My research is in the area of mathematical logic. I am specifically interested in topics at the intersection between computability theory and proof theory (reverse mathematics, constructive mathematics, intuitionistic arithmetic and analysis, type theory). Most of my work has been in classical reverse mathematics, mainly in the reverse mathematics of order theory. I am currently interested in some questions about constructive mathematics and in the interaction bewteen constructive mathematics and reverse mathematics.

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1 Overview of my research

Reverse mathematics [Fri75] is a tool in mathematical logic to assess the *computational* strength of theorems in core areas of ordinary mathematics. I regard reverse mathematics as a formalization of *computable mathematics* (the same way Aczel's constructive Zermelo-Fraenkel set theory is a formalization of constructive mathematics). By computable mathematics I mean a whole variety of mathematical investigations that apply recursion theory (Turing computability theory) and classical logic to the analysis of ordinary mathematics (e.g., recursive mathematics, computable structure theory, computable analysis). In reverse mathematics we search for the minimal subsystem of classical second-order arithmetic needed to establish a given theorem. The most important subsystems of reverse mathematics are known as the *Big Five* and have a natural recursion-theoretic interpretation: they assert the closure under certain operations.

- RCA_0 (Recursive Comprehesion Axiom): Every computable set of natural numbers exists
- WKL_0 (Weak Königs Lemma): Every infinite binary tree has an infinite path
- ACA_0 (Arithmetical Comprehension Axiom): The Turing jump of every set of natural numbers exists
- ATR_0 (Arithmetical Transfinite Recursion): The transfinite iteration of the Turing jump of every set of natural numbers exists
- $\mathbf{\Pi}^1_1$-CA_0 ($\mathbf{\Pi}^1_1$ Comprehension Axiom): Every set of natural numbers defined by a Π^1_1-formula exists

Main Theme: Most mathematical theorems are provable in the base system RCA_0 or equivalent over RCA_0 to one of the remaining four systems. For instance, the Baire Category Theorem BCT for complete separable metric spaces "The intersection of countably many open dense sets on a complete separable metric space is dense" is provable in RCA_0 and the Heine-Borel Covering Lemma HBCL "Every covering of the closed interval $[0,1]$ by a sequence of open intervals has a finite subcovering" is equivalent to WKL_0 over RCA_0. Therefore, from a reverse mathematics point of view, BCT is computationally weaker than HBCL.

Simpson [Sim09] classifies hundreds of theorems from all branches of mathematics into the Big Five catalogue. Although the Big Five seem to exhaust all of mathematics, there are quite a number of exceptions. The best known is Ramsey's Theorem for pairs and two colors (cf. Hirschfeldt [Hir15]).

1.1 Reverse mathematics and order theory

Order theory has broad applications in all areas of mathematics and computer science. In my past research I have studied a large number of theorems in the context of reverse mathematics. Here is a brief review.

Linear extensions. Every partial order has a linear extension. This is known as Szpilrajn's Theorem. In [FM12] Marcone and I study certain linearization theorems of the form "every τ-like partial order has a τ-like linear extension", for $\tau \in \{\omega, \omega^*, \zeta, \omega + \omega^*\}$. The notion of τ-likeness is intended to capture a structural property of the order type τ. Being ω-like (ω^*-like) means that every element has finitely many predecessors (successors), while being ζ-like means that every interval is finite. Finally, being $\omega + \omega^*$-like means that every element has either finitely many predecessors or finitely many successors.

We show that some of these statements (linearizability of ω and ζ) are natural equivalents of $\mathsf{B}\boldsymbol{\Sigma}^0_2$, the bounding principle, also known as collection in first-order arithmetic, for Σ^0_2 formulas. For $\tau = \omega + \omega^*$ we obtain an equivalence with ACA_0. We use RCA_0 as base system.

Initial intervals. Partial orders can be studied by looking at their initial intervals (downward closed sets). In [FM14] Marcone and I show that Bonnet's theorem "A partial order has no infinite antichains iff every initial interval is a finite union of ideals" is equivalent to ACA_0, and so is the statement that every well-partial order is a finite union of ideals. Bonnet's proof is based on a theorem by Erdös and Tarski "A partial order with no infinite strong antichains has no arbitrarily large finite strong antichains", where a strong antichain is a set of pairwise incompatible elements, i.e., elements with no common upper bound. We show that these two theorems have indeed the same reverse mathematics strength ACA_0. Bonnet proved that the *number* of initial intervals is related to *density*: "A partial order is scattered (no copy of the rationals) and has no infinite antichains iff it has countably many initial intervals". We show that the forward direction is equivalent to ATR_0 and that the inverse direction is provable in WKL_0 but not in RCA_0. We use RCA_0 as base system.

