LINEAR EXTENSIONS OF PARTIAL ORDERS
AND REVERSE MATHEMATICS

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Abstract. We introduce the notion of $\tau$-like partial order, where $\tau$ is one of the linear order types $\omega$, $\omega^*$, $\omega + \omega^*$, and $\zeta$. For example, being $\omega$-like means that every element has finitely many predecessors, while being $\zeta$-like means that every interval is finite. We consider statements of the form “any $\tau$-like partial order has a $\tau$-like linear extension” and “any $\tau$-like partial order is embeddable into $\tau^*$” (when $\tau$ is $\zeta$ this result appears to be new). Working in the framework of reverse mathematics, we show that these statements are equivalent either to $B\Sigma_2^0$ or to $ACA_0$ over the usual base system $\text{RCA}_0$.

1. Introduction

Szpilrajn’s Theorem ([Szp30]) states that any partial order has a linear extension. This theorem raises many natural questions, where in general we search for properties of the partial order which are preserved by some or all its linear extensions. For example it is well-known that a partial order is a well partial order if and only if all its linear extensions are well-orders.

A question which has been widely considered is the following: given a linear order type $\tau$, is it the case that any partial order that does not embed $\tau$ can be extended to a linear order that also does not embed $\tau$? If the answer is affirmative, $\tau$ is said to be extendible, while $\tau$ is weakly extendible if the same holds for any countable partial order. For instance, the order types of the natural numbers, of the integers, and of the rationals are extendible. Bonnet ([Bon69]) and Jullien ([Jul69]) characterized all countable extendible and weakly extendible linear order types respectively.

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We are interested in a similar question: given a linear order type \( \tau \) and a property characterizing \( \tau \) and its suborders, is it true that any partial order which satisfies that property has a linear extension which also satisfies the same property? In our terminology: does any \( \tau \)-like partial order have a \( \tau \)-like linear extension? Here we address this question for the linear order types \( \omega, \omega^* \) (the inverse of \( \omega \), \( \omega + \omega^* \) and \( \zeta \) (the order of integers). So, from now on, \( \tau \) will denote one of these.

**Definition 1.1.** Let \((P, \leq_P)\) be a countable partial order. We say that \(P\) is

- \(\omega\)-like if every element of \(P\) has finitely many predecessors;
- \(\omega^*\)-like if every element of \(P\) has finitely many successors;
- \(\omega + \omega^*\)-like if every element of \(P\) has finitely many predecessors or finitely many successors;
- \(\zeta\)-like if for every pair of elements \(x, y \in P\) there exist only finitely many elements \(z\) with \(x <_P z <_P y\).

The previous definition resembles Definition 2.3 of Hirschfeldt and Shore ([HS07]), where linear orders of type \(\omega, \omega^*\) and \(\omega + \omega^*\) are introduced. The main difference is that the order properties defined by Hirschfeldt and Shore are meant to uniquely determine a linear order type up to isomorphism, whereas our definitions apply to partial orders in general and do not determine an order type. Notice also that, for instance, an \(\omega\)-like partial order is also \(\omega + \omega^*\)-like and \(\zeta\)-like.

We introduce the following terminology:

**Definition 1.2.** We say that \(\tau\) is linearizable if every \(\tau\)-like partial order has a linear extension which is also \(\tau\)-like.

With this definition in hand, we are ready to formulate the results we want to study:

**Theorem 1.3.** The following hold:

1. \(\omega\) is linearizable;
2. \(\omega^*\) is linearizable;
3. \(\omega + \omega^*\) is linearizable;
4. \(\zeta\) is linearizable.

A proof of the linearizability of \(\omega\) can be found in Fraïssé’s monograph ([Fra00, §2.15]), where the result is attributed to Milner and Pouzet. (2) is similar to (1) and the proof of (3) easily follows from (1) and (2). The linearizability of \(\zeta\) is apparently a new result (for a proof see Lemma 3.2 below).

In this paper we study the statements contained in Theorem 1.3 from the standpoint of reverse mathematics (the standard reference is [Sim09]), whose goal is to characterize the axiomatic assumptions needed to prove mathematical theorems. We assume the reader is familiar with systems such as \(\text{RCA}_0\) and \(\text{ACA}_0\). The reverse mathematics
of weak extendibility is studied in [DHLS03] and [Mon06]. The existence of maximal linear extensions of well partial orders is studied from the reverse mathematics viewpoint in [MS11].

