Bounded Modified Realizability

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Abstract

We define a notion of realizability, based on a new assignment of formulas, which does not care for precise witnesses of existential statements, but only for bounds for them. The novel form of realizability supports a very general form of the FAN theorem, refutes Markov's principle but meshes well with some classical principles, including the lesser limited principle of omniscience and weak König's lemma. We discuss some applications, as well as some previous results in the literature.

1 Introduction

The realizability method, invented by Stephen Kleene in 1945 [8], is a way of making explicit the constructive content of arithmetical sentences. It is closely related with intuitionistic logic, being reminiscent of the Brouwer-Heyting-Kolmogorov interpretation of the intuitionistic logical words (see [15]

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for a modern overview of realizability). In particular, Kleene's realizability and related notions (e.g., modified realizability due to Kreisel in [13]) decides disjunctions and provides precise witnesses for existential statements. In this paper, we shift attention from the constructive content of arithmetical sentences to the program of extracting bounding information from (semi-intuitionistic) proofs in arithmetic. The emphasis on bounds is, by no means, a novel idea: Ulrich Kohlenbach has, for some years now, been urging a shift of attention to numerical bounds (cf. [12] for a recent statement).

Bounded modified realizability, as we call the novel realizability notion, is based on an analysis of formulas which *always* disregards precise existential witnesses and decisions concerning disjunctive statements. In this respect, it is different from the *monotone realizability* introduced by Kohlenbach in [10], which is still a notion based on ordinary (modified) realizability. Our new form of realizability is inspired by work realized in [5] on a newly found *functional* interpretation (in the sense of Gödel [6]), but it is simpler than it in one crucial aspect. The new functional interpretation relies on *intensional* majorizability relations, regulated by rules, instead of axioms (with the effect of inducing the failure of the deduction theorem). This feature is mandatory because it prevents the functional interpretation from analyzing bounded quantifications of non-zero type, thereby managing to treat them as computationally empty. However, in the case of realizability, this *intensional* move is unnecessary because the ordinary majorizability relation is already treated as computationally empty by the realizability interpretation. As a consequence, in the framework herein presented, there are no intensional notions and the deduction theorem does hold.

This paper is organized as follows. In Section 2 we present the underlying framework of the paper. The central result of the work is the Soundness Theorem of Section 3. In this section, we also prove a Characterization Theorem for the notion of bounded modified realizability and study the adjunction of some conspicuous principles to our theories. We extend our analysis to the arithmetical setting in the Section 4 and lay out a commented list of some extra principles that mesh well with the new notion of realizability.

2 General framework

Except for the absence of the intensional majorizability relations, we follow the general framework of [5]. Accordingly, we let $\mathcal{L}^{\omega}_{\leq}$ be a language in all finite types (based on a given ground type 0) with a distinguished binary relation symbol $\leq_0$ (infixing between terms of type 0) and distinguished constants m of type $0 \to (0 \to 0)$ and z of type 0 (the constant z is needed to ensure that each finite type is inhabited by at least one closed term). The theory $\mathsf{IL}^{\omega}_{\leq}$ is intuitionistic logic in all finite types (see [2] for the Hilbert-type deduction system that we use) with axioms stating that $\leq_0$ is reflexive, transitive, and with the axioms

$$\mathsf{A}_1 \;:\; x \leq_0 \mathrm{m}(x,y) \wedge y \leq_0 \mathrm{m}(x,y)$$

$$\mathsf{A}_2 \;:\; x \leq_0 x' \wedge y \leq_0 y' \to \mathrm{m}(x,y) \leq_0 \mathrm{m}(x',y')$$

Equality does not pose particular problems for habitual realizability notions, and it is usually defined extensionally. However, bounded modified realizability does not seem to realize full extensionality (although it does realize important cases of it; see 1 of Subsection 4.3). Due to this feature, we opt for a treatment of equality based on the minimal alternative described by Troelstra in the end of section 3.1 and the beginning of section 3.3 of [14]. In the minimal alternative, there is a symbol of equality for terms of type 0 only. Its axioms are

$$\mathsf{E}_1 \;:\; x =_0 x$$

$$\mathsf{E}_2 \;:\; x =_0 y \wedge \phi[x/w] \to \phi[y/w]$$

where ϕ is an atomic formula with a distinguished type 0 variable w. In order to characterize the behavior of the logical constants (combinators) Π and Σ, we must also add

$$\mathsf{E}_{\Pi,\Sigma} \;:\; \phi[\Pi(x,y)/w] \leftrightarrow \phi[x/w], \quad \phi[\Sigma(x,y,z)/w] \leftrightarrow \phi[xz(yz)/w],$$

where ϕ is a an atomic formula with a distinguished variable w, and x, y and z are variables of appropriate type.

In the language $\mathcal{L}^{\omega}_{\leq}$ we can define Bezem's strong majorizability relation $\leq^*_\rho$ for type ρ and state its main properties (cf. [3]). The proofs are simple and well-known: see [5] for a recent reference. The relations are defined by induction on the types:

(a) $x \leq^*_0 y := x \leq_0 y$

(b) $x \leq^*_{\rho\to\sigma} y := \forall u^\rho, v^\rho (u \leq^*_\rho v \to xu \leq^*_\sigma yv \wedge yu \leq^*_\sigma yv))$

Lemma 1. $\mathsf{IL}^{\omega}_{\leq}$ *proves*

(i) $x \leq^* y \rightarrow y \leq^* y.$

(ii) $x \leq^* y \wedge y \leq^* z \rightarrow x \leq^* z,$

(iii) $x \leq y \wedge y \leq^* z \rightarrow x \leq^* z,$

where the relation $\leq_\sigma$ *is the pointwise "less than or equal to" relation: It is* $\leq_0$ *for type 0, and* $x \leq_{\rho\to\sigma} y$ *is defined recursively by* $\forall u^\rho (xu \leq_\sigma yu)$.

