1).

Within $\mathsf{I}\Sigma^0_0$, one can show that $\mathsf{I}\Sigma^0_{n+1}$ is equivalent to $\mathsf{S}\Pi^0_n$, strong collection for Π^0_n-formulas (see [HP93, Ch. I Sect. 2(b)]). We first assume $\mathsf{I}\Sigma^0_1$ and prove $\mathsf{S}\Pi^0_1$, that is,

$$(\forall a)(\exists d)(\forall x < a)\big(\exists y\forall z\theta(x,y,z) \to (\exists y < d)(\forall z)\theta(x,y,z)\big),$$

where θ is Σ^0_0. Let $f(x,s)$ be the least $y \leq s$ such that $(\forall z < s)\theta(x,y,z)$, and s if there is no such a y. By Σ^0_1-induction one can show that if $(\exists y)(\forall z)\theta(x,y,z)$ then $f(x) = \lim_{s\to\infty} f(x,s) =$ the least y such that $(\forall z)\theta(x,y,z)$.

Define a coloring $C\colon \mathbb{N} \to 2^a$ as follows. Let $D\colon \mathbb{N}\times\mathbb{N} \to 2$ be the function of Lemma 5.6 and set $C(s) = \langle D(f(x,s),s)\colon x < a\rangle$. By (4) there exists d such that $gs(H) \leq d$ for arbitrarily large C-homogeneous sets H. We claim that d is as desired. Let $x < a$ and suppose that $f(x)$ exists. Fix s such that $f(x,t) = f(x)$ for all $t > s$. We aim to prove that $D(f(x), \cdot)$ has arbitrarily large homogeneous sets H with $gs(H) \leq d$ and therefore it must be $f(x) < d$. Let k be given. By the assumption, there exists a C-homogeneous set G of size $s + k + 1$ and $gs(G) \leq d$. Let H consist of the last k elements of G. Then $|H| = k$, $gs(H) \leq d$ and $s < t$ for all $t \in H$. Suppose G is C-homogeneous for color i. Then for all $t \in H$ we have that $i(x) = C(t)(x) = D(f(x,t),t) = D(f(x),t)$ and hence H is homogeneous for $D(f(x), \cdot)$.

It remains to show that (4) implies $\mathsf{I}\Sigma^0_1$ over RCA^*_0. We can apply the argument above to prove $\mathsf{S}\Pi^0_0$, that is,

$$(\forall a)(\exists d)(\forall x < a)\big(\exists y \theta(x,y,z) \to (\exists y < d)\theta(x,y,z)\big),$$

where θ is Σ^0_0. Define $f(x,s)$ to be the least $y < s$ such that $\theta(x,y)$ if y exists and s otherwise. Now Σ^0_0-induction is sufficient to show that if $\exists y \theta(x,y)$ then $f(x) = \lim_{s \to \infty} f(x,s) =$ the least y such that $\theta(x,y)$. Define C as before by using D and show that d is as desired. ☐

6. Finite Brown's Lemma

Theorem 6.1 (Brown's lemma, finite)**.** *Let $f \colon \mathbb{N} \to \mathbb{N}$. Then for all $r > 0$ there exists n such that every coloring $C \colon n \to r$ has a C-homogeneous set H with $|H| > f(gs(H))$.*

Proof. From Brown's Lemma by using König's Lemma. ☐

Theorem 6.2. *The finite version of Brown's Lemma is provable in* RCA_0.

Proof. The proof is by Σ^0_1-induction on the number of colors (see Brown [Bro81, Fact 3]). Here we follow the proof given in [LR04, Theorem 10.33]. Let $f \colon \mathbb{N} \to \mathbb{N}$ be given. Without loss of generality we may assume that f is nondecreasing, for otherwise we can define $g(n) = \sum_{i \leq n} f(i)$. The proof is by Σ^0_1-induction on r.

For $r = 1$, let $n_1 = f(1) + 1$. Suppose n_r works for r. We claim that $n_{r+1} = (r+1)f(n_r) + 1$ works for $r + 1$. Let $C \colon n_{r+1} \to r + 1$ be any coloring. Set $H_i = \{x < n_{r+1} \colon C(x) = i\}$ for $i \leq r$. We may assume that $|H_i| \leq f(gs(H_i))$ for every $i \leq r$, for otherwise we are done. We may also assume that $gs(H_i) \leq n_r$ for every $i \leq r$, for otherwise there exists an interval of size n_r which avoids some color i, and the conclusion follows by induction. As f is nondecreasing, we have that $f(gs(H_i)) \leq f(n_r)$ for all $i \leq r$. Therefore, $|H_i| \leq f(n_r)$ for every $i \leq r$, and

$$n_{r+1} = \sum_{i \leq r} |H_i| \leq (r+1) f(n_r),$$

a contradiction. □

We define Brown numbers as follows.