Our main result is that the linearizability of $\tau$ is equivalent over $\text{RCA}_0$ to the $\Sigma^0_2$ bounding principle $\text{B}\Sigma^0_2$, when $\tau \in \{\omega, \omega^*, \zeta\}$, and to $\text{ACA}_0$ when $\tau = \omega + \omega^*$. For more details on $\text{B}\Sigma^0_2$, including an apparently new equivalent (simply asserting that a finite union of finite sets is finite), see §2 below.

The linearizability of $\omega$ appears to be the first example of a genuine mathematical theorem (actually appearing in the literature for its own interest, and not for its metamathematical properties) that turns out to be equivalent to $\text{B}\Sigma^0_2$.

To round out our reverse mathematics analysis, we also consider a notion closely related to linearizability:

**Definition 1.4.** We say that $\tau$ is **embeddable** if every $\tau$-like partial order $P$ embeds into $\tau$, that is there exists an order preserving map from $P$ to $\tau$.\[1\]

It is rather obvious that $\tau$ is linearizable if and only if $\tau$ is embeddable. Let us notice that $\text{RCA}_0$ easily proves that embeddable implies linearizable. Not surprisingly, the converse is not true. In fact, we show that embeddability is strictly stronger when $\tau \in \{\omega, \omega^*, \zeta\}$, and indeed equivalent to $\text{ACA}_0$. The only exception is given by $\omega + \omega^*$, for which both properties are equivalent to $\text{ACA}_0$.

We use the following definitions in $\text{RCA}_0$.

**Definition 1.5 (RCA).** Let $\leq$ denote the usual ordering of natural numbers. The linear order $\omega$ is $(\mathbb{N}, \leq)$, while $\omega^*$ is $(\mathbb{N}, \geq)$.

Let $\{P_i: i \in Q\}$ be a family of partial orders indexed by a partial order $Q$. The **lexicographic sum** of the $P_i$ along $Q$, denoted by $\sum_{i \in Q} P_i$, is the partial order on the set $\{(i, x): i \in Q \land x \in P_i\}$ defined by

$$(i, x) \leq (j, y) \iff i < Q j \lor (i = j \land x \leq P_i y).$$

The sum $\sum_{i<n} P_i$ can be regarded as the lexicographic sum along the $n$-element chain. In particular $P_0 + P_1$ is the lexicographic sum along the 2-element chain (and we have thus defined $\omega + \omega^*$ and $\zeta = \omega^* + \omega$).

Similarly, the **disjoint sum** $\bigoplus_{i<n} P_i$ is the lexicographic sum along the $n$-element antichain.

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\[1\]To formalize this definition in $\text{RCA}_0$, we need to fix a canonical representative of the order type $\tau$, which we do in Definition 1.5.
2. $\Sigma_0^0$ BOUNDING AND FINITE UNION OF FINITE SETS

Let us recall that $B\Sigma_2^0$ (standing for $\Sigma_2^0$ bounding, and also known as $\Sigma_2^0$ collection) is the scheme:

$$(B\Sigma_2^0) \quad (\forall i < n)(\exists m)\phi(i, n, m) \implies (\exists k)(\forall i < n)(\exists m < k)\phi(i, n, m),$$

where $\phi$ is any $\Sigma_0^2$ formula.

It is well-known that RCA$_0$ does not prove $B\Sigma_2^0$, which is strictly weaker than $\Sigma_2^0$ induction. Neither of WKL$_0$ and $B\Sigma_2^0$ implies the other and Hirst ([Hir87], for a widely available proof see [CJS01, Theorem 2.11]) showed that $RT_2^1$ (Ramsey theorem for pairs and two colors) implies $B\Sigma_2^0$.

A few combinatorial principles are known to be equivalent to $B\Sigma_2^0$ over RCA$_0$.

Hirst ([Hir87], for a widely available proof see [CJS01, Theorem 2.10]) showed that, over RCA$_0$, $B\Sigma_2^0$ is equivalent to the infinite pigeonhole principle, i.e. the statement

$$(RT_{1<\infty}) \quad \forall n)(\forall f : N \to n)(\exists A \subseteq N \text{ infinite})(\exists c < n)(\forall m \in A)(f(m) = c).$$

(The notation arises from viewing the infinite pigeonhole principle as Ramsey theorem for singletons and an arbitrary finite number of colors.)