For convenience and clarity, the language $\mathcal{L}^{\omega}_{\leq}$ includes the primitive syntactic device of *bounded quantifications*, i.e. quantifications of the form $\forall x \leq^* t$ and $\exists x \leq^* t$, for terms t not containing the variable x. *Bounded formulas* are those formulas in which every quantifier is bounded. The theory $\mathsf{IL}^{\omega}_{\leq}$ has also the following schematic axioms:

$$\mathsf{B}_\forall \ : \ \forall x \leq^* t A(x) \leftrightarrow \forall x (x \leq^* t \rightarrow A(x))$$

$$\mathsf{B}_\exists \ : \ \exists x \leq^* t A(x) \leftrightarrow \exists x (x \leq^* t \wedge A(x)),$$

Definition 1. *We define, by induction on the type, the functional* m_ρ *of type* $\rho \to (\rho \to \rho)$ *according to the following clauses:*

(a) m_0 *is* m

(b) $\mathrm{m}_{\rho\to\sigma}(x, y) := \lambda u^\rho.\mathrm{m}_\sigma(xu, yu)$

It is well known that the combinators Π and Σ enjoy the property of *combinatorial completeness* whereby, given any term t with a distinguished variable u, there is a term $\lambda u.t$ whose free variables are those of t except for u, such that $(\lambda u.t)(q)$ is (in the sense of allowing the pertinent substitutions of one term for the other) $t[q/u]$. We are relying on the combinatorial completeness feature in the above definition.

Lemma 2. $\mathsf{IL}^{\omega}_{\leq}$ *proves*

(i) $x \leq^*_\rho x \wedge y \leq^*_\rho y \rightarrow x \leq^*_\rho \mathrm{m}(x, y) \wedge y \leq^*_\rho \mathrm{m}(x, y)$

(ii) $\mathrm{m}_\rho \leq^* \mathrm{m}_\rho$.

We now adapt the notion of majorizability theory introduced in [5] to our present situation:

Definition 2. *Consider a fixed language* $\mathcal{L}^\omega_\leq$. *A theory* T^ω *in* $\mathcal{L}^\omega_\leq$ *is called a* majorizability theory *for* $\mathcal{L}^\omega_\leq$ *if it extends* $\mathsf{IL}^\omega_\leq$ *and, for every constant* c^ρ, *there is a closed term* t^ρ *such that* $\mathsf{T}^\omega \vdash c \leq^*_\rho t$.

Lemma 3. *Let* T^ω *be a majorizability theory. For every closed term* t^ρ *of* $\mathcal{L}^\omega_\leq$ *there exists another closed term* $\tilde{t}^\rho$ *of* $\mathcal{L}^\omega_\leq$, *such that*

$$\mathsf{T}^\omega \vdash t \leq^* \tilde{t}.$$

Within the context of a language $\mathcal{L}^\omega_\leq$, we say that a term $\tilde{t}$ is a majorant of an (open) term t if it is a term with the same (free) variables as t and $\mathsf{T}^\omega \vdash \lambda \underline{w}.t \leq^* \lambda \underline{w}.\tilde{t}$. It is an easy consequence of Lemma 3 that, for majorizability theories, every (open) term has a majorant. A term t is called *monotone* if it is self-majorizing. To say that a functional f is monotone is to assume that $f \leq^* f$. In the sequel, we shall often quantify over monotone functionals. We abbreviate the quantifications $\forall f(f \leq^* f \to A(f))$ and $\exists f(f \leq^* f \wedge A(f))$ by $\tilde{\forall} f A(f)$ and $\tilde{\exists} f A(f)$, respectively.

An underlined term $\underline{t}$ stands for a (possibly empty) tuple of terms $t_1, t_2, \ldots, t_k$. We use the underlined notation for tuples in several contexts. For instance, $\forall \underline{x} \leq^* \underline{t}$ stands for the string $\forall x_1 \leq^* t_1 \forall x_2 \leq^* t_2 \ldots \forall x_k \leq^* t_k$, where $x_1, x_2, \ldots, x_k$ and $t_1, t_2, \ldots, t_k$ are sequences of variables, respectively terms, with matching types. After a while, we will no longer underline terms. It will be clear from the context when we are referring to a tuple of terms, instead of a single term.

3 The new realizability notion

In this section, we define bounded modified realizability and prove corresponding soundness and characterization theorems.

3.1 The soundness theorem

In order to define the bounded modified realizability we need the following syntactic notion:

Definition 3. *A formula of* $\mathcal{L}^\omega_\leq$ *is called* $\tilde{\exists}$-free *if it is built from atomic formulas by means of conjunction, disjunction, implication, bounded quantifications and monotone universal quantifications, i.e., quantifications of the form* $\tilde{\forall} a(\ldots)$.

This notion reminds the well-known notion of $\exists$-free formula (cf. [16]), but do observe that it allows disjunctions. We now proceed with the definition of the new realizability notion. We chose to define it in a slightly unfamiliar way. Instead of saying what are realizing tuples of functionals, we associate to each formula of the language an existential formula. One should see the existential tuples of this quantifier as the familiar places for the (purported) realizers of the given formula.

Definition 4. *To each formula A of the language $\mathcal{L}^{\omega}_{\leq}$ we associate formulas $(A)^{\mathrm{br}}$ and A_{br} of the same language so that $(A)^{\mathrm{br}}$ is of the form $\tilde{\exists}\underline{b}A_{\mathrm{br}}(\underline{b})$, with $A_{\mathrm{br}}(\underline{b})$ a $\tilde{\exists}$-free formula.*

1. *$(A)^{\mathrm{br}}$ and $(A)_{\mathrm{br}}$ are simply A, for atomic formulas A.*

If we have already interpretations for A and B given by $\tilde{\exists}\underline{b}A_{\mathrm{br}}(\underline{b})$ and $\tilde{\exists}\underline{d}B_{\mathrm{br}}(\underline{d})$ (respectively) then, we define

