Burgess, John, *Fixing Frege*, Princeton and Oxford: Princeton University Press, 2005, pp. xii + 257, US$39.95 (cloth).

Frege's system in the *Grundgesetze der Arithmetik* is inconsistent: Russell *dixit*. The main characteristic of the Fregean system is the presence of an operator that applies to second-order unary variables F to form a term $\ddagger F$ that can be substituted for object variables. This operator - the extension operator - obeys the (in)famous Law V of the *Grundgesetze*: $\forall F \forall G(\ddagger F = \ddagger G \leftrightarrow \forall x(Fx \leftrightarrow Gx))$. Russell's solution of the paradox, the theory of types, dropped the extension operator and, with this move, directed the development of the logicism away from it. Frege's original system, with its extension operator, entered a torpor of eighty years. After all, who (save in Australia) is interested in inconsistent systems? We may say without exaggeration that a single book, *Frege's Conception of Numbers as Objects* (Aberdeen: Scots Philosophical Monographs, 1983), authored by Crispin Wright, initiated a flurry of interest in Fregean-like (neo-Fregean) systems that continues to this very day. Wright's book drew attention to the *way* Frege developed arithmetic within the inconsistent system (this bit was foreshadowed by Charles Parsons) and asked the inspiring question below. Frege's development of arithmetic consists of two steps. First, Frege uses the extension operator to *define* a cardinality operator # that obeys Hume's Principle: $\forall F \forall G(\#F = \#G \leftrightarrow F \approx G)$, where $\approx$ is the equinumerosity relation. Second, the development of arithmetic can proceed from Hume's Principle without relying on the extension operator any longer (this became known as *Frege's Theorem*). Consider Frege's framework, drop the extension operator and replace it by a *primitive* cardinality operator ruled by Hume's Principle. Is this system (*Frege's Arithmetic*) consistent? The answer is yes - Burgess and Allen Hazen proved it in reviews (!) of Wright's book - and neo-logicism was born.

Fixing Frege surveys and evaluates the neo-Fregean systems produced since Wright's seminal work. Burgess discusses the scope and limits of the proposed systems (in terms of how much mathematics can be developed in them), their axiomatic basis (unveiling the existential commitments of the axioms), and the character of their definitions (do they involve arbitrary choices, artificial devices or ignore anything that seems to be an ingredient of the intuitive notion?). The book is matchless for its informativeness, clarity and attention to subtle points. There is something for everybody, from the novice to the experienced logician. In addition, the book is sprinkled by interesting technical open problems (most of them accessible, I believe) that may attract the attention of the more technically minded. *Per contra*, Burgess states that his book aims merely to characterize, not resolve, philosophical issues, leaving the ultimate philosophical evaluation of the neo-Fregean strategies to the reader. This is not necessarily a bad thing. My complain lies elsewhere. At the very end of the book, Burgess openly confesses his skeptical leanings about neo-Fregean approaches, advancing a (grammatical!) objection that, in his words, 'has a devastating effect' on many of the theories surveyed. By his own admission, in writing the book Burgess embarked on a Sisyphean philosophical task.

The book is divided into three chapters. The opening chapter describes (in modern language) the formal system of the *Grundgesetze* and outlines Frege's strategy for developing arithmetic. Burgess introduces quite exuberant notations (a mark of the book) in the discussions of abstract operators (of which both the extension and the cardinality operators are particular cases). Fortunately, there are tables that help the reader move through the notational jungle. Russell's paradox and Russell's way out are discussed, as well as an offhand, spurious fix, by Frege himself. The second chapter treats variants of the Fregean system with predicative comprehension. The focus here lies on the extension operator, since the predicative restriction bars inconsistency. The last chapter concerns impredicative theories and it is asked whether there are principled grounds for accepting or rejecting particular abstraction operators. A self-serving criterion is to accept only those abstraction operators that do not lead to contradiction. Surprisingly, even this rather weak criterion is inappropriate because there are pairs of abstraction operators, each one consistent in isolation, but inconsistent when taken together. This is George Boolos's famous bad company objection, which Burgess discusses in detail.