Chong, Lempp and Yang ([CLY10]) showed that a combinatorial principle $\PART$ about infinite $\omega + \omega^*$ linear orders, introduced by Hirschfeldt and Shore ([HS07, §4]), is also equivalent to $B\Sigma_2^0$. More recently, Hirst ([Hir12]) also proved that $B\Sigma_2^0$ is equivalent to a statement apparently similar to Hindman’s theorem, but much weaker from the reverse mathematics viewpoint.

We consider the statement that a finite union of finite sets is finite:

$$(FUF) \quad (\forall i < n)(X_i \text{ is finite}) \implies \bigcup_{i<n} X_i \text{ is finite.}$$

Here “$X$ is finite” means $(\exists m)(\forall x \in X)(x < m)$. This statement can be viewed as a second-order version of $\Pi_0$ regularity, which in the context of first-order arithmetic is known to be equivalent to $\Sigma_2$ bounding (see e.g. [?, Theorem 2.23.4]).

**Lemma 2.1.** Over RCA$_0$, $B\Sigma_2^0$ is equivalent to FUF.

**Proof.** First notice that FUF follows immediately from the instance of $B\Sigma_2^0$ relative to the $\Pi_1^0$, and hence $\Sigma_2^0$, formula $\forall x \in X_i(x < m)$.

For the other direction we use Hirst’s result recalled above: it suffices to prove that FUF implies $RT_{1<\infty}$. Let $f : N \to n$ be given. Define for each $i < n$ the set $X_i = \{m : f(m) = i\}$. Clearly $\bigcup_{i<n} X_i = N$ is infinite. By FUF, there exists $i < n$ such that $X_i$ is infinite. Now $X_i$ is an infinite homogeneous set for $f$. \qed
3. Linearizable types

Notice that Szpilrajn’s Theorem is easily seen to be computably true (see [Dow98, Observation 6.1]) and provable in $\text{RCA}_0$. We use this fact several times without further notice.

We start by proving that $\mathcal{B}\Sigma^0_2$ suffices to establish the linearizability of $\omega$, $\omega^*$ and $\zeta$.

**Lemma 3.1.** $\text{RCA}_0$ proves that $\mathcal{B}\Sigma^0_2$ implies the linearizability of $\omega$ and $\omega^*$.

**Proof.** We argue in $\text{RCA}_0$ and, by Lemma 2.1, we may assume $\text{FUF}$. Let us consider first $\omega$. So let $P$ be an $\omega$-like partial order which, to avoid trivialities, we may assume to be infinite. We recursively define a sequence $z_n \in P$ by letting $z_n$ be the least (w.r.t. the usual ordering of $\mathbb{N}$) $x \in P$ such that $(\forall i < n)(x \not\leq_P z_i)$.

We show by $\Sigma^0_1$ induction that $z_n$ is defined for all $n \in \mathbb{N}$. Suppose that $z_i$ is defined for all $i < n$. We want to prove $(\exists x \in P)(\forall i < n)(x \not\leq_P z_i)$. Define $X_i = \{x \in P : x \leq_P z_i\}$ for $i < n$. Since $P$ is $\omega$-like, each $X_i$ is finite. By $\text{FUF}$, $\bigcup_{i<n} X_i$ is also finite. The claim follows from the fact that $P$ is infinite.

Now define for each $n \in \mathbb{N}$ the finite set $P_n = \{x \in P : x \leq_P z_n \land (\forall i < n)(x \not\leq_P z_i)\}$.

It is not hard to see that the $P_n$’s form a partition of $P$, and that if $x \leq_P y$ with $x \in P_i$ and $y \in P_j$, then $i \leq j$. Then let $L$ be a linear extension of the lexicographic sum $\sum_{n \in \omega} P_n$. $L$ is clearly a linear order and extends $P$ by the remark above. To prove that $L$ is $\omega$-like, note that the set of $L$-predecessors of an element of $P_n$ is included in $\bigcup_{i \leq n} P_i$, which is finite, by $\text{FUF}$ again.