2. *$(A \wedge B)^{\mathrm{br}}$ is $\tilde{\exists}\underline{b}, \underline{d}(A_{\mathrm{br}}(\underline{b}) \wedge B_{\mathrm{br}}(\underline{d}))$,*
3. *$(A \vee B)^{\mathrm{br}}$ is $\tilde{\exists}\underline{b}, \underline{d}(A_{\mathrm{br}}(\underline{b}) \vee B_{\mathrm{br}}(\underline{d}))$,*
4. *$(A \to B)^{\mathrm{br}}$ is $\tilde{\exists}\underline{f}\tilde{\forall}\underline{b}(A_{\mathrm{br}}(\underline{b}) \to B_{\mathrm{br}}(\underline{f}\underline{b}))$.*

For bounded quantifiers we have:

5. *$(\forall x \leq^* t\, A(x))^{\mathrm{br}}$ is $\tilde{\exists}\underline{b}\forall x \leq^* t\, A_{\mathrm{br}}(\underline{b}, x)$,*
6. *$(\exists x \leq^* t\, A(x))^{\mathrm{br}}$ is $\tilde{\exists}\underline{b}\exists x \leq^* t\, A_{\mathrm{br}}(\underline{b}, x)$.*

And for unbounded quantifiers we define

7. *$(\forall x A(x))^{\mathrm{br}}$ is $\tilde{\exists}\underline{f}\tilde{\forall}a\forall x \leq^* a\, A_{\mathrm{br}}(\underline{f}a, x)$.*
8. *$(\exists x A(x))^{\mathrm{br}}$ is $\tilde{\exists}a, \underline{b}\exists x \leq^* a\, A_{\mathrm{br}}(\underline{b}, x)$.*

Let us insist again and say that the tuples above may be empty. It also is understood, for instance, that $(A \to B)_{\mathrm{br}}$ is $\tilde{\forall}\underline{b}(A_{\mathrm{br}}(\underline{b}) \to B_{\mathrm{br}}(\underline{f}\underline{b}))$. Similarly for the other clauses. As usual, the case of negation is a particular case of the implication:

9. $(\neg A)^{\mathrm{br}}$ is $\tilde{\forall}\underline{b}\neg A_{\mathrm{br}}(\underline{b})$.

By inspection on the above clauses it is clear that,

Lemma 4 (Monotonicity Lemma). *Let* $(A(x))^{\mathrm{br}}$ *be* $\tilde{\exists}\underline{b}A_{\mathrm{br}}(\underline{b}, \underline{x})$. *The following* monotonicity property *holds:*

$$\mathsf{T}^\omega \vdash \underline{b} \leq^* \underline{b}' \wedge A_{\mathrm{br}}(\underline{b}, \underline{x}) \to A_{\mathrm{br}}(\underline{b}', \underline{x}).$$

Proposition 1. *If* A *is a* $\tilde{\exists}$*-free formula then* $(A)^{\mathrm{br}}$ *is* A_{br} *and they are equivalent to* A.

Proof. The proof is by a straightforward induction on the complexity of the $\tilde{\exists}$-free formula. Note that we cannot say that $(A)^{\mathrm{br}}$ is exactly A because $\tilde{\forall}aA(a)$ is syntactically an abbreviation of $\forall a(a \leq^* a \to A(a))$ and hence, by definition, $(\tilde{\forall}aA(a))^{\mathrm{br}}$ is $\tilde{\forall}a\forall x \leq^* a\,(x \leq^* x \to A_{\mathrm{br}}(x))$. The latter formula is, of course, equivalent to $\tilde{\forall}aA_{\mathrm{br}}(a)$. □

For simplicity, in the next corollary and in the Soundness Theorem, we adopt as definition of $x \leq^*_{\rho\to\sigma} y$ the formula $\tilde{\forall}v^\rho\forall u \leq^*_\rho v(xu \leq^*_\sigma yv \wedge yu \leq^*_\sigma yv)$. Note that this definition is given by a $\tilde{\exists}$-free formula, and that (by (i) of Lemma 1) it is equivalent to the original one. With this proviso, the following is an immediate consequence of Proposition 1.

Corollary 1. *Given any type* ρ, *the formula* $(x \leq^*_\rho y)^{\mathrm{br}}$ *is* $(x \leq^*_\rho y)_{\mathrm{br}}$ *and they are equivalent to* $x \leq^*_\rho y$.

We state the Soundness Theorem in a strong form that already incorporates the principles which are automatically realized by the bounded modified realizability (*vide* the Characterization Theorem ahead). Two of these principles are related to similar principles that arise in the discussion of Kreisel's modified realizability (see [16]). The third principle is characteristic of the bounded modified realizability.

I. The *Bounded Choice Principle*

$$\mathsf{bAC}^{\rho,\tau}: \ \forall x^\rho\exists y^\tau A(x,y) \to \tilde{\exists}f^{\rho\to\tau}\tilde{\forall}b^\rho\forall x \leq^*_\rho b\exists y \leq^*_\tau fb\,A(x,y),$$

where A is an arbitrary formula of the language $\mathcal{L}^\omega_{\leq^*}$. The standard Axiom of Choice does not seem to be realizable in general. Still, in 3.2 we will see that a *monotone* version of the Axiom of Choice is interpreted by the Bounded Modified Realizability. For other discussions concerning choice, see Subsections 4.3 and 4.4.

II. The *Bounded Independence of Premises Principle*

$$\mathsf{bIP}^{\rho}_{\tilde{\exists}\mathrm{free}} : \ (A \to \exists y^{\rho} B(y)) \to \tilde{\exists} b^{\rho}(A \to \exists y \leq^{*}_{\rho} b\, B(y)),$$

where A is a $\tilde{\exists}$-free formula and B is an arbitrary formula. Even though we have stated $\mathsf{bIP}^{\rho}_{\tilde{\exists}\mathrm{free}}$ for a single variable y only, the *tuple* version is an easy consequence of the principle as stated. We will generalize this principle to a wider class of formulas A in the end of this section.

