My research is in the area of mathematical logic. Most of my work so far is in classical reverse mathematics, especially in the reverse mathematics of order theory (well-quasi-orders, scattered partial orders, etc). Reverse mathematics is a research program in logic and the foundations of mathematics introduced by Friedman [Fri75, Fri76] in order to provide a proof-theoretic analysis of ordinary mathematics. For a general overview and comprehensive introduction to the subject see [Sim09]. For a more computability-theoretic introduction see Hirschfeldt's monograph [Hir15].

Overview of my research

Linear extensions. It is well-known that every partial order has a linear extension (Szpilrajn's Theorem). In [FM12] Marcone and I consider particular instances of "linearizability". For this purpose we define the notion of τ-like partial order, where τ is one of the linear order types ω, ω^* (the inverse order of ω), ζ (the order of the integers) and $\omega + \omega^*$.

Being ω-like (ω^*-like) means that every element has finitely many predecessors (successors), while being ζ-like means that every interval is finite. Being $\omega + \omega^*$-like means that every element has either finitely many predecessors or finitely many successors.

For each τ we consider the statement "every τ-like partial order has a τ-like linear extension". These statements turn out to be equivalent either to $\mathsf{B}\mathbf{\Sigma}^0_2$ or to ACA_0 over the usual base system RCA_0.

Theorem 1 (Frittaion and Marcone 2012)**.** *Over* RCA_0*, the following are pairwise equivalent:*

(1) $\mathsf{B}\mathbf{\Sigma}^0_2$;

(2) *Every ω-like partial order has an ω-like linear extension;*

(3) *Every ζ-like linear extension has a ζ-like linear extension.*

The linearizability of ω can be found in Fraïssé's monograph ([Fra00, §2.15]), where the result is attributed to Milner and Pouzet. The linearizability of ζ is apparently a new result.

Theorem 2 (Frittaion and Marcone 2012)**.** *Over* RCA_0*, the following are equivalent:*

(1) ACA_0;

(2) *Every $\omega + \omega^*$-like partial order has an $\omega + \omega^*$-like linear extension.*

Initial intervals. In [FM14] Marcone and I study several theorems about the connection between a partial order and the set of its initial intervals. We need a few definitions.

Definition 3. An *initial interval* is a downward closed set. An *ideal* is an initial interval I such that any two elements $x, y \in I$ have a common upper bound $z \in I$. An *antichain* is a set of incomparable elements. A *strong antichain* is a set of pairwise incompatible elements, i.e. elements with no common upper bound. A partial order is *scattered* if it does not contain copies of the rationals. Finally, a partial order P is a *well-partial-order* (*wpo*) if for every function $f\colon \mathbb{N} \to P$ there exist $x < y$ such that $f(x) \leq_P f(y)$.

Theorem (Bonnet)**.** *A partial order has no infinite antichains if and only if every initial interval is a finite union of ideals.*

Theorem (Bonnet)**.** *A partial order is scattered and has no infinite antichains if and only if it has countably many initial intervals.*

Theorem (Erdös and Tarski)**.** *A partial order with no infinite strong antichains has no arbitrarily large finite strong antichains.*

The main results are the following.

Theorem 4 (Frittaion and Marcone 2014)**.** *Over* RCA_0*, the following are pairwise equivalent:*

(1) ACA_0*;*

(2) *Every partial order with no infinite strong antichains does not contain arbitrarily large finite strong antichains;*

(3) *If a partial order has no infinite antichains then every initial interval is a finite union of ideals;*

(4) *Every well-partial order is a finite union of ideals.*

Theorem 5 (Frittaion and Marcone 2014)**.** *Over* ACA_0*, the following are equivalent:*

(1) ATR_0*;*

(2) *Every scattered partial order with no infinite antichains has countably many initial intervals;*

(3) *Every scattered linear order has countably many initial intervals.*

Theorem 6 (Frittaion and Marcone 2014)**.** *The following statements are provable in* WKL_0 *but not in* RCA_0*:*

(1) *Every partial order with an infinite antichain contains an initial interval that is not a finite union of ideals;*

(2) *Every partial order with countably many initial intervals has no infinite antichains.*

The unprovability in RCA_0 of statements (1) and (2) follows from the following result which shows the failure in REC of both statements.