For $\omega^*$, repeat the same construction using $\geq_P$ in place of $\leq_P$, and let $L$ be a linear extension of $\sum_{n \in \omega} P_n$. □

**Lemma 3.2.** $\text{RCA}_0$ proves that $\mathcal{B}\Sigma^0_2$ implies the linearizability of $\zeta$.

**Proof.** In $\text{RCA}_0$ assume $\text{FUF}$. Let $P$ be a $\zeta$-like partial order, which we may again assume to be infinite. It is convenient to use the notation $[x, y]_P = \{z \in P : x \leq_P z \leq_P y \lor y \leq_P z \leq_P x\}$, so that $[x, y]_P \neq \emptyset$ if and only if $x$ and $y$ are comparable.

We define by recursion a sequence $z_n \in P$ by letting $z_n$ be the least (w.r.t. the ordering of $\mathbb{N}$) $x \in P$ such that

$$x \not\in \bigcup_{i,j<n} [z_i, z_j]_P.$$ 

As before, since $P$ is infinite and $\zeta$-like, one can prove using $\Sigma^0_1$ induction and $\text{FUF}$ that $z_n$ is defined for every $n \in \mathbb{N}$. It is also easy
to prove that
\[ P = \bigcup_{i,j \in \mathbb{N}} [z_i, z_j]_P. \]
Define for each \( n \in \mathbb{N} \) the set
\[ P_n = \bigcup_{i < n} [z_i, z_n]_P \setminus \bigcup_{i,j < n} [z_i, z_j]_P. \]
By FUF, the \( P_n \)'s are finite. Moreover, they clearly form a partition of \( P \). Note also that \( z_n \in P_n \) and every element of \( P_n \) is comparable with \( z_n \). Furthermore, every interval \([x, y]_P\) is included in some \([z_i, z_j]_P\).

Notice that the same holds for any partial order extending \( \leq_P \).

We now extend \( \leq_P \) to a partial order \( \preceq_P \) such that any linear extension of \((P, \preceq_P)\) is \( \zeta \)-like. We say that \( n \) is left if \( z_n \leq_P z_i \) for some \( i < n \); otherwise, we say that \( n \) is right. Notice that, since \( z_n \in P_n \), \( n \) is right if and only if \( z_i \leq_P z_n \) for some \( i < n \) or \( z_n \) is incomparable with every \( z_i \) with \( i < n \).

The order \( \preceq_P \) places \( P_n \) below or above every \( P_i \) with \( i < n \) depending on whether \( n \) is left or right. Formally, for \( x, y \in P \) such that \( x \in P_n \) and \( y \in P_m \) let
\[ x \preceq_P y \iff (n = m \land x \leq_P y) \lor (n < m \land m \text{ is right}) \lor (m < n \land n \text{ is left}). \]
We claim that \( \preceq_P \) extends \( \leq_P \). Let \( x \leq_P y \) with \( x \in P_n \) and \( y \in P_m \). If \( n = m \), \( x \preceq_P y \) by definition. Suppose now that \( n < m \), so that we need to prove that \( m \) is right. As \( x \in P_n \), \( z_i \leq_P x \) for some \( i \leq n \). Since \( y \in P_m \), \( y \) is comparable with \( z_m \). Suppose that \( z_m <_P y \). Then \( y \leq_P z_j \) for some \( j < m \), and so \( z_i \leq_P x \leq_P y \leq_P z_j \) with \( i, j < m \), contrary to \( y \in P_m \). It follows that \( y \preceq_P z_m \) and thereby \( z_i \preceq_P z_m \) with \( i < m \). Therefore, \( m \) is right, as desired. The case \( n > m \) (where we need to prove that \( n \) is left) is similar.

We claim that \((P, \preceq_P)\) is still \( \zeta \)-like. To see this, it is enough to show that for all \( i, j < n \)
\[ \{x \in P : z_i \preceq_P x \preceq_P z_j \} \subseteq \bigcup_{k < n} P_k \]
and apply FUF. Let \( x \in P_k \) be such that \( z_i \preceq_P x \preceq_P z_j \). Suppose, for a contradiction, that \( k \geq n \) and hence that \( i, j < k \). By the definition of \( \preceq_P \), \( z_i \preceq_P x \) implies that \( k \) is right. At the same time, \( x \preceq_P z_j \) implies that \( k \) is left, a contradiction.

Now let \( L \) be any linear extension of \((P, \preceq_P)\) and hence of \((P, \preceq_P)\).
We claim that \( L \) is \( \zeta \)-like. To prove this, we show that for all \( i, j \in \mathbb{N} \)
\[ \{x \in P : z_i \leq_L x \leq_L z_j \} = \{x \in P : z_i \preceq_P x \preceq_P z_j \}. \]
One inclusion is obvious because \( \leq_L \) extends \( \preceq_P \). For the converse, observe that the \( z_n \)'s are \( \preceq_P \)-comparable with any other element.