III. The *Majorizability Axioms*

$$\mathsf{MAJ}^{\rho} : \ \forall x^{\rho} \exists y^{\rho} (x \leq^{*}_{\rho} y).$$

Note that in the presence of this principle, every universal (resp. existential) quantification is equivalent to a monotone universal (resp. existential) quantification followed by a bounded quantification. An application of this fact is that, in the presence of MAJ^{ω}, we can allow plain universal quantifications in the definition of $\tilde{\exists}$-free formulas.

We use bAC^{ω}, $\mathsf{bIP}^{\omega}_{\tilde{\exists}\mathrm{free}}$ and MAJ^{ω}, respectively, for the aggregate of each of the above principles over all types. Before we go on with the Soundness Theorem, it is worth observing that a vast generalization of Brouwer's FAN theorem follows from the above three principles.

Proposition 2. *The theory* $\mathsf{IL}^{\omega}_{\leq} + \mathsf{bAC}^{\omega} + \mathsf{bIP}^{\omega}_{\tilde{\exists}\mathrm{free}} + \mathsf{MAJ}^{\omega}$ *proves the* Bounded Collection Principle

$$\mathsf{bBC}^{\rho,\tau} : \ \tilde{\forall} c(\forall z \leq^{*} c^{\rho} \exists y^{\tau} A(y,z) \to \tilde{\exists} b \forall z \leq^{*} c \exists y \leq^{*} b A(y,z)),$$

where A is an arbitrary formula.

Observation 1. *In the context of analysis, Brouwer's theorem is the case $\rho = 1$, $\tau = 0$. Note that the usual formulation of Brouwer's* FAN *theorem in, for instance, section 1.9.24 of [16], differs from the statement above in that it concerns continuity, as opposed to majorizability. In Section 4, we will discuss some related principles already considered by Kohlenbach in [10].*

Observation 2. *The case $\rho = \tau = 0$ extends the familiar* bounded collection principle *of arithmetic.*

Proof. We repeat an argument presented in [5] that also works in the present setting. Let c monotone be fixed. Assume that $\forall z(z \leq^{*} c^{\rho} \to \exists y^{\tau} A(y,z))$. By $\mathsf{bIP}^{\omega}_{\tilde{\exists}\mathrm{free}}$ we get $\forall z \tilde{\exists} b(z \leq^{*} c^{\rho} \to \exists y \leq^{*} bA(y,z))$, which by bAC^{ω} yields

$$\tilde{\exists} f \tilde{\forall} a \forall z \leq^* a \tilde{\exists} b \leq^* fa(z \leq^* c^\rho \to \exists y \leq^* bA(y, z)).$$

In turn, by (ii) of Lemma 1, this implies

$$\tilde{\exists} f \tilde{\forall} a \forall z \leq^* a(z \leq^* c^\rho \to \exists y \leq^* faA(y, z)).$$

Taking $a := c$, we get $\tilde{\exists} b \forall z \leq^* c \exists y \leq^* bA(y, z)$. □

The main theorem of this paper is:

Theorem 1 (Soundness). *Consider a fixed language* $\mathcal{L}^\omega_\leq$*. Let* T^ω *be a majorizability theory for* $\mathcal{L}^\omega_\leq$ *and assume that* $(A(\underline{z}))^{\mathrm{br}}$ *is* $\tilde{\exists} \underline{b} A_{\mathrm{br}}(\underline{b}, \underline{z})$*, where* $A(\underline{z})$ *is an arbitrary formula of* $\mathcal{L}^\omega_\leq$ *with its free variables as displayed. If*

$$\mathsf{IL}^\omega_\leq + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\mathrm{free}} + \mathsf{MAJ}^\omega \vdash A(\underline{z}),$$

then there are closed monotone terms $\underline{t}$ *of appropriate types such that*

$$\mathsf{T}^\omega \vdash \tilde{\forall} \underline{a} \forall \underline{z} \leq^* \underline{a}\, A_{\mathrm{br}}(\underline{t}\underline{a}, \underline{z}).$$

Proof. The proof proceeds by induction on the length of the derivation of A. For the sake of simplicity, we will not explicitly include the parameters z. The non-logical axioms are universal and, hence, are realized by the empty tuple. Let us consider the axioms for bounded quantifiers. We assume that each of the bounded principles $\mathsf{B}_\forall$ and $\mathsf{B}_\exists$ is a shorthand for two separate principles: the left-to-right and the right-to-left implications.

$\mathsf{B}_\forall$. $\forall x \leq^* t\, A(x) \leftrightarrow \forall x(x \leq^* t \to A(x))$. Assume that $(A(x))^{\mathrm{br}}$ is $\tilde{\exists} \underline{b} A_{\mathrm{br}}(\underline{b}, x)$. For the left-to-right implication we need to define monotone closed terms $\underline{q}$ such that

$$\tilde{\forall} \underline{b}(\forall x \leq^* t\, A_{\mathrm{br}}(\underline{b}, x) \to \tilde{\forall} c \forall x \leq^* c(x \leq^* t \to A_{\mathrm{br}}(\underline{q}\underline{b}c, x)))$$

Of course, the terms $\underline{q} := \lambda \underline{u} v.\underline{u}$ do the job. Concerning the right-to-left implication, we need monotone terms $\underline{q}$ such that

$$\tilde{\forall} \underline{b}(\tilde{\forall} c \forall x \leq^* c\,(x \leq^* t \to A_{\mathrm{br}}(\underline{b}c, x)) \to \forall x \leq^* t\, A_{\mathrm{br}}(\underline{q}\underline{b}, x))$$

In this case define $\underline{q} := \lambda \underline{u}.\underline{u}\tilde{t}$ where $\tilde{t}$ is a monotone majorant of t (whose existence is guaranteed in the majorizability theory T^ω). It is easy to see that the defined terms $\underline{q}$ are monotone and do the job.