Definition 6.3. For $f\colon \mathbb{N} \to \mathbb{N}$ and $r > 0$ let $B_f(r)$ be the least natural number n such that every r-coloring of n has a homogeneous set H with $|H| > f(gs(H))$.

Remember that the superexponential function $2_k(n)$ is defined by $2_0(n) = n$ and $2_{k+1}(n) = 2^{2_k(n)}$. Let $2_r = 2_r(1)$. The proof of Theorem 6.2 gives superexponential upper bounds n_r for $B_f(r)$. For instance, if $f(d) = 2^d$, then $n_r \geq 2_r$.

Theorem 6.4 (Ardal [Ard10])**.** *Let $f(d) = md$ with $m > 0$. Then for all $r > 0$, $B_f(r) \leq r(2^{mr} - mr) + 1$.*

Recall that, for $f\colon \mathbb{N} \to \mathbb{N}$, BL_f is the statement "For all $r > 0$ there exists n such that every $C\colon n \to r$ has a C-homogeneous set H such that $|H| > f(gs(H))$.

Theorem 6.5. *The statement $(\forall m > 0)\mathsf{BL}_{d \mapsto md}$ is provable in* EFA.

Proof. The reader can check that the proofs of Lemmas 10, 11, 12, 13, 14, 22, 23 and Theorems 15, 24 of [Ard10] can be formalized in EFA. □

We aim to show that RCA_0^* does not prove the finite version of Brown's lemma by proving that $B_f(r)$ is not elementary recursive for $f(d) = 2^d$.

Proposition 6.6. *The function $g(n) = 2_{\log_2(n)}$ is not elementary recursive.*

Proof. Note that for $n = 2_{k+2}$ we have $2_k(n) \leq 2_{\log_2(n)}$. Therefore there is no k such that $2_{\log_2(n)} < 2_k(n)$ for all n. □

Theorem 6.7. *Let $f(d) = 2^d$. Then for all $r > 0$, $B_f(r) > 2_{\log_2(r)}$.*

Proof. To ease notation, write $B(r)$ for $B_f(r)$ with $f(d) = 2^d$. For all $s \geq 0$ we define a number $n_s \geq 2_s$ and a coloring $C_s\colon n_s \to 2^s$ witnessing $B(2^s) > n_s$.

In general, for $f\colon \mathbb{N} \to \mathbb{N}$ nondecreasing, $B_f(r) > n$ iff there exists a coloring $C\colon n \to r$ such that for all $i < r$ the finite set $H = \{x < n\colon C(x) = i\}$ satisfies

$$(*) \qquad |I| \leq f(gs(I)) \text{ for every interval } I \text{ of } H.$$

Notice that $(*)$ is shift invariant. For finite sets $A < B$ let $d(A, B) = \min B - \max A$. Note that if A satisfies $(*)$ and $m \cdot |A| \leq f(d)$, then the set $H = \bigcup_{l<m} A_l$ satisfies $(*)$, where $A_0 < A_1 < \ldots < A_{m-1}$, each A_l is a shift of A and $d(A_l, A_{l+1}) = d$.

We ensure that for all s and for all $i < 2^s$ the set $\{x < n_s \colon C_s(x) = i\}$ satisfies $(*)$ with $f(d) = 2^d$. For $s = 0$, let $n_0 = 2$ and $C_0 = 00$. For $s = 1$, let $n_1 = 2^4 = 16$ and

$$C_1 = 0011001100110011.$$

Suppose we have defined n_s and $C_s \colon n_s \to 2^s$. Let $n_{s+1} = 2^{s+1} \cdot 2^{n_0+n_1+\ldots+n_s+1}$. Define $C_{s+1} \colon n_{s+1} \to 2^{s+1}$ by

$$C_{s+1} = C_s D_s C_s D_s \ldots C_s D_s,$$

where $C_s D_s$ is repeated 2^{n_s} times and D_s is a copy of C_s with the color $i < 2^s$ replaced by $i+2^s$. By induction one can show that $n_{s+1} = 2n_s 2^{n_s}$.

Claim. For all s and for all $i < 2^s$ the set $H_{i,s} = \{x < n_s \colon C_s(x) = i\}$ satisfies $(*)$.

