LINEAR EXTENSIONS OF PARTIAL ORDERS AND REVERSE MATHEMATICS

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Abstract. We introduce the notion of τ-like partial order, where τ is one of the linear order types ω, ω^*, $\omega+\omega^*$, and ζ. For example, being ω-like means that every element has finitely many predecessors, while being ζ-like means that every interval is finite. We consider statements of the form "any τ-like partial order has a τ-like linear extension" and "any τ-like partial order is embeddable into τ" (when τ is ζ this result appears to be new). Working in the framework of reverse mathematics, we show that these statements are equivalent either to $\mathsf{B}\mathbf{\Sigma}^0_2$ or to ACA_0 over the usual base system RCA_0.

1. Introduction

Szpilrajn's Theorem ([Szp30]) states that any partial order has a linear extension. This theorem raises many natural questions, where in general we search for properties of the partial order which are preserved by some or all its linear extensions. For example it is well-known that a partial order is a well partial order if and only if all its linear extensions are well-orders.

A question which has been widely considered is the following: given a linear order type τ, is it the case that any partial order that does not embed τ can be extended to a linear order that also does not embed τ? If the answer is affirmative, τ is said to be extendible, while τ is weakly extendible if the same holds for any countable partial order. For instance, the order types of the natural numbers, of the integers, and of the rationals are extendible. Bonnet ([Bon69]) and Jullien ([Jul69]) characterized all countable extendible and weakly extendible linear order types respectively.

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We are interested in a similar question: given a linear order type τ and a property characterizing τ and its suborders, is it true that any partial order which satisfies that property has a linear extension which also satisfies the same property? In our terminology: does any τ-like partial order have a τ-like linear extension? Here we address this question for the linear order types ω, ω^* (the inverse of ω), $\omega+\omega^*$ and ζ (the order of integers). So, from now on, τ will denote one of these.

Definition 1.1. Let $(P, \leq_P)$ be a countable partial order. We say that P is

- ω-*like* if every element of P has finitely many predecessors;
- ω^*-*like* if every element of P has finitely many successors;
- $\omega+\omega^*$-*like* if every element of P has finitely many predecessors or finitely many successors;
- ζ-*like* if for every pair of elements $x, y \in P$ there exist only finitely many elements z with $x <_P z <_P y$.

The previous definition resembles Definition 2.3 of Hirschfeldt and Shore ([HS07]), where linear orders of type ω, ω^* and $\omega + \omega^*$ are introduced. The main difference is that the order properties defined by Hirschfeldt and Shore are meant to uniquely determine a linear order type up to isomorphism, whereas our definitions apply to partial orders in general and do not determine an order type. Notice also that, for instance, an ω-like partial order is also $\omega + \omega^*$-like and ζ-like.

We introduce the following terminology:

Definition 1.2. We say that τ is *linearizable* if every τ-like partial order has a linear extension which is also τ-like.

With this definition in hand, we are ready to formulate the results we want to study:

Theorem 1.3. *The following hold:*

(1) ω *is linearizable;*
(2) ω^* *is linearizable;*
(3) $\omega + \omega^*$ *is linearizable;*
(4) ζ *is linearizable.*

A proof of the linearizability of ω can be found in Fraïssé's monograph ([Fra00, §2.15]), where the result is attributed to Milner and Pouzet. (2) is similar to (1) and the proof of (3) easily follows from (1) and (2). The linearizability of ζ is apparently a new result (for a proof see Lemma 3.2 below).

In this paper we study the statements contained in Theorem 1.3 from the standpoint of reverse mathematics (the standard reference is [Sim09]), whose goal is to characterize the axiomatic assumptions needed to prove mathematical theorems. We assume the reader is familiar with systems such as RCA_0 and ACA_0. The reverse mathematics of weak extendibility is studied in [DHLS03] and [Mon06]. The existence of maximal linear extensions of well partial orders is studied from the reverse mathematics viewpoint in [MS11].