$\mathsf{B}_\exists$. $\exists x \leq^* t\, A(x) \leftrightarrow \exists x(x \leq^* t \wedge A(x))$. Assume that $(A(x))^{\mathrm{br}}$ is $\tilde{\exists} \underline{b} A_{\mathrm{br}}(\underline{b}, x)$. For the left-to-right implication we need to define monotone terms $q, \underline{s}$ such that

$$\tilde{\forall}\underline{b}(\exists x \leq^* t\, A_{\rm br}(\underline{b}, x) \to \exists x \leq^* q\underline{b}(x \leq^* t \wedge A_{\rm br}(\underline{s}\underline{b}, x)))$$

Clearly the terms $q := \lambda\underline{u}.\tilde{t}$ where $\tilde{t}$ is a monotone majorant of t (whose existence is guaranteed in the majorizability theory T^ω) and $\underline{s} := \lambda\underline{u}.\underline{u}$ do the job. Concerning the reverse direction, we must define terms $\underline{q}$ such that

$$\tilde{\forall}c, \underline{b}(\exists x \leq^* c\,(x \leq^* t \wedge A_{\rm br}(\underline{b}, x)) \to \exists x \leq^* t\, A_{\rm br}(\underline{q}c\underline{b}, x)).$$

Just define $\underline{q} := \lambda u\underline{v}.\underline{v}$.

For the logical axioms and rules assume that $(A)^{\rm br}$ is $\tilde{\exists}\underline{b}A_{\rm br}(\underline{b})$, $(B)^{\rm br}$ is $\tilde{\exists}\underline{c}B_{\rm br}(\underline{c})$ and $(C)^{\rm br}$ is $\tilde{\exists}\underline{d}C_{\rm br}(\underline{d})$.

1. $A, A \to B \Rightarrow B$. By induction hypothesis we have monotone terms $\underline{t}$ and $\underline{q}$ such that $A_{\rm br}(\underline{t})$ and $\tilde{\forall}\underline{b}(A_{\rm br}(\underline{b}) \to B_{\rm br}(\underline{q}\underline{b}))$. Clearly $\underline{s} := \underline{q}(\underline{t})$ is monotone and $B_{\rm br}(\underline{s})$ holds.

2. $A \to B, B \to C \Rightarrow A \to C$. By induction hypothesis we have monotone terms $\underline{t}$ and $\underline{q}$ such that $\tilde{\forall}\underline{b}(A_{\rm br}(\underline{b}) \to B_{\rm br}(\underline{t}\underline{b}))$ and $\tilde{\forall}\underline{c}(B_{\rm br}(\underline{c}) \to C_{\rm br}(\underline{q}\underline{c}))$. Clearly $\underline{s} := \lambda\underline{u}.\underline{q}(\underline{t}\underline{u})$ is monotone and $\tilde{\forall}\underline{b}(A_{\rm br}(\underline{b}) \to C_{\rm br}(\underline{s}\underline{b}))$ holds.

3a. $A \vee A \to A$. We need monotone terms $\underline{t}$ such that

$$\tilde{\forall}\underline{b}, \underline{c}(A_{\rm br}(\underline{b}) \vee A_{\rm br}(\underline{c}) \to A_{\rm br}(\underline{t}\underline{b}\underline{c})).$$

By monotonicity (Lemmas 2 and 4), it is clear that the terms $\underline{t} := \lambda\underline{u}\underline{v}.\mathrm{m}(\underline{u}, \underline{v})$ do the job.

3b. $A \to A \wedge A$. Trivial.

4. $A \to A \vee B$ and $A \wedge B \to A$. Trivial.

5. $A \vee B \to B \vee A$ and $A \wedge B \to B \wedge A$. Trivial.

6. $A \to B \Rightarrow C \vee A \to C \vee B$. By induction hypothesis we have monotone terms $\underline{t}$ such that $\tilde{\forall}\underline{b}(A_{\rm br}(\underline{b}) \to B_{\rm br}(\underline{t}\underline{b}))$. We must define monotone terms $\underline{q}$ and $\underline{s}$ such that

$$\tilde{\forall}\underline{c}, \underline{b}(C_{\rm br}(\underline{c}) \vee A_{\rm br}(\underline{b}) \to C_{\rm br}(\underline{q}\underline{c}\underline{b}) \vee B_{\rm br}(\underline{s}\underline{c}\underline{b})).$$

It is clear that $\underline{q} := \lambda\underline{u}, \underline{v}.\underline{u}$ and $\underline{s} := \lambda\underline{u}, \underline{v}.\underline{v}$ do the job.

7. $(A \wedge B) \to C \Rightarrow A \to (B \to C)$ and $A \to (B \to C) \Rightarrow (A \wedge B) \to C$. Due to *currying* the terms that realize one of the formulas also realize the other.

8. $\bot \to A$. Trivial.

9. $A \to B(z) \Rightarrow A \to \forall z B(z)$. By induction hypothesis there are monotone terms $\underline{t}$ such that $\tilde{\forall}a\forall z \leq^* a\,\tilde{\forall}\underline{b}(A_{\rm br}(\underline{b}) \to B_{\rm br}(\underline{t}a\underline{b}, z))$. Let $q := \lambda\underline{u}v.tv\underline{u}$. Clearly q is monotone and realizes the conclusion.

10. $\forall x A(x) \to A(t)$. We must define monotone terms $\underline{q}$ such that

$$\tilde{\forall}\underline{f}(\tilde{\forall}a\forall x \leq^* a\, A_{\rm br}(\underline{f}a, x) \to A_{\rm br}(\underline{qf}, t)).$$

Let $\tilde{t}$ be a monotone majorant of t (whose existence is guaranteed in the majorizability theory T^ω), and define $\underline{q} := \lambda \underline{f}.\underline{f}\tilde{t}$. These terms do the job.

11. $A(t) \to \exists x A(x)$. We need to define monotone closed terms $q, \underline{r}$ such that

$$\tilde{\forall}\underline{b}(A_{\rm br}(\underline{b}, t) \to \exists x \leq^* q\underline{b}\, A_{\rm br}(\underline{rb}, x)).$$

Let $\tilde{t}$ be a monotone majorant of t in the majorizability theory T^ω (whose existence is guaranteed in the majorizability theory T^ω), and define $q := \lambda\underline{u}.\tilde{t}$ and $\underline{r} := \lambda\underline{v}.\underline{v}$.