Theorem 7 (Frittaion and Marcone 2014)**.** *There exists a recursive partial order with an infinite recursive antichain such that every recursively enumerable initial interval is the downward closure of a finite set.*

Scattered partial orders. In my PhD thesis [Fri14] I study the reverse mathematics of several theorems about scattered partial orders.

Hausdorff's theorem and generalizations. A well-known theorem by Hausdorff [Hau08] states that the class of scattered linear orders is the least class which contains the empty set, singletons and is closed under lexicographic sums along $\mathbb{Z}$.

Theorem 8 (Frittaion 2014)**.** *Over* ACA_0*, Hausdorff's theorem is equivalent to* ATR_0*.*

Another characterization of countable scattered linear orders is given by the following (see [Fra00, §5.3.2]):

Theorem. *Every countable linear order is scattered if and only if there exists a countable ordinal that does not embed into it.*

Notice that the left-to-right direction is not true in general: for instance ω_1 is a scattered linear order and any countable ordinal embeds into it.

Theorem 9 (Frittaion 2014)**.** ATR_0 *proves the for every countable scattered linear order there exists a countable ordinal that does not embed into it.*

The right-to-left direction can be easily proved in RCA_0. It is open whether the interesting direction reverses to ATR_0. If not, any proof in a weaker system must avoid the fact that any scattered linear order embeds into some power of $\mathbb{Z}$, because this turns out to be equivalent to ATR_0 (over ACA_0) (see [Clo89]).

The following two theorems generalize Hausdorff's theorem to scattered FAC partial orders and to countable FAC partial orders respectively.

Theorem ([AB99])**.** *The class of scattered partial orders with no infinite antichains is the least class which contains wpo's, reverse wpo's, and is closed under lexicographic sums and extensions.*

Theorem ([ABC$^+$12])**.** *The class of countable partial orders with no infinite antichains is the least class which contains countable linear orders, wpo's, reverse wpo's, and is closed under lexicographic sums and extensions.*

The formalization of these theorems requires the coding of the corresponding classes of partial orders. A natural way to encode iterations of lexicographic sums is by labelled well-founded trees where the labels are partial orders. For instance, a lexicographic sum $\sum_{x\in P} P_x$ can be represented by a tree of height 1, where the root is labelled with P and each child $\langle x\rangle$ is labelled with P_x for all $x \in P$. In general, one can define a *code* to be a set $\{P_\sigma \colon \sigma \in T\}$, where T is a well-founded tree and each P_σ is a partial order on the children of σ. Given a code T, it is possible to define within ACA_0 a partial order, denoted $\sum_{\sigma\in T} P_\sigma$, which provably in ATR_0 is isomorphic to the partial order obtained by iterating all the sums along the tree from the bottom up. If $\mathcal{B}$ is a class of countable partial orders, we write $P \in \mathcal{C}(\mathcal{B})$ if and only if there exists a code T labelled with partial orders from $\mathcal{B}$ such that P is isomorphic to an extension of $\sum_{\sigma\in T} P_\sigma$. Let $\mathcal{S}$ be the class of countable wpo's and reverse wpo's, and $\mathcal{F}$ be the class of countable linear orders, countable wpo's and countable reverse wpo's.

Theorem 10 (Frittaion 2014)**.** ACA_0 *proves:*

- *If $P \in \mathcal{C}(\mathcal{S})$ then P is scattered and has no infinite antichains;*
- *If $P \in \mathcal{C}(\mathcal{F})$ then P has no infinite antichains.*

On the other hand, $\mathbf{\Pi}^1_2$-CA_0 *proves:*

- *If P is scattered and has no infinite antichains then $P \in \mathcal{C}(\mathcal{S})$;*
- *If P has no infinite antichains then $P \in \mathcal{C}(\mathcal{F})$.*

These statements are both $\mathbf{\Pi}^1_3$ and by general arguments they cannot imply $\mathbf{\Pi}^1_2$-CA_0. No reversals are known.