Our main result is that the linearizability of τ is equivalent over RCA_0 to the $\mathbf{\Sigma}^0_2$ bounding principle $\mathsf{B}\mathbf{\Sigma}^0_2$ when $\tau \in \{\omega, \omega^*, \zeta\}$, and to ACA_0 when $\tau = \omega + \omega^*$. For more details on $\mathsf{B}\mathbf{\Sigma}^0_2$, including an apparently new equivalent (simply asserting that a finite union of finite sets is finite), see §2 below.

The linearizability of ω appears to be the first example of a genuine mathematical theorem (actually appearing in the literature for its own interest, and not for its metamathematical properties) that turns out to be equivalent to $\mathsf{B}\mathbf{\Sigma}^0_2$.

To round out our reverse mathematics analysis, we also consider a notion closely related to linearizability:

Definition 1.4. We say that τ is *embeddable* if every τ-like partial order P embeds into τ, that is there exists an order preserving map from P to τ.[1]

It is rather obvious that τ is linearizable if and only if τ is embeddable. Let us notice that RCA_0 easily proves that embeddable implies linearizable. Not surprisingly, the converse is not true. In fact, we show that embeddability is strictly stronger when $\tau \in \{\omega, \omega^*, \zeta\}$, and indeed equivalent to ACA_0. The only exception is given by $\omega + \omega^*$, for which both properties are equivalent to ACA_0.

We use the following definitions in RCA_0.

Definition 1.5 (RCA_0)**.** Let $\leq$ denote the usual ordering of natural numbers. The linear order ω is $(\mathbb{N}, \leq)$, while ω^* is $(\mathbb{N}, \geq)$.

Let $\{P_i \colon i \in Q\}$ be a family of partial orders indexed by a partial order Q. The *lexicographic sum* of the P_i along Q, denoted by $\sum_{i \in Q} P_i$, is the partial order on the set $\{(i, x) \colon i \in Q \wedge x \in P_i\}$ defined by

$$(i, x) \leq (j, y) \iff i <_Q j \vee (i = j \wedge x \leq_{P_i} y).$$

The *sum* $\sum_{i<n} P_i$ can be regarded as the lexicographic sum along the n-element chain. In particular $P_0 + P_1$ is the lexicographic sum along the 2-element chain (and we have thus defined $\omega + \omega^*$ and $\zeta = \omega^* + \omega$).

Similarly, the *disjoint sum* $\bigoplus_{i<n} P_i$ is the lexicographic sum along the n-element antichain.

[1] To formalize this definition in RCA_0, we need to fix a canonical representative of the order type τ, which we do in Definition 1.5.

2. $\mathbf{\Sigma}^0_2$ BOUNDING AND FINITE UNION OF FINITE SETS

Let us recall that $\mathsf{B}\mathbf{\Sigma}^0_2$ (standing for $\mathbf{\Sigma}^0_2$ bounding, and also known as $\mathbf{\Sigma}^0_2$ collection) is the scheme:

$$(\mathsf{B}\mathbf{\Sigma}^0_2) \quad (\forall i < n)(\exists m)\varphi(i, n, m) \implies (\exists k)(\forall i < n)(\exists m < k)\varphi(i, n, m),$$

where φ is any $\mathbf{\Sigma}^0_2$ formula.

It is well-known that RCA_0 does not prove $\mathsf{B}\mathbf{\Sigma}^0_2$, which is strictly weaker than $\mathbf{\Sigma}^0_2$ induction. Neither of WKL_0 and $\mathsf{B}\mathbf{\Sigma}^0_2$ implies the other and Hirst ([Hir87], for a widely available proof see [CJS01, Theorem 2.11]) showed that RT^2_2 (Ramsey theorem for pairs and two colors) implies $\mathsf{B}\mathbf{\Sigma}^0_2$.

A few combinatorial principles are known to be equivalent to $\mathsf{B}\mathbf{\Sigma}^0_2$ over RCA_0.