Scattered partial orders. Recall that a partial orders is *scattered* if it does not contain a copy of the rationals. In my thesis [Fri14] I prove that a well-known theorem by Hausdorff "The class of scattered linear orders is the least class which contains the empty set, singletons and is closed under lexicographic sums along $\mathbb{Z}$" is equivalent to ATR_0 over ACA_0. This theorem can be generalized to partial orders [AB99]: The class of scattered partial orders with no infinite antichains is the least class which contains well-partial orders, and is closed under reverse partial orders, lexicographic sums, and partial extensions. I prove that in $\mathbf{\Pi}^1_2$-CA_0 one can show that a scattered partial order with no infinite antichains has a certain representation in terms of well-partial orders. This requires a coding with well-founded trees similar to the coding of Borel sets in reverse mathematics. The statement is $\mathbf{\Pi}^1_3$ and by general arguments it cannot imply $\mathbf{\Pi}^1_2$-CA_0. No reversals are known.

Well-scattered partial orders. Well-partial orders (wpo) are well-known in order theory and in computer science and admit several characterizations. The most important, especially for applications in computer science, is in terms of bad sequences. Say that a sequence $(x_n)_{n\in\mathbb{N}}$ on a partial order P is bad if $x_n \not\leq_P x_m$ for all $n < m$. One can define a wpo by saying that it does not contain bad sequences. Cholak, Marcone and Solomon [CMS04] show that different characterizations for wpos have different reverse mathematics strength. More precisely, passing from one definition to another requires different strength. In my thesis I consider well-scattered partial orders (wspo), a generalization of wpos. One can define a wspo by saying that it does not admit bad $\mathbb{Q}$-sequences, where a bad $\mathbb{Q}$-sequence on a partial order P is a function $f\colon \mathbb{Q} \to P$ such that $f(x) \not\leq_P f(y)$ for all $x <_{\mathbb{Q}} y$. Surprisingly, wpos and wspos have similar characterizations. In the case of wpos, the principle CAC (Chain Antichain), a consequence of Ramsey's theorem for pairs and two colors, is needed to prove that some definitions are equivalent. The case of wspos is similar but not as clear. Erdös and Rado [ER52, Theorem 4] proved that every coloring $f\colon [\mathbb{Q}]^2 \to 2$ admits either an infinite 0-homogeneous set or a dense 1-homogeneous set. I show that passing from some definitions of wspo to others requires semi-transitive versions of Erdös and Rado theorem for r colors, denoted by st-ER^2_r. It turns out that st-ER^2_3 is equivalent to CAC + st-ER^2_2, but the relation between CAC and st-ER^2_2 is open. Recall on the other hand that CAC is equivalent to the semitransitive version of Ramsey's theorem for pairs and r colors for any standard number $r \geq 2$.

Coloring rationals. Patey and I [FP17] study Erdös and Rado theorem on colorings of rationals, denoted ER^2_2. My conjecture is that ER^2_2 is strictly between ACA_0 and RT^2_2. In our paper we obtain a separation of ER^2_2 from $\mathsf{RT}^2_{<\infty}$ under computable reducibility. Given two Π^1_2 statements P and Q, we say that P is *computably reducible* to Q if every P-instance X_0 computes a Q-instance X_1 such that for every solution Y to X_1, $Y \oplus X_0$ computes a solution to X_0. We show that ER^2_2 does not computably reduce to $\mathsf{RT}^2_{<\infty}$.

Added note. Dzhafarov and Patey [DP17] showed that $\mathsf{RT}^2_2 + \mathsf{WKL}_0$ does not imply ER^2_2. In particular, ER^2_2 is strictly stronger than RT^2_2. It is still open whether ER^2_2 is weaker than ACA_0.

1.2 Reverse mathematics and combinatorial number theory

Brown's lemma [Bro68]) is a well-known theorem in combinatorial number theory. It asserts that piecewise syndetic sets of natural numbers are partition regular, that is whenever we partition a piecewise syndetic set into finitely many sets, at least one set must be piecewise syndetic. Brown's lemma BL is closely related to van der Waerden's theorem VDWT. The latter asserts that in every partition of the natural numbers into finitely many sets, at least one set contains arbitrarily long arithmetic progressions.

In [Fri17a] I study the strength of Brown's lemma and its finite version over the base system RCA_0^*: this system is to RCA_0 as elementary function arithmetic EFA is to primitive recursive arithmetic PRA. Indeed, RCA_0^* and RCA_0 are conservative extensions of EFA and PRA respectively for Π^0_2-formulas. I show that BL is equivalent to induction for Σ^0_2 formulas, and that VDWT is equivalent to bounding for Σ^0_2 formulas, both over RCA_0^*. On the other hand, the finite version of Brown's lemma, which is reminiscent of the Paris-Harrington principle, is provable in RCA_0 but not in RCA_0^*.