We can now state and prove our reverse mathematics results.
Theorem 3.3. Over RCA₀, the following are pairwise equivalent:

1. $\mathbf{B}\Sigma^0_2$;
2. $\omega$ is linearizable;
3. $\omega^*$ is linearizable;
4. $\zeta$ is linearizable.

Proof. Lemma 3.1 gives (1) $\implies$ (2) and (1) $\implies$ (3). The implication (1) $\implies$ (4) is Lemma 3.2.

To show (2) $\implies$ (1), we assume linearizability of $\omega$ and prove FUF.

So let \( \{X_i : i < n\} \) be a finite family of finite sets. We define \( P = \bigoplus_{i<n}(X_i + \{m_i\}) \), where the \( m_i \)'s are distinct and every \( X_i \) is regarded as an antichain. \( P \) is $\omega$-like, and so by (2) there exists an $\omega$-like linear extension \( L \) of \( P \). Let \( m_j \) be the \( L \)-maximum of \( \{m_i : i < n\} \). Then \( \bigcup_{i<n}X_i \) is included in the set of \( L \)-predecessors of \( m_j \), and is therefore finite because \( L \) is $\omega$-like.

The implication (3) $\implies$ (1) is analogous. For (4) $\implies$ (1), prove FUF by using the partial order \( \bigoplus_{i<n}(\{\ell_i\} + X_i + \{m_i\}) \).

We now show that the linearizability of $\omega + \omega^*$ requires ACA₀.

Theorem 3.4. Over RCA₀, the following are equivalent:

1. ACA₀;
2. $\omega + \omega^*$ is linearizable.

Proof. We begin by proving (1) $\implies$ (2). Let \( P \) be an $\omega + \omega^*$-like partial order. In ACA₀ we can define the set \( P_0 \) of the elements having finitely many predecessors. So \( P_1 = P \setminus P_0 \) consists of elements having finitely many successors. Clearly, \( P_0 \) is $\omega$-like and \( P_1 \) is $\omega^*$-like. Since ACA₀ is strong enough to prove $\mathbf{B}\Sigma^0_2$, by Lemma 3.1, \( P_0 \) has an $\omega$-like linear extension \( L_0 \) and \( P_1 \) has an $\omega^*$-like linear extension \( L_1 \). Since \( P_0 \) is downward closed and \( P_1 \) is upward closed, it is not difficult to check that the linear order \( L = L_0 + L_1 \) is $\omega + \omega^*$-like and extends \( P \).

For the converse, let \( f : \mathbb{N} \to \mathbb{N} \) be a one-to-one function. We set out to define an $\omega + \omega^*$-like partial order \( P \) such that any $\omega + \omega^*$-like linear extension of \( P \) encodes the range of \( f \). To this end, we use an $\omega + \omega^*$-like linear order \( A = \{a_n : n \in \mathbb{N}\} \) given by the false and true stages of \( f \). Recall that \( n \in \mathbb{N} \) is said to be true (for \( f \)) if \((\forall m > n)(f(m) > f(n))\) and false otherwise, and note that the range of \( f \) is $\Delta^0_1$ definable from any infinite set of true stages.

The idea for \( A \) comes from the well-known construction of a computable linear order such that any infinite descending sequence computes $\emptyset'$. This construction can be carried out in RCA₀ (see [MSH, Lemma 4.2]). Here, we define \( A \) by letting \( a_n \leq a_m \) if and only if either

\[ f(k) < f(n) \text{ for some } n < k \leq m, \text{ or} \]

\[ m \leq n \text{ and } f(k) > f(m) \text{ for all } m < k \leq n. \]
It is not hard to see that $A$ is a linear order. Moreover, if $n$ is false, then $a_n$ has finitely many predecessors and infinitely many successors. Similarly, if $n$ is true, then $a_n$ has finitely many successors and infinitely many predecessors. In particular, $A$ is an $\omega + \omega^*$-like linear order.