12. $A(z) \to B \Rightarrow \exists z A(z) \to B$. It is easy to see that the terms that realize the left-hand side also realize the right-hand side.

Finally, we need to check the principles bAC^ω, $\mathsf{bIP}^\omega_{\tilde{\exists}\text{free}}$ and MAJ^ω:

bAC^ω. In order to realize bAC^ω, an easy computation shows that it is enough to define monotone terms t and $\underline{q}$ such that,

$$\tilde{\forall}f, \underline{g}(\tilde{\forall}a\forall x \leq^* a\, \exists y \leq^* fa\, A_{\rm br}(\underline{g}a, x, y) \to \tilde{\forall}a\forall x \leq^* a\, \exists y \leq^* tf\underline{g}a\, A_{\rm br}(\underline{q}f\underline{g}a, x, y)).$$

Clearly the projections $t := \lambda f, \underline{g}.f$ and $\underline{q} := \lambda f, \underline{g}.\underline{g}$ do the job.

$\mathsf{bIP}^\omega_{\tilde{\exists}\text{free}}$. It is straightforward to see that one needs monotone terms t and $\underline{q}$ such that,

$$\tilde{\forall}b, \underline{c}((A \to \exists y \leq^* b\, B_{\rm br}(\underline{c}, y)) \to (A \to \exists y \leq^* tb\underline{c}\, B_{\rm br}(\underline{q}b\underline{c}, y))),$$

for A an $\exists$-free formula. The projections $t := \lambda u, \underline{v}.u$ and $\underline{q} := \lambda u, \underline{v}.\underline{v}$ do the job.

MAJ^ω. For each type ρ we need a monotone term $t^{\rho\to\rho}$ such that,

$$\tilde{\forall}a^\rho\forall x \leq^*_\rho a\exists y \leq^* ta\, (x \leq^*_\rho y).$$

The identity functional $t := \lambda u^\rho.u$ does the job. $\Box$

3.2 The characterization theorem

There is a proof of the following result in [5] (in a slightly different setting):

Proposition 3 (Monotone Axiom of Choice). *The theory* $\mathsf{IL}^\omega_\leq + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\text{free}}$ *proves*

$$(\tilde{\forall}\underline{a}\tilde{\forall}\underline{b}\tilde{\forall}\underline{b}' \leq^* \underline{b}[A(\underline{a},\underline{b}') \to A(\underline{a},\underline{b})] \wedge \tilde{\forall}\underline{a}\tilde{\exists}\underline{b}A(\underline{a},\underline{b})) \to \tilde{\exists}\underline{f}\tilde{\forall}\underline{a}A(\underline{a},\underline{f}(\underline{a})),$$

where A is an arbitrary formula of the language $\mathcal{L}^{\omega}_{\leq}$.

Theorem 2 (Characterization). *Let A be an arbitrary formula of $\mathcal{L}^{\omega}_{\leq}$. Then*

$$\mathsf{IL}^{\omega}_{\leq^*} + \mathsf{bAC}^{\omega} + \mathsf{bIP}^{\omega}_{\tilde{\exists}\text{free}} + \mathsf{MAJ}^{\omega} \vdash A \leftrightarrow (A)^{\text{br}}.$$

Proof. The proof is by induction on the complexity of the formula A. There is nothing to prove concerning atomic formulas. Suppose that $\mathsf{IL}^{\omega}_{\leq^*} + \mathsf{bAC}^{\omega} + \mathsf{bIP}^{\omega}_{\tilde{\exists}\text{free}} + \mathsf{MAJ}^{\omega}$ proves the equivalences $B \leftrightarrow \tilde{\exists}\underline{b}B_{\text{br}}(\underline{b})$ and $C \leftrightarrow \tilde{\exists}\underline{c}C_{\text{br}}(\underline{c})$.

The cases when A is $B \wedge C$ or $B \vee C$ are straightforward. Let A be $\forall x \leq^* tB(x)$. In this case, $(A)^{\text{br}}$ is (by definition) $\exists \underline{b} \forall x \leq^* t\, B_{\text{br}}(\underline{b}, x)$. On the other hand, by induction hypothesis, A is equivalent to $\forall x \leq^* t\, \tilde{\exists}\underline{b}B_{\text{br}}(\underline{b}, x)$. By bBC^{ω} (see Proposition 2) and the monotonicity of B_{br} (see Lemma 4), we can switch the quantifier $\forall x \leq^* t$ with the quantifiers $\tilde{\exists}\underline{b}$, as wanted. Similarly, let A be $\forall x B(x)$. In this case, A_{br} is (by definition) $\tilde{\exists}\underline{f}\tilde{\forall}a\forall x \leq^* a\, B_{\text{br}}(\underline{f}a, x)$. On the other hand, by induction hypothesis, A is equivalent to $\forall x\tilde{\exists}\underline{b}B_{\text{br}}(\underline{b}, x)$, and hence (by MAJ^{ω}) to $\tilde{\forall}a\forall x \leq^* a\, \tilde{\exists}\underline{b}B_{\text{br}}(\underline{b}, x)$. In turn, by bBC^{ω}, this is equivalent to $\tilde{\forall}a\tilde{\exists}\underline{b}\forall x \leq^* a\, B_{\text{br}}(\underline{b}, x)$. Now, we apply the Monotone Axiom of Choice (cf. Proposition 3) to get the equivalent $\tilde{\exists}\underline{f}\tilde{\forall}a\forall x \leq^* a\, B_{\text{br}}(\underline{f}a, x)$, as desired. The cases of the existential quantifier and the bounded existential quantifier are straightforward.