By induction on s we prove that for all $i < 2^s$:
(1) $|H_{i,s}| = n_s/2^s$;
(2) $n_s = \max H_{i,s} - \min H_{i,s} + n_0 + \ldots + n_{s-1} + 1$;
(3) $H_{i,s}$ satisfies $(*)$;
For $s = 0$ this is true. Suppose it is true for s and let $m = 2^{n_s}$. Fix $i < 2^{s+1}$. By construction either $i < 2^s$ and $H_{i,s+1} = \bigcup_{l<m} A_l$, where $A_0 < A_1 < \ldots < A_{m-1}$ and $A_l = H_{i,s} + 2ln_s$, or $i = j + 2^s$ with $j < 2^s$ and $H_{i,s+1} = \bigcup_{l<m} A_l$, where $A_0 < A_1 < \ldots < A_{m-1}$ and $A_l = H_{j,s} + (2l + 1)n_s$. Suppose $i < 2^s$. The argument for $i = j + 2^s$ is similar. Then

$$|H_{i,s+1}| = m \cdot |H_{i,s}| \overset{(1)}{=} m \cdot \frac{n_s}{2^s} = 2^{n_s} \cdot \frac{n_s}{2^s} = \frac{n_{s+1}}{2^{s+1}}.$$

Let $a = \min H_{i,s+1}$ and $b = \max H_{i,s+1}$. We aim to show that $n_{s+1} = b - a + n_0 + \ldots + n_s + 1$. Write n_{s+1} as $a + (b - a) + (n_{s+1} - b)$. Now

$$a = \min H_{i,s+1} = \min H_{i,s}$$

and

$$n_{s+1} - b = n_{s+1} - \max H_{i,s+1} = (n_s - \max H_{i,s}) + n_s.$$

It follows that

$$n_{s+1} = \min H_{i,s} + (b - a) + n_s - \max H_{i,s} + n_s =$$

$$b-a+n_s+(n_s+\min H_{i,s}-\max H_{i,s}) \overset{(2)}{=} b-a+n_s+(n_0+\ldots+n_{s-1}+1) =$$
$$b - a + n_0 + \ldots n_s + 1.$$

It remains to show that $H_{i,s+1}$ satisfies $(*)$. By induction $H_{i,s}$ satisfies $(*)$ and so every A_l satisfies $(*)$ because $(*)$ is invariant under shift. Since $m \cdot |A_0| = |H_{i,s+1}| = 2^{n_s} \cdot \frac{n_s}{2^s} = 2^{n_0+\ldots+n_s+1}$, it is sufficient to show that $d(A_l, A_{l+1}) = d(A_0, A_1) = n_0 + n_1 + \ldots + n_s + 1$. Now

$$d(A_0, A_1) = n_s + \min H_{i,s} + (n_s - \max H_{i,s}) \stackrel{(2)}{=} n_s + (n_0 + \ldots + n_{s-1} + 1) = n_0 + \ldots + n_s + 1.$$

This completes the proof of the claim. By induction it is easy to prove that $n_s \geq 2_s$. It follows that

$$B(r) \geq B(2^{\log_2(r)}) > n_{\log_2(r)} \geq 2_{\log_2(r)},$$

as desired. □

Notice that Theorem 6.7 is provable in $\mathsf{I}\Sigma^0_1$.

Corollary 6.8. *The statement $\mathsf{BL}_{d \mapsto 2^d}$ is not provable in EFA. In particular, RCA_0^* does not prove the finite version of Brown's lemma.*

Proof. Let $f(d) = 2^d$. The function $B_f(r)$ is Δ_0-definable and so EFA proves BL_f iff $B_f(r)$ is provably recursive in EFA iff $B_f(r)$ is elementary recursive. By Theorem 6.7 the function $B_f(r)$ is not elementary recursive.

The second part follows from the fact that the statement BL_f is Π_2 and RCA_0^* is conservative over EFA for Π_2-sentences. □

As the finite van der Waerden's theorem is already provable in RCA_0 and presumably in RCA_0^*, the question whether Brown's lemma implies van der Waerden's theorem can be settled only over a very weak system of arithmetic.

Question. What is the relationship between (the finite versions of) Brown's lemma and van der Waerden's theorem over a suitable bounded fragment of second-order arithmetic?

Appendix A.

The classical proof of Brown's lemma (see for instance [Bro71, Lemma 1] or [LR04, Theorem 10.32]) is based on the following fact.

Lemma A.1. *Let $r \in \mathbb{N}$. If $S \subseteq r^{<\mathbb{N}}$ is infinite, then there exists $g\colon \mathbb{N} \to r$ such that for all k there is $\sigma \in S$ with $g \restriction k \subseteq \sigma$.*

Here we use the following notation. Given $r \in \mathbb{N}$, $r^{<\mathbb{N}}$ denotes the set of finite sequences of natural numbers in $\{0, \ldots, r-1\}$. Given $k \in \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{N}$, $g \restriction k$ denotes the finite sequence $\langle g(0), \ldots, g(k-1)\rangle$. By $\sigma \subseteq \tau$ we mean that τ is an extension of σ.