1.3 Reverse mathematics and termination analysis

Noetherian spaces. Goubault-Larrecq [GL07] introduced the study of Noetherian spaces in the context of infinite-state verification problems. Noetherian spaces arise in algebraic geometry. In fact, the Zariski topology of a Noetherian ring is Noetherian. On the other hand, they constitute a topological version of well-quasi orders (wqo) and so provide a more general framework for termination analysis. The relationship with well-quasi orders is that a quasi-order Q is a wqo if and only if the Alexandroff topology of Q is Noetherian, where the open sets of the Alexandroff topology are the upward closed sets of Q.

In [FHM$^+$16] Hendtlass, Marcone, Van der Meeren, Shafer and I extend the framework introduced by Dorais for countable second-countable topological spaces to uncountable second-countable topological spaces and analyze results by Goubault-Larrecq [GL07] concerning the relationship between a wqo Q and various topologies on $\mathcal{P}(Q)$, the power set of Q. Given a quasi-order Q, one can define two quasi-orders $\mathcal{P}^\flat(Q)$ and $\mathcal{P}^\sharp(Q)$ on $\mathcal{P}(Q)$ by letting

- $A \leq^\flat B$ iff $(\forall a \in A)(\exists b \in B)(a \leq_Q b)$;
- $A \leq^\sharp B$ iff $(\forall b \in B)(\exists a \in A)(a \leq_Q b)$.

The $\leq^\flat$ quasi-order is known in computer science as the Hoare quasi-order. If Q is a wqo then $\mathcal{P}^\bullet(Q)$ need not be a wqo, where $\bullet \in \{\flat, \sharp\}$, but the upper topology on $\mathcal{P}^\bullet(Q)$ is Noetherian. In general, given a quasi-order Q, the basic closed sets of the upper topology of Q are the downward closures of finite subsets of Q. Similar results apply to $\mathcal{P}_\mathrm{f}(Q)$, the set of finite subsets of Q. We provide the following equivalences with ACA_0 over RCA_0:

(1) If Q is a wqo, then $\mathcal{A}(\mathcal{P}^\flat_\mathrm{f}(Q))$ is Noetherian

(2) If Q is a wqo, then $\mathcal{U}(\mathcal{P}^\flat_\mathrm{f}(Q))$ is Noetherian

(3) If Q is a wqo, then $\mathcal{U}(\mathcal{P}^\sharp_\mathrm{f}(Q))$ is Noetherian

(4) If Q is a wqo, then $\mathcal{U}(\mathcal{P}^\flat(Q))$ is Noetherian

(5) If Q is a wqo, then $\mathcal{U}(\mathcal{P}^\sharp(Q))$ is Noetherian

Size-change termination. Lee, Jones and Ben-Amram [LJB01] introduced the notion of size-change termination for first-order functional programs. Size-change analysis provides a general method for automated termination proofs and has been applied to higher-order programs, logic programs, and term rewrite systems. Size-change termination is based on the notion of size-change graph. Given a first-order functional program P one associates to every call $f \to g$ a bipartite graph which describes the relation between source and target parameter values.

In [FSY17] Steila, Yokoyama and I analyze the SCT criterion [LJB01, Theorem 4]. The original proof of the SCT criterion is based on Ramsey's theorem for pairs. We show that this is far from optimal and pinpoint the exact reverse mathematics strength of the SCT criterion to induction for Σ^0_2 formulas. To do so, we introduce and study a corollary of Ramsey's theorem for pairs, called Triangle Ramsey's theorem. It states that for any coloring of pairs of natural numbers in k colors, there is a color i and some $x \in \mathbb{N}$ such that the triangle $\{x, y, z\}$ is homogeneous for color i for infinitely many pairs y, z. We show that this corollary of Ramsey's theorem implies the SCT criterion and that the SCT criterion implies the Strong Pigeonhole Principle. From these (and some further) results we are able to conclude that both the SCT criterion and the Triangle Ramsey's theorem are equivalent to Σ^0_2-induction over RCA_0.

In [FSYP17] Pelupessy, Steila, Yokoyama and I continue the study of size-change termination and analyze the soundness of the SCT method [LJB01, Theorem 1]. In particular, we prove that a particular instance of the statement "Every SCT program is terminating" is equivalent to the well-foundedness of ω^{ω^ω} over RCA_0.