Well-scattered partial orders. Bonnet and Pouzet provide four equivalent conditions for a partial order to be well-scattered.

Definition 11. A *well-scattered* partial order (*wspo*) is a partial order P such that for all $f\colon \mathbb{Q} \to P$ there exist $x <_{\mathbb{Q}} y$ such that $f(x) \leq_P f(y)$.

Remember that a partial order is *scattered* if it does not contain a copy of the rationals.

Theorem (Bonnet and Pouzet, 1969)**.** *Let P be a partial order. The following are equivalent:*

(A) *P is a wspo;*

(B) *P is scattered and has no infinite antichains;*

(C) *Every linear extension of P is scattered;*

(D) *For every function $f\colon \mathbb{Q} \to P$ there exists an infinite set $A \subseteq \mathbb{Q}$ such that $x <_{\mathbb{Q}} y$ implies $f(x) \leq_P f(y)$ for all $x, y \in A$.*

Well-partial orders have a similar well-known characterization.

Theorem (Folklore)**.** *Let P be a partial order. Then the following are equivalent:*

(a) *P is a wpo;*

(b) *P is well-founded and has no infinite antichains;*

(c) *Every linear extension of P is well-founded;*

(d) *For every function $f\colon \mathbb{N} \to P$ there exists an infinite set $A \subseteq \mathbb{N}$ such that $x < y$ implies $f(x) \leq_P f(y)$ for all $x, y \in A$.*

We have two statements for each equivalence. For instance, we denote by (ab) the statement "Every wpo is well-founded and has no infinite antichains". In [CMS04] Cholak, Marcone and Solomon study the reverse mathematics of these statements with respect to wpo's. In my thesis I improve some of their results and study the corresponding statements for wspo's.

Theorem 12 ([CMS04, **?**]). *The following holds:*

(1) *The statements* (*ab*)*,* (*ac*)*,* (*da*)*,* (*db*) *and* (*dc*) *are provable in* RCA_0*;*

(2) *The statements* (*ca*) *and* (*cb*) *are provable* WKL_0 *but not in* WWKL_0*;*

(3) *Over* RCA_0*, each of the statements* (*ad*)*,* (*ba*)*,* (*bc*) *and* (*bd*) *is equivalent to* CAC*;*

(4) *Over* RCA_0*,* (*cd*) *is between* $\mathsf{CAC} + \mathsf{WKL}_0$ *and* CAC*.*

The strength of (*ca*), (*cb*) and (*cd*) is open.

In the case of wspo's many statements follow from the following theorem (see [ER52, Theorem 4]):

Theorem (Erdös and Rado 1952)**.** *For every coloring* $c\colon [\mathbb{Q}]^2 \to 2$ *there exists either an infinite* 0*-homogeneous set or a dense* 1*-homogeneous set.*

Let ER^2_2 be the above statement. The proof of ER^2_2 can be formalized in ACA_0. On the other hand, ER^2_2 implies RT^2_2. The reverse mathematics strength of this statement is still open (see also my work with Patey [FP15]). The principle CAC (for Chain Antichain) is a consequence of RT^2_2 and is equivalent to a semitransitive version of RT^2_2 (see [HS07]), denoted by $\mathsf{st\text{-}RT}^2_2$. Actually, for all $r \geq 2$, CAC is equivalent to $\mathsf{st\text{-}RT}^2_r$, the semitransitive version of Ramsey's theorem for pairs and r colors. In the case of wspo's I introduce a semitransitive version of Erdös-Rado theorem for pairs and r colors, denoted by $\mathsf{st\text{-}ER}^2_r$. Here, $\mathsf{st\text{-}ER}^2_3$ is equivalent to CAC+$\mathsf{st\text{-}ER}^2_2$, but the relation between CAC and $\mathsf{st\text{-}ER}^2_2$ is open.