Classical proof of Brown's lemma. By induction on r. If $r = 1$, there is nothing to prove. Let $C\colon \mathbb{N} \to r+1$. We say that $\sigma \in \mathbb{N}^{<\mathbb{N}}$ is a factor of C, and write $\sigma \in S(C)$, if $\sigma \subseteq \langle C(y), C(y+1), C(y+2), \ldots\rangle$ for some y. Let S be the set of factors which avoid the color r, that is, $S = S(C) \cap r^{<\mathbb{N}}$.

We may assume that $\{x \in \mathbb{N}\colon C(x) = r\}$ is not syndetic, otherwise we are done. We claim that for every d there is $\sigma \in S$ of length d. Suppose not. Let d be such that every $\sigma \in S(C)$ of length d does not avoid r. Then it is easy to see that $\{x \in \mathbb{N}\colon C(x) = r\}$ is d-syndetic, contrary to our assumption. Therefore S is infinite.

By Lemma A.1, there exists $g\colon \mathbb{N} \to r$ such that for all n there exists $\sigma \in S$ with $g \upharpoonright n \subseteq \sigma$. By induction, let $i < r$ such that $\{x \in \mathbb{N}\colon g(x) = i\}$ is piecewise syndetic and let d be a witness. We claim that $\{x \in \mathbb{N}\colon C(x) = i\}$ is piecewise d-syndetic. Fix n. There exists a set F of size n and gaps bounded by d such that $g(x) = i$ for all $x \in F$. Let $k > F$. By the assumption on g, let $\sigma \in S$ such that $g \upharpoonright k \subseteq \sigma$. Now, σ is a factor of C, say $\sigma \subseteq \langle C(y), C(y+1), \ldots\rangle$. Hence, for all $x < k$ we have that $g(x) = C(y+x)$. It easily follows that $y + F$ is as desired (size n, gaps bounded by d, C-homogeneous for color i). □

A loose formalization of the above proof goes through ACA_0 plus Π^1_1-induction. In particular, Lemma A.1 is provable in ACA_0 and actually equivalent to ACA_0.

Proposition A.2 (RCA_0). *The following are equivalent:*

(1) ACA_0
(2) *for any $r \in \mathbb{N}$, if $S \subseteq r^{<\mathbb{N}}$ is infinite, then there exists $g\colon \mathbb{N} \to r$ such that for all n there is $\sigma \in S$ with $g \upharpoonright n \subseteq \sigma$.*
(3) *if $S \subseteq 2^{<\mathbb{N}}$ is infinite, then there exists $g\colon \mathbb{N} \to 2$ such that for all n there is $\sigma \in S$ with $g \upharpoonright n \subseteq \sigma$.*

Proof. (1) implies (2). Define a tree $T \subseteq r^{<\mathbb{N}}$ by letting $\tau \in T$ iff $(\exists \sigma \in S)\tau \subseteq \sigma$. T is an infinite finitely branching tree. By König's Lemma, there exists an infinite path g. Then g is as desired. (2) implies (3) is trivial.

For (3) implies (1), it is enough to show that the range of every one-to-one function $f\colon \mathbb{N} \to \mathbb{N}$ exists (see Simpson [Sim09, Lemma III.1.3]). Let $f\colon \mathbb{N} \to \mathbb{N}$ be given. Define $S \subseteq 2^{<\mathbb{N}}$ by letting $\sigma \in S$ if and only if $(\forall y < |\sigma|)(\sigma(y) = 1 \leftrightarrow (\exists x < |\sigma|) f(x) = y)$. Then S is infinite. By (3), let g be such that every initial segment of g is an initial segment of some $\sigma \in S$. We claim that $y \in \mathrm{range}(f)$ iff $g(y) = 1$. Suppose $g(y) = 1$ and let $g \upharpoonright y + 1 \subseteq \sigma$ with $\sigma \in S$. Then $\sigma(y) = 1$ and so $y \in \mathrm{range}(f)$.

Suppose that $y \in \text{range}(f)$ and let $f(x) = y$. Let $n > x, y$ and $\sigma \in S$ such that $g \restriction n \subseteq \sigma$. Then $\sigma(y) = 1$ and so $g(y) = 1$. $\square$

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Mathematical Institute, Tohoku University, Japan
E-mail address: frittaion@math.tohoku.ac.jp
URL: http://www.math.tohoku.ac.jp/~frittaion/