Hirst ([Hir87], for a widely available proof see [CJS01, Theorem 2.10]) showed that, over RCA_0, $\mathsf{B}\mathbf{\Sigma}^0_2$ is equivalent to the infinite pigeonhole principle, i.e. the statement

$(\mathsf{RT}^1_{<\infty})$

$$(\forall n)(\forall f : \mathbb{N} \to n)(\exists A \subseteq \mathbb{N} \text{ infinite})(\exists c < n)(\forall m \in A)(f(m) = c).$$

(The notation arises from viewing the infinite pigeonhole principle as Ramsey theorem for singletons and an arbitrary finite number of colors.)

Chong, Lempp and Yang ([CLY10]) showed that a combinatorial principle PART about infinite $\omega + \omega^*$ linear orders, introduced by Hirschfeldt and Shore ([HS07, §4]), is also equivalent to $\mathsf{B}\mathbf{\Sigma}^0_2$. More recently, Hirst ([Hir12]) also proved that $\mathsf{B}\mathbf{\Sigma}^0_2$ is equivalent to a statement apparently similar to Hindman's theorem, but much weaker from the reverse mathematics viewpoint.

We consider the statement that a finite union of finite sets is finite:

$$(\mathsf{FUF}) \qquad (\forall i < n)(X_i \text{ is finite}) \implies \bigcup_{i<n} X_i \text{ is finite}.$$

Here "X is finite" means $(\exists m)(\forall x \in X)(x < m)$. This statement can be viewed as a second-order version of Π_0 regularity, which in the context of first-order arithmetic is known to be equivalent to Σ_2 bounding (see e.g. [**?**, Theorem 2.23.4]).

Lemma 2.1. *Over* RCA_0, $\mathsf{B}\mathbf{\Sigma}^0_2$ *is equivalent to* FUF.

Proof. First notice that FUF follows immediately from the instance of $\mathsf{B}\mathbf{\Sigma}^0_2$ relative to the $\mathbf{\Pi}^0_1$, and hence $\mathbf{\Sigma}^0_2$, formula $(\forall x \in X_i)(x < m)$.

For the other direction we use Hirst's result recalled above: it suffices to prove that FUF implies $\mathsf{RT}^1_{<\infty}$. Let $f : \mathbb{N} \to n$ be given. Define for each $i < n$ the set $X_i = \{m : f(m) = i\}$. Clearly $\bigcup_{i<n} X_i = \mathbb{N}$ is infinite. By FUF, there exists $i < n$ such that X_i is infinite. Now X_i is an infinite homogeneous set for f. □

3. Linearizable types

Notice that Szpilrajn's Theorem is easily seen to be computably true (see [Dow98, Observation 6.1]) and provable in RCA_0. We use this fact several times without further notice.

We start by proving that $\mathsf{B}\mathbf{\Sigma}^0_2$ suffices to establish the linearizability of ω, ω^* and ζ.

Lemma 3.1. *RCA_0 proves that $\mathsf{B}\mathbf{\Sigma}^0_2$ implies the linearizability of ω and ω^*.*

Proof. We argue in RCA_0 and, by Lemma 2.1, we may assume FUF. Let us consider first ω. So let P be an ω-like partial order which, to avoid trivialities, we may assume to be infinite. We recursively define a sequence $z_n \in P$ by letting z_n be the least (w.r.t. the usual ordering of $\mathbb{N}$) $x \in P$ such that $(\forall i < n)(x \not\leq_P z_i)$.

We show by $\mathbf{\Sigma}^0_1$ induction that z_n is defined for all $n \in \mathbb{N}$. Suppose that z_i is defined for all $i < n$. We want to prove $(\exists x \in P)(\forall i < n)(x \not\leq_P z_i)$. Define $X_i = \{x \in P \colon x \leq_P z_i\}$ for $i < n$. Since P is ω-like, each X_i is finite. By FUF, $\bigcup_{i<n} X_i$ is also finite. The claim follows from the fact that P is infinite.

Now define for each $n \in \mathbb{N}$ the finite set

$$P_n = \{x \in P \colon x \leq_P z_n \land (\forall i < n)(x \not\leq_P z_i)\}.$$

It is not hard to see that the P_n's form a partition of P, and that if $x \leq_P y$ with $x \in P_i$ and $y \in P_j$, then $i \leq j$. Then let L be a linear extension of the lexicographic sum $\sum_{n\in\omega} P_n$. L is clearly a linear order and extends P by the remark above. To prove that L is ω-like, note that the set of L-predecessors of an element of P_n is included in $\bigcup_{i\leq n} P_i$, which is finite, by FUF again.