It remains to study implication. Suppose that A is $B \to C$. Assume $(A)^{\text{br}}$. By definition, we have $\tilde{\exists}\underline{f}\tilde{\forall}\underline{b}(B_{\text{br}}(\underline{b}) \to C_{\text{br}}(\underline{f}\underline{b}))$. Using the induction hypothesis twice, it is clear that this implies $B \to C$. Conversely, suppose $B \to C$. By the induction hypothesis (twice), we have $\tilde{\exists}\underline{b}B_{\text{br}}(\underline{b}) \to \tilde{\exists}\underline{c}C_{\text{br}}(\underline{c})$. Intuitionistic logic yields $\tilde{\forall}\underline{b}(B_{\text{br}}(\underline{b}) \to \tilde{\exists}\underline{c}C_{\text{br}}(\underline{c}))$. By $\mathsf{bIP}^{\omega}_{\tilde{\exists}\text{free}}$ and the monotonicity of C_{br} (cf. Lemma 4), we get $\tilde{\forall}\underline{b}\tilde{\exists}\underline{c}(B_{\text{br}}(\underline{b}) \to C_{\text{br}}(\underline{c}))$. By an application of the Monotone Axiom of Choice (cf. Proposition 3) we conclude $(A)^{\text{br}}$. □

3.3 Conspicuous logical forms

We study logical forms which are equivalent, imply or are implied by their own $^{\text{br}}$ associates (this parallels similar studies in ordinary realizability, e.g. in [16]). Proposition 1 gives a special role to $\tilde{\exists}$-free formulas. Let us define two superclasses of the class of $\tilde{\exists}$-free formulas.

Definition 5. *Let $\mathcal{L}^{\omega}_{\trianglelefteq}$ be a fixed language. We define the classes of formulas Γ_{br} and Π_{br} according to the following clauses:*

i. *Atomic formulas are in Γ_{br} and Π_{br}.*

ii. *The class Γ_{br} is closed under conjunctions, disjunctions, bounded quantifications, monotone universal quantifications and existential quantifications.*

iii. *The class Π_{br} is closed under conjunctions, disjunctions, bounded quantifications and universal quantifications.*

iv. *If A is in Γ_{br} and B is in Π_{br} then $(A \to B)$ is in Π_{br} and $(B \to A)$ is in Γ_{br}.*

Note that $\tilde{\exists}$-free formulas are simultaneously in Γ_{br} and Π_{br}.

Proposition 4. *Let A be in Γ_{br}. Then $\mathsf{IL}^{\omega}_{\trianglelefteq} \vdash (A)^{\mathrm{br}} \to A$. Let B be in Π_{br}. Then $\mathsf{IL}^{\omega}_{\trianglelefteq} \vdash B \to (B)^{\mathrm{br}}$. Moreover, in the latter case, $(B)^{\mathrm{br}}$ is B_{br}.*

Proof. We prove the claims simultaneously, by induction on the complexity of formulas. The atomic case is clear, as well as conjunction, disjunction and existential bounded quantifications (since, modulo logical equivalence, these commute with the $^{\mathrm{br}}$-transformation). The cases of existential quantifications regarding the Γ_{br}-class and of universal quantification regarding the Π_{br} class pose no difficulty. Let us look at the monotone universal quantifier regarding the class Γ_{br}. Suppose that A is $\tilde{\forall} a C(a)$, with $C(a)$ in Γ_{br}. Then $(A)^{\mathrm{br}}$ is $\tilde{\exists} \underline{f} \tilde{\forall} a \forall x \leq^* a (x \leq^* x \to C_{\mathrm{br}}(\underline{f} a, x))$. By induction hypothesis, this entails A.

It remains to study *implication*. Suppose that A is in Γ_{br} and B is in Π_{br}. By induction hypothesis, $(A \to B)^{\mathrm{br}}$ is $\tilde{\forall} \underline{b} (A_{\mathrm{br}}(\underline{b}) \to B_{\mathrm{br}})$ and this is implied by $A \to B$. On the other hand, $(B \to A)^{\mathrm{br}}$ is $\tilde{\exists} \underline{b} (B_{\mathrm{br}} \to A_{\mathrm{br}}(\underline{b}))$. By induction hypothesis, this implies $B \to A$. $\square$

If one adds a principle of the form Π_{br} to our theory, the above proposition shows that such a principle is trivially realized within a theory that includes it. Therefore, we have the following version of the Soundness Theorem:

Theorem 3 (Soundness, Specific Extension). *Consider a fixed language $\mathcal{L}^{\omega}_{\trianglelefteq}$. Let T^{ω} be a majorizability theory for $\mathcal{L}^{\omega}_{\trianglelefteq}$ and assume that $(A(\underline{z}))^{\mathrm{br}}$ is $\tilde{\exists} \underline{b} A_{\mathrm{br}}(\underline{b}, \underline{z})$, where $A(\underline{z})$ is an arbitrary formula of $\mathcal{L}^{\omega}_{\trianglelefteq}$ with its free variables as displayed. If*

$$\mathsf{IL}^\omega_\leq + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde\exists\text{free}} + \mathsf{MAJ}^\omega + \Delta \vdash A(\underline{z}),$$

where Δ *is a collection of* $\Pi_{\rm br}$ *sentences, then there are closed monotone terms* $\underline{t}$ *of appropriate types such that*

$$\mathsf{T}^\omega + \Delta \vdash \tilde\forall \underline{a} \forall \underline{z} \leq^* \underline{a}\, A_{\rm br}(\underline{t}\underline{a}, \underline{z}).$$

The following is an immediate consequence of the above theorem and of Proposition 4:

Corollary 2. *Consider a language* $\mathcal{L}^\omega_\leq$ *and* T^ω *a majorizability theory for it. Let* Δ *be a collection of* $\Pi_{\rm br}$ *sentences. Then, the theory* $\mathsf{IL}^\omega_\leq + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde\exists\text{free}} + \mathsf{MAJ}^\omega + \Delta$ *is conservative over* $\mathsf{T}^\omega + \Delta$ *with respect to sentences in* $\Gamma_{\rm br}$*.*

We finish this section with some notes on the classes $\Pi_{\rm br}$ and $\Gamma_{\rm br}$ in the presence of the principles $\mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde\exists\text{free}} + \mathsf{MAJ}^\omega$.