Now let $P = A \oplus B$ where $B = \{b_n : n \in \mathbb{N}\}$ is a linear order of order type $\omega^*$, defined by letting $b_n \leq b_m$ if and only if $n \geq m$. It is clear that $P$ is an $\omega + \omega^*$-like partial order. By hypothesis, there exists an $\omega + \omega^*$-like linear extension $L$ of $P$. We claim that $n$ is a false stage if and only if it satisfies the $\Pi_1^0$ formula $(\forall m)(a_n <_L b_m)$.

In fact, if $n$ is false and $b_m \leq_L a_n$, then $b_m$ has infinitely many successors in $L$, since $a_n$ has infinitely many successors in $P$ and a fortiori in $L$. On the other hand, $b_m$ has infinitely many predecessors in $P$, and hence also in $L$, contradiction. Likewise, if $n$ is true and $a_n <_L b_m$ for all $m$, then $a_n$ has infinitely many successors as well as infinitely many predecessors in $L$, which is a contradiction again.

Therefore, the set of false stages is $\Delta_1^0$, and so is the set of true stages, which thus exists in $\text{RCA}_0$. This completes the proof. □

4. Embeddable types

We turn our attention to embeddability. As noted before, $\text{RCA}_0$ suffices to prove that “$\tau$ is embeddable” implies “$\tau$ is linearizable”. The converse is true in $\text{ACA}_0$. Actually, embeddability is equivalent to $\text{ACA}_0$. We thus prove the following.

Theorem 4.1. The following are pairwise equivalent over $\text{RCA}_0$:

1. $\text{ACA}_0$;
2. $\omega$ is embeddable;
3. $\omega^*$ is embeddable;
4. $\zeta$ is embeddable;
5. $\omega + \omega^*$ is embeddable;

Proof. We first show that (1) implies the other statements. Since $B\Sigma_2^0$ is provable in $\text{ACA}_0$, it follows from Theorem 3.3 that $\text{ACA}_0$ proves the linearizability of $\omega$, $\omega^*$ and $\zeta$. By Theorem 3.4, $\text{ACA}_0$ proves the linearizability of $\omega + \omega^*$. We now claim that in $\text{ACA}_0$ “$\tau$ is linearizable” implies “$\tau$ is embeddable” for each $\tau$ we are considering. The key fact is that the property of having finitely many predecessors (successors) in a partial order, as well as having exactly $n \in \mathbb{N}$ predecessors (successors), is arithmetical. Analogously, for a set, and hence for an interval, being finite or having size exactly $n \in \mathbb{N}$ is arithmetical too. (All these properties are in fact $\Sigma_2^0$.)

We consider explicitly the case of $\omega + \omega^*$ (the other cases are similar). So let $L$ be a $\omega + \omega^*$-like linear extension of a given $\omega + \omega^*$-like partial order. We want to show that $L$ is embeddable into $\omega + \omega^*$. Define
f : L → ω + ω* by

\[
\begin{align*}
 f(x) &= \begin{cases} 
 (0, \{ \{ y \in L : y <_L x \} \} ) & \text{if } x \text{ has finitely many predecessors}, \\
 (1, \{ \{ y \in L : x <_L y \} \} ) & \text{otherwise}.
\end{cases}
\end{align*}
\]

It is easy to see that \( f \) preserves the order.

For the reversals, notice that (5) → (1) immediately follows from Theorem 3.4.

As the others are quite similar, we only prove (2) → (1) with a construction similar to that used in the proof of Theorem 3.1 in [FH90]. Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be a given one-to-one function. We want to prove that the range of \( f \) exists. We fix an antichain \( A = \{ a_m : m \in \mathbb{N} \} \) and elements \( b^n_j \) for \( n \in \mathbb{N} \) and \( j \leq n \). The partial order \( P \) is obtained by putting for each \( n \in \mathbb{N} \) the \( n + 1 \) elements \( b^n_j \) below \( a_{f(n)} \). Formally, \( b^n_j \leq_P a_m \) when \( f(n) \leq m \), and there are no other comparabilities.

\( P \) is clearly an \( \omega \)-like partial order. Apply the hypothesis and obtain an embedding \( h : P \rightarrow \omega \). Now, we claim that \( m \) belongs to the range of \( f \) if and only if \( (\exists n < h(a_m))(f(n) = m) \). One implication is trivial. For the other, suppose that \( f(n) = m \). By construction, \( a_m \) has at least \( n + 1 \) predecessors in \( P \), and thus it must be \( h(a_m) > n \).

\[ \square \]

**References**


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