The first chapter also includes a nice tour of formal mathematical systems, from the very weak arithmetical theories to the very strong set-theoretic systems with large cardinality assumptions. Two stopping places are relevant for the discussion of the predicative theories. The first place is the very starting point of the tour: Robinson's theory of arithmetic Q. Burgess shows that Q is interpretable in the predicative theories surveyed. The significance of this result is that it gives a measure of how much mathematics the theories support. Concerning this point, Burgess takes some time explaining the Solovay-Nelson-Wilkie result that Peano arithmetic with the induction scheme restricted to *bounded* formulae is interpretable in Q (an explicit proof of the result is not provided, but the techniques for proving it, at least in a mitigated *local* form, are available in the book, as Burgess observes). Even though this *bounded* theory is not very strong, non-trivial parts of arithmetic and analysis are *interpretable* in it. The second stopping place concerns *Gentzen's arithmetic* (in Burgess' felicitous terminology). This theory has the minimal amount of arithmetic needed to prove Gentzen's cut-elimination theorem, a central technical result for providing finitistic proofs of consistency results. Burgess shows (and this is one of the highlights of the book) that Gentzen's arithmetic proves the consistency of the predicative systems considered. *Ergo*, by Gödel's second incompleteness theorem, Gentzen's arithmetic is not interpretable in these systems, thereby establishing an upper bound on what can be interpreted in the predicative theories.

Gentzen's arithmetic (or whereabouts) seems to be the stopping point for the predicative strategy *strictu sensu*. Burgess also mentions, in a quite dismissive way, strategies that are based on predicative strategies *given* the notion of finiteness. These strategies interpret theories well beyond Gentzen's arithmetic (e.g., first-order Peano arithmetic), but they are classified as anti-Fregean. Nevertheless, they have shown that taking finiteness for granted and proceeding predicatively thereafter is essentially the same thing as working predicatively *from* the Frege-Dedekind (impredicative) definition of the natural number concept. Proof-theoretically, the impredicativity of the Frege-Dedekind notion of natural number is quite mild compared to the impredicativity of full-blown second-order logic. This logical distinction is a blind spot in the philosophy of mathematics, and it remains to be seen whether or not it is of significance.

The last chapter concerns stronger, impredicative, theories. Burgess considers refinements of Frege's theorem, as well as the piecemeal use of abstraction operators to introduce mathematical objects (pursued by Bob Hale and other collaborators of Wright's Scottish school of neo-Fregeanism). A general theory of abstraction due to Kit Fine is also sketched. The theory can be seen as giving a global answer to Boolos's bad company objection, one that revolves around the notion of *non-inflating* equivalence relations (in the sense that there are no more equivalence classes of concepts than there are objects) which obey certain invariance requirements under the action of permuting objects. There follows a survey of Frege-inspired set theories (in the sense that the notions of set and element are subordinated to the notion of the subsumption of an object under a concept) where the extension operator is subjected to restrictions based on the un-Fregean idea of limitation of size. The technical part of the book ends up with the characterization of the set theoretic system needed to have a perfect match between the standard (Tarskian) and the intuitive (in a natural sense) notions of second-order validity. I fully agree with Burgess that this match should be more widely known.

Fixing Frege has an interest beyond neo-Fregean studies. It condenses in a single place many important technical tools that any worker in the philosophy of mathematics should carry in his bag. It is bound to be a reference for years to come. But make no mistake: under the veil of the technicalities lie deep philosophical issues. The profession of skepticism at the end of the book concerns, after all, the central issue of the meaningfulness of second-order quantification. Burgess resists giving Frege a grain of salt on this matter. For the believer the road ahead is clear, if hazardous: make second-order quantification palatable, even without salt.

I have only found a slip in the book worth mentioning. In the description of the theory WKL_0 in the first chapter, the recursive inseparability concerns recursively enumerable sets, not their complements.

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