2 Current work and future directions

Conservation results in constructive mathematics. In [Fri17b] I study a realizability notion introduced by Goodman [Goo78] to prove that intuitionistic finite-type arithmetic augmented with the axiom of choice is conservative over first-order intuitionistic arithmetic HA (Heyting arithmetic). This is a classical result in the metamathematics of constructivism. However, it is not quite well-understood. Indeed there are many proofs of this result, although complete and rigorous proofs are surprisingly rare. In my opinion Goodman's proof [Goo78] remains *the* proof to this day. In [Fri17b] I show how a suitable extensional version of Goodman realizability allows to prove that also the addition of extensionality results in a conservative extension of Heyting arithmetic. This was proved by Beeson in a rather sketchy way. A recent paper by Benno van den Berg and Lotte van Slooten [vdBvS17] gives yet another proof of Goodman's theorem and provides a rigorous proof of its extensional version.

Goodman's proof shows that the adding (indices for) partial arithmetical functions does not increase the strength of HA. In fact, I claim that the theory obtained from HA by adding a predicate $A(a, x, y)$, whose intended meaning is "the value of the partial function with index a on input x is well defined and equal to y", written $ax \downarrow y$, and axioms of the form

$$\exists a \forall \underline{x} \, \big(\exists y \psi(\underline{x}, y) \to \exists y (a\underline{x} \downarrow y \wedge \psi(\underline{x}, y)) \;\; \& \;\; \forall y (a\underline{x} \downarrow y \to \psi(\underline{x}, y))\big)$$

for all formulas $\psi(x_1, \ldots, x_n, y)$ in the the language of HA, is conservative over HA. Goodman realizability combines Kleene recursive realizability with forcing, where the forcing conditions are finite approximations of Skolem partial functions for existential formulas $\exists y \psi(x_1, \ldots, x_n, y)$, that is, finite sequences p such that $\forall \underline{x} \left(\underline{x} \in \mathrm{dom}(p) \rightarrow \psi(\underline{x}, p(\underline{x})) \right)$. It would be interesting to prove this claim and generalize it to second- and higher-order arithmetic.

Choice principles in intuitionistic finite-type arithmetic. In intuitionistic set theory and finite-type arithmetic, one can consider the following choice principles.

Axiom of choice AC:

$$\forall x \exists y \varphi(x, y) \rightarrow \exists f \forall x \varphi(x, f(x))$$

Axiom of relativized dependent choice RDC:

$$\forall x(\varphi(x) \rightarrow \exists y(\varphi(y) \wedge \psi(x, y)) \rightarrow \forall x(\varphi(x) \rightarrow \exists (x_n)_{n \in \mathbb{N}}(x_0 = x \wedge \forall n \psi(x_n, x_{n+1})))$$

It is not difficut to see that over classical finite-type arithmetic PA^ω, AC implies RDC. Indeed, in a classical setting, RDC is equivalent to the more familiar axiom of dependent choice DC. It follows from [Koh92] that, over PA^ω, RDC is weaker than AC. To my knowledge, is still unknown whether in the context of intuitionistic finite-type arithmetic HA^ω, AC implies RDC. This goes back to at least [HK66]. Interestingly, over HA^ω, the axiom of (countable) choice CAC implies collection principle for all formulas and RDC implies induction for all formulas. Both proofs use only quantifier-free induction. This nicety suggests an interesting analogy: CAC is to RDC as collection is to induction. It is not difficult to see that most proof interpretations the realize AC do not use induction, and maybe one can use this to show that AC does not imply RDC over HA^ω. Anyway, this has not been done yet.

Uniformization theorems in reverse mathematics. In recent times there has been an increasing interest in the reverse mathematics community in so called uniformization theorems [HM11, DHS12, Dor14, Fuj15, FK15, Kuy17, HM17]. Most of the results show that, for an appropriate notion of uniformity, if a theorem P has a certain syntactic form and is provable in a certain (semi-)constructive formal system *Constr*, then its uniform version is provable in a certain classical formal system *Class*. The systems under considerations are (semi-)constructive systems based on finite-type arithmetic well-known in the proof-theory of constructive mathematics and classical systems from reverse mathematics. Similarly, for appropriate notions of uniform reduction (usually formalizing Weihrauch reducibility), they show that if $P \rightarrow Q$ is provable in a certain (semi-)constructive formal system *Constr*, then the existence of a uniform reduction of Q to P can be proved in a certain classical formal system *Class*.

I would like to further investigate this theme and try to obtain sharper results. Except for [Kuy17], the existing results are more or less direct applications of well-known proof interpretations. Moreover, the notion of uniformity used in systems of finite-type arithmetic appears to be too strong. In fact, uniformity asks for the existence of a total functional. For instance, a total functional witnessing the $\forall(\rightarrow \exists)$ statement $\forall f(\exists x(fx =$

$0) \to \exists y(fy = 0))$ cannot exist provably in RCA_0^ω (Feferman's μ-operator), but obviously there is a partial computable functional that given f searches for x such that $fx = 0$. Idea: investigate finite-type theories for partial functionals.

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