Theorem 13 (Frittaion 2014)**.** *The following holds:*

(1) *The statements* (AB)*,* (AC)*,* (DA)*,* (DB) *and* (DC) *are provable in* RCA_0*;*

(2) *The statements* (CA) *and* (CB) *are provable in* WKL_0 *but not in* WWKL_0*;*

(3) *Over* RCA_0*,* (BD) *is equivalent to* $\mathsf{st\text{-}ER}^2_3$*;*

(4) *Over* RCA_0*,* (BA) *is between* $\mathsf{st\text{-}ER}^2_3$ *and* $\mathsf{st\text{-}ER}^2_2$*;*

(5) *Over* RCA_0*,* (BC) *is equivalent to* $\mathsf{st\text{-}ER}^2_2$*;*

(6) *Over* RCA_0*,* (AD) *is between* $\mathsf{st\text{-}ER}^2_3$ *and* CAC*;*

(7) *Over* RCA_0*,* (CD) *is between* $\mathsf{st\text{-}ER}^2_3$+$\mathsf{WKL}_0$ *and* CAC*;*

The strength of several statements is open.

Erdös and Rado on coloring of rationals. In [FP15] Patey and I obtain partial results about the strength of the aforementioned theorem by Erdös and Rado on colorings of rationals.

Theorem (Erdös and Rado 1952)**.** *Every coloring* $c\colon [\mathbb{Q}]^2 \to 2$ *admits either an infinite* 0*-homogeneous set or a dense* 1*-homogeneous set.*

It is known that ER^2_2 is between ACA_0 and RT^2_2. I conjecture that ER^2_2 is strictly between ACA_0 and RT^2_2. In this paper we obtain a separation of ER^2_2 from $\mathsf{RT}^2_{<\infty}$ under computable reducibility.

Definition 14 (Computable reducibility)**.** Fix two Π^1_2 statements P and Q. P is *computably reducible* to Q if every P-instance X_0 computes a Q-instance X_1 such that for every solution Y to X_1, $Y \oplus X_0$ computes a solution to X_0.

Theorem 15 (Frittaion and Patey 2015)**.** ER^2_2 *is not computably reducible to* $\mathsf{RT}^2_{<\infty}$*.*

We also consider the infinite pigeonhole principle for rationals "Every finite coloring of the rationals contains a dense homogeneous set", denoted by $\mathsf{ER}^1_{<\infty}$, and show that:

Theorem 16 (Frittaion and Patey 2015)**.** *Over* RCA_0*,*

(1) $\mathsf{ER}^2_2 \vee \mathsf{I}\Sigma^0_2$ *implies* $\mathsf{ER}^1_{<\infty}$*.*

(2) $\mathsf{ER}^1_{<\infty}$ *implies* $\mathsf{RT}^1_{<\infty}$*.*

(3) $\mathsf{RT}^1_{<\infty}$ *does not imply* $\mathsf{ER}^1_{<\infty}$*.*

Well-quasi-orders and Noetherian spaces. The study of Noetherian spaces as a generalization of wqo's was started by Goubault-Larrecq [GL07] in the context of infinite-state verification problems. A topological space is Noetherian if every open set is compact. Equivalently, if there are no strictly increasing ascending sequences of open sets. Noetherian spaces originate in algebraic geometry, as the Zariski topology of a Noetherian ring is Noetherian. The relationship with wqo's is the following: a quasi-order Q is a wqo if and only if $\mathcal{A}(Q)$, the Alexandroff topology of Q, is Noetherian, where the open sets of the Alexandroff topology are the upward closed sets of Q.

In [FHM+16] Hendtlass, Marcone, Van der Meeren, Shafer and I extend the framework introduced by Dorais for countable second-countable topological spaces to uncountable second-countable topological spaces and analyze results by Goubault-Larrecq [GL07] concerning the relationship between a wqo Q and various topologies on $\mathcal{P}(Q)$, the power set of Q. Given a quasi-order Q, one defines quasi-orders $\mathcal{P}^\flat(Q)$ and $\mathcal{P}^\sharp(Q)$ on $\mathcal{P}(Q)$ by letting

- $A \leq^\flat B$ iff $(\forall a \in A)(\exists b \in B)(a \leq_Q b)$;
- $A \leq^\sharp B$ iff $(\forall b \in B)(\exists a \in A)(a \leq_Q b)$.