Proposition 5. *Consider a language* $\mathcal{L}^\omega_\leq$*. Over the theory* $\mathsf{IL}^\omega_\leq + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde\exists\text{free}} + \mathsf{MAJ}^\omega$ *every formula is equivalent to a formula in* $\Gamma_{\rm br}$*, and every formula in* $\Pi_{\rm br}$ *is equivalent to an* $\tilde\exists$*-free formula.*

Proof. By the Characterization Theorem, every formula A is equivalent (modulo the theory $\mathsf{IL}^\omega_\leq + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde\exists\text{free}}$) to $(A)^{\rm br}$. This latter formula has the form $\tilde\exists \underline{b} A_{\rm br}(\underline{b})$, where $A_{\rm br}(\underline{b})$ is a $\tilde\exists$-free formula. Now, note that the class $\Gamma_{\rm br}$ is closed under existential monotone quantifications. The proof that formulas in $\Pi_{\rm br}$ are equivalent to $\tilde\exists$-free formulas is by induction on the complexity of the formula. We only need to check the conditional clause. Suppose that $A \in \Gamma_{\rm br}$ and $B \in \Pi_{\rm br}$. Since A is equivalent to $\tilde\exists \underline{b} A_{\rm br}(\underline{b})$ if follows that $A \to B$ is intuitionistically equivalent to $\tilde\forall \underline{b}(A(\underline{b})_{\rm br} \to B)$. This formula is equivalent to a $\tilde\exists$-free formula if B is. □

Corollary 3. *The theory* $\mathsf{IL}^\omega_\leq + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde\exists\text{free}} + \mathsf{MAJ}^\omega$ *proves the following principle of independence of premises* $\mathsf{bIP}^\omega_{\Pi_{\rm br}}$*:*

$$(A \to \exists y B(y)) \to \tilde\exists b (A \to \exists y \leq^* b\, B(y)),$$

where A *is in* $\Pi_{\rm br}$*,* B *is an arbitrary formula and* y *is of arbitrary type.*

Note that negated formulas are in $\Pi_{\rm br}$ modulo $\mathsf{IL}^\omega_\leq + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde\exists\text{free}} + \mathsf{MAJ}^\omega$ (use the above Proposition). Therefore, the scheme $\mathsf{bIP}^\omega_\neg$:

$$(\neg A \to \exists y B(y)) \to \tilde\exists b (\neg A \to \exists y \leq^* b\, B(y)),$$

is provable in $\mathsf{IL}^\omega_\leq + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde\exists\text{free}} + \mathsf{MAJ}^\omega$ for arbitrary formulas A and B.

4 The role of some arithmetical principles

As an application of Theorem 3, we will discuss some arithmetical principles whose realizations follow from themselves. We must start our discussion by extending the general framework based on $\mathsf{IL}^{\omega}_{\leq}$ to arithmetic.

4.1 Theories of arithmetic

The arithmetical theories that concern us here are: HA^{ω}, $\mathsf{PRA}^{\omega}_{i}$ and $\mathsf{G}_n\mathsf{A}^{\omega}_{i}$ ($n \geq 2$). The first one is an intuitionistic expansion of Gödel's quantifier-free calculus T with quantifiers ranging over each finite type. The theory $\mathsf{PRA}^{\omega}_{i}$ differs from HA^{ω} because it only has the "predicative" recursors $\hat{\mathrm{R}}_{\sigma}$ due to Kleene (see [2] for a description of these recursors) and, correspondingly, induction in the following restricted form:

$$A(0) \wedge \forall x(A(x) \to A(\mathrm{S}x)) \to \forall x A(x),$$

where A is a quantifier-free formula. It is known that the type 1 closed terms of the language of $\mathsf{PRA}^{\omega}_{i}$ define the primitive recursive functions. The theories $\mathsf{G}_n\mathsf{A}^{\omega}_{i}$ ($n \geq 2$) were introduced by Kohlenbach (see [9] for details). They form a sequence of increasing strength, closely related to the levels of Grzegorczyk's hierarchy of primitive recursive functions. The principle of induction present is the restricted one above.

These theories must be set up within the general framework of Section 2. Accordingly, the treatment of equality in these theories is the minimal one due to Troelstra in [14] (note the treatment of the impredicative recursors). We define the usual less than or equal numerical relation $\leq_0$, and the usual term $\max^{0\to(0\to0)}$, giving the maximum of two numbers. Under these theories, $\leq_0$ is a reflexive and transitive relation and max satisfies the axioms A_1 and A_2. It is clear that we may suppose that $\leq_0$ and max are primitive symbols of the language, and may take 0 as the distinguished constant of type zero.

For the sake of completeness, we must say that all the above theories of arithmetic include: (a) a minimization functional μ_b of type $(0 \to 1) \to 1$ such that $\mu_b f^{0\to1} n^0 =_0 \min_0 k \leq_0 n(fnk =_0 0)$ if such a $k \leq_0 n$ exists, and $=_0 0$ otherwise; and (b) a maximization functional M of type $1 \to 1$ satisfying the equations $Mf0 =_0 f0$ and $Mf(n+1) =_0 \max_0(Mfn, f(n+1))$. We usually write f^M instead of Mf. Note that $f \leq^*_1 f^M$.

The following result is due to Howard [7] for HA^{ω}, and to Kohlenbach [9] for the other theories:

Proposition 6. *The theories* HA^ω, PRA^ω *and* $\mathsf{G}_n\mathsf{A}_i^\omega$ *($n \geq 2$) are majorizability theories.*

The soundness theorem has an arithmetical extension:

Theorem 4 (Soundness, Arithmetical Extension). *Let* T^ω *be one of the theories* HA^ω, PRA_i^ω *or* $\mathsf{G}_n\mathsf{A}_i^\omega$ *($n \geq 2$), and assume* $(A(\underline{z}))^{\mathrm{br}} := \tilde{\exists}\underline{b}A_{\mathrm{br}}(\underline{b},\underline{z})$, *where* $A(\underline{z})$ *is an arbitrary formula of* $\mathcal{L}_\leq^\omega$ *with its free variables as displayed. If*

$$\mathsf{T}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\mathrm{free}} + \mathsf