For ω^*, repeat the same construction using $\geq_P$ in place of $\leq_P$, and let L be a linear extension of $\sum_{n\in\omega^*} P_n$. □

Lemma 3.2. *RCA_0 proves that $\mathsf{B}\mathbf{\Sigma}^0_2$ implies the linearizability of ζ.*

Proof. In RCA_0 assume FUF. Let P be a ζ-like partial order, which we may again assume to be infinite. It is convenient to use the notation $[x, y]_P = \{z \in P \colon x \leq_P z \leq_P y \lor y \leq_P z \leq_P x\}$, so that $[x, y]_P \neq \emptyset$ if and only if x and y are comparable.

We define by recursion a sequence $z_n \in P$ by letting z_n be the least (w.r.t. the ordering of $\mathbb{N}$) $x \in P$ such that

$$x \notin \bigcup_{i,j<n} [z_i, z_j]_P.$$

As before, since P is infinite and ζ-like, one can prove using $\mathbf{\Sigma}^0_1$ induction and FUF that z_n is defined for every $n \in \mathbb{N}$. It is also easy to prove that

$$P = \bigcup_{i,j \in \mathbb{N}} [z_i, z_j]_P.$$

Define for each $n \in \mathbb{N}$ the set

$$P_n = \bigcup_{i<n} [z_i, z_n]_P \setminus \bigcup_{i,j<n} [z_i, z_j]_P.$$

By FUF, the P_n's are finite. Moreover, they clearly form a partition of P. Note also that $z_n \in P_n$ and every element of P_n is comparable with z_n. Furthermore, every interval $[x, y]_P$ is included in some $[z_i, z_j]_P$. Notice that the same holds for any partial order extending $\leq_P$.

We now extend $\leq_P$ to a partial order $\preceq_P$ such that any linear extension of $(P, \preceq_P)$ is ζ-like. We say that n is left if $z_n \leq_P z_i$ for some $i < n$; otherwise, we say that n is right. Notice that, since $z_n \in P_n$, n is right if and only if $z_i \leq_P z_n$ for some $i < n$ or z_n is incomparable with every z_i with $i < n$.

The order $\preceq_P$ places P_n below or above every P_i with $i < n$ depending on whether n is left or right. Formally, for $x, y \in P$ such that $x \in P_n$ and $y \in P_m$ let

$$x \preceq_P y \iff (n = m \wedge x \leq_P y) \vee (n < m \wedge m \text{ is right}) \vee (m < n \wedge n \text{ is left}).$$

We claim that $\preceq_P$ extends $\leq_P$. Let $x \leq_P y$ with $x \in P_n$ and $y \in P_m$. If $n = m$, $x \preceq_P y$ by definition. Suppose now that $n < m$, so that we need to prove that m is right. As $x \in P_n$, $z_i \leq_P x$ for some $i \leq n$. Since $y \in P_m$, y is comparable with z_m. Suppose that $z_m <_P y$. Then $y \leq_P z_j$ for some $j < m$, and so $z_i \leq_P x \leq_P y \leq_P z_j$ with $i, j < m$, contrary to $y \in P_m$. It follows that $y \leq_P z_m$ and thereby $z_i \leq_P z_m$ with $i < m$. Therefore, m is right, as desired. The case $n > m$ (where we need to prove that n is left) is similar.

We claim that $(P, \preceq_P)$ is still ζ-like. To see this, it is enough to show that for all $i, j < n$

$$\{x \in P \colon z_i \preceq_P x \preceq_P z_j\} \subseteq \bigcup_{k<n} P_k$$

and apply FUF. Let $x \in P_k$ be such that $z_i \prec_P x \prec_P z_j$. Suppose, for a contradiction, that $k \geq n$ and hence that $i, j < k$. By the definition of $\preceq_P$, $z_i \prec_P x$ implies that k is right. At the same time, $x \prec_P z_j$ implies that k is left, a contradiction.

Now let L be any linear extension of $(P, \preceq_P)$ and hence of $(P, \leq_P)$. We claim that L is ζ-like. To prove this, we show that for all $i, j \in \mathbb{N}$

$$\{x \in P \colon z_i \leq_L x \leq_L z_j\} = \{x \in P \colon z_i \preceq_P x \preceq_P z_j\}.$$

One inclusion is obvious because $\leq_L$ extends $\preceq_P$. For the converse, observe that the z_n's are $\preceq_P$-comparable with any other element. $\square$

We can now state and prove our reverse mathematics results.

Theorem 3.3. *Over* RCA_0*, the following are pairwise equivalent:*

(1) $\mathsf{B}\mathbf{\Sigma}^0_2$;
(2) ω *is linearizable;*
(3) ω^* *is linearizable;*
(4) ζ *is linearizable.*

Proof. Lemma 3.1 gives (1) $\to$ (2) and (1) $\to$ (3). The implication (1) $\to$ (4) is Lemma 3.2.

To show (2) $\to$ (1), we assume linearizability of ω and prove FUF. So let $\{X_i : i < n\}$ be a finite family of finite sets. We define $P = \bigoplus_{i<n}(X_i + \{m_i\})$, where the m_i's are distinct and every X_i is regarded as an antichain. P is ω-like, and so by (2) there exists an ω-like linear extension L of P. Let m_j be the L-maximum of $\{m_i : i < n\}$. Then $\bigcup_{i<n} X_i$ is included in the set of L-predecessors of m_j, and is therefore finite because L is ω-like.

The implication (3) $\to$ (1) is analogous. For (4) $\to$ (1), prove FUF by using the partial order $\bigoplus_{i<n}(\{\ell_i\} + X_i + \{m_i\})$. □

We now show that the linearizability of $\omega + \omega^*$ requires ACA_0.

Theorem 3.4. *Over* RCA_0*, the following are equivalent:*

(1) ACA_0;
(2) $\omega + \omega^*$ *is linearizable.*

Proof. We begin by proving (1) $\to$ (2). Let P be an $\omega+\omega^*$-like partial order. In ACA_0 we can define the set P_0 of the elements having finitely many predecessors. So $P_1 = P \setminus P_0$ consists of elements having finitely many successors. Clearly, P_0 is ω-like and P_1 is ω^*-like. Since ACA_0 is strong enough to prove $\mathsf{B}\mathbf{\Sigma}^0_2$, by Lemma 3.1, P_0 has an ω-like linear extension L_0 and P_1 has an ω^*-like linear extension L_1. Since P_0 is downward closed and P_1 is upward closed, it is not difficult to check that the linear order $L = L_0 + L_1$ is $\omega + \omega^*$-like and extends P.

For the converse, let $f : \mathbb{N} \to \mathbb{N}$ be a one-to-one function. We set out to define an $\omega+\omega^*$-like partial order P such that any $\omega+\omega^*$-like linear extension of P encodes the range of f. To this end, we use an $\omega+\omega^*$-like linear order $A = \{a_n : n \in \mathbb{N}\}$ given by the false and true stages of f. Recall that $n \in \mathbb{N}$ is said to be true (for f) if $(\forall m > n)(f(m) > f(n))$ and false otherwise, and note that the range of f is $\mathbf{\Delta}^0_1$ definable from any infinite set of true stages.

The idea for A comes from the well-known construction of a computable linear order such that any infinite descending sequence computes $\emptyset'$. This construction can be carried out in RCA_0 (see [MS11, Lemma 4.2]). Here, we define A by letting $a_n \leq a_m$ if and only if either

$$f(k) < f(n) \text{ for some } n < k \leq m, \text{ or}$$
$$m \leq n \text{ and } f(k) > f(m) \text{ for all } m < k \leq n.$$

It is not hard to see that A is a linear order. Moreover, if n is false, then a_n has finitely many predecessors and infinitely many successors. Similarly, if n is true, then a_n has finitely many successors and infinitely many predecessors. In particular, A is an $\omega + \omega^*$-like linear order.

Now let $P = A \oplus B$ where $B = \{b_n : n \in \mathbb{N}\}$ is a linear order of order type ω^*, defined by letting $b_n \leq b_m$ if and only if $n \geq m$. It is clear that P is an $\omega + \omega^*$-like partial order. By hypothesis, there exists an $\omega + \omega^*$-like linear extension L of P. We claim that n is a false stage if and only if it satisfies the $\mathbf{\Pi}^0_1$ formula $(\forall m)(a_n <_L b_m)$.

In fact, if n is false and $b_m \leq_L a_n$, then b_m has infinitely many successors in L, since a_n has infinitely many successors in P and a fortiori in L. On the other hand, b_m has infinitely many predecessors in P, and hence also in L, contradiction. Likewise, if n is true and $a_n <_L b_m$ for all m, then a_n has infinitely many successors as well as infinitely many predecessors in L, which is a contradiction again.

Therefore, the set of false stages is $\mathbf{\Delta}^0_1$, and so is the set of true stages, which thus exists in RCA_0. This completes the proof. □

4. Embeddable types

We turn our attention to embeddability. As noted before, RCA_0 suffices to prove that "τ is embeddable" implies "τ is linearizable". The converse is true in ACA_0. Actually, embeddability is equivalent to ACA_0. We thus prove the following.

Theorem 4.1. *The following are pairwise equivalent over* RCA_0*:*

1. ACA_0*;*
2. ω *is embeddable;*
3. ω^* *is embeddable;*
4. ζ *is embeddable;*
5. $\omega + \omega^*$ *is embeddable;*

Proof. We first show that (1) implies the other statements. Since $\mathsf{B}\mathbf{\Sigma}^0_2$ is provable in ACA_0, it follows from Theorem 3.3 that ACA_0 proves the linearizability of ω, ω^* and ζ. By Theorem 3.4, ACA_0 proves the linearizability of $\omega+\omega^*$. We now claim that in ACA_0 "τ is linearizable" implies "τ is embeddable" for each τ we are considering. The key fact is that the property of having finitely many predecessors (successors) in a partial order, as well as having exactly $n \in \mathbb{N}$ predecessors (successors), is arithmetical. Analogously, for a set, and hence for an interval, being finite or having size exactly $n \in \mathbb{N}$ is arithmetical too. (All these properties are in fact $\mathbf{\Sigma}^0_2$.)

We consider explicitly the case of $\omega+\omega^*$ (the other cases are similar). So let L be a $\omega + \omega^*$-like linear extension of a given $\omega + \omega^*$-like partial order. We want to show that L is embeddable into $\omega + \omega^*$. Define

$f\colon L \to \omega + \omega^*$ by

$$f(x) = \begin{cases} (0, |\{y \in L\colon y <_L x\}|) & \text{if } x \text{ has finitely many predecessors,} \\ (1, |\{y \in L\colon x <_L y\}|) & \text{otherwise.} \end{cases}$$

It is easy to see that f preserves the order.

For the reversals, notice that (5) $\to$ (1) immediately follows from Theorem 3.4.

As the others are quite similar, we only prove (2) $\to$ (1) with a construction similar to that used in the proof of Theorem 3.1 in [FH90]. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a given one-to-one function. We want to prove that the range of f exists. We fix an antichain $A = \{a_m\colon m \in \mathbb{N}\}$ and elements b^n_j for $n \in \mathbb{N}$ and $j \leq n$. The partial order P is obtained by putting for each $n \in \mathbb{N}$ the $n+1$ elements b^n_j below $a_{f(n)}$. Formally, $b^n_j \leq_P a_m$ when $f(n) \leq m$, and there are no other comparabilities.

P is clearly an ω-like partial order. Apply the hypothesis and obtain an embedding $h\colon P \to \omega$. Now, we claim that m belongs to the range of f if and only if $(\exists n < h(a_m))(f(n) = m)$. One implication is trivial. For the other, suppose that $f(n) = m$. By construction, a_m has at least $n+1$ predecessors in P, and thus it must be $h(a_m) > n$. □

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