The $\leq^\flat$ quasi-order is known in computer science as the Hoare quasi-order. If Q is a wqo then $\mathcal{P}^\bullet(Q)$ need not be a wqo, where $\bullet \in \{\flat, \sharp\}$. However, Goubault-Larrecq proved that "If Q is a wqo then $\mathcal{U}(\mathcal{P}^\bullet(Q))$, the upper topology on $\mathcal{P}^\bullet(Q)$, is Noetherian". In general, given a quasi-order Q, the basic closed sets of the upper topology of Q are the downward closures of finite subsets of Q. Similar results apply to $\mathcal{P}_{\mathrm{f}}(Q)$, the set of finite subsets of Q. We show that these statement are equivalent to ACA_0 within RCA_0.

Theorem 17 (Frittaion, Hendtlass, Marcone, van der Meeren, Shafer 2016)**.** *The following are equivalent over* RCA_0*:*

(1) ACA_0.

(2) *If* Q *is a wqo, then* $\mathcal{A}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ *is Noetherian.*

(3) *If* Q *is a wqo, then* $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(Q))$ *is Noetherian.*

(4) *If* Q *is a wqo, then* $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(Q))$ *is Noetherian.*

(5) *If* Q *is a wqo, then* $\mathcal{U}(\mathcal{P}^\flat(Q))$ *is Noetherian.*

(6) *If* Q *is a wqo, then* $\mathcal{U}(\mathcal{P}^\sharp(Q))$ *is Noetherian.*

Brown's lemma. In [Fri16] I study the strength of Brown's lemma and its finite version. Brown's lemma (see [Bro68]) asserts that piecewise syndetic sets are partition regular, that is whenever we partition a piecewise syndetic set into finitely many sets, at least one set must be piecewise syndetic.

Definition 18. An infinite set $X \subseteq \mathbb{N}$ is *piecewise syndetic* if there exists $d \in \mathbb{N}$ such that X contains arbitrarily large finite sets with gaps bounded by d, where a set has gaps bounded by d if the difference between any two consecutive elements is $\leq d$.

Theorem (Brown 1968)**.** *Every coloring* $c\colon \mathbb{N} \to r$ *has a* c*-homogeneous piecewise syndetic set.*

Brown's lemma is related to van der Waerden's theorem.

Theorem (Van der Waerden's theorem, infinite version)**.** *Every coloring* $c\colon \mathbb{N} \to r$ *has a* c*-homogeneous set with arbitrarily long arithmetic progressions.*

Definition 19. Let $H \subseteq \mathbb{N}$ be finite. Define the *gap size* of H, denoted $gs(H)$, as the largest difference between two consecutive elements of H. In other words, the gap size of H is the least d such that H has gaps bounded by d. If $|H| \leq 1$ let $gs(H) = 1$.

Theorem 20 (Brown's Lemma, finite)**.** *Let* $f\colon \mathbb{N} \to \mathbb{N}$*. Then for all* $r > 0$ *there exists* n *such that every coloring* $c\colon n \to r$ *has a* c*-homogeneous set* H *with* $|H| > f(gs(H))$*.*

The finite version of Brown's lemma is reminiscent of the Paris-Harrington principle. We can think of $|H| > f(gs(H))$ as a largeness condition on H.

Definition 21. For $f\colon \mathbb{N} \to \mathbb{N}$, let BL_f be the statement "For all $r > 0$ there exists n such that every $c\colon n \to r$ has a c-homogeneous set H such that $|H| > f(gs(H))$.

Theorem 22 (Frittaion 2016)**.** *The following holds:*

(1) *Over* RCA_0^**, Brown's lemma is equivalent to* $\mathsf{I}\Sigma^0_2$*.*

(2) *Over* RCA_0^**, the infinite version of van der Waerden's theorem is equivalent to* $\mathsf{B}\Sigma^0_2$*.*

(3) *The finite version of Brown's lemma is provable in* RCA_0

(4) *If* $f(d) = d$*, then* BL_f *is provable in* RCA_0^*

(5) *If* $f(d) = 2^d$*, then* BL_f *is not provable in* RCA_0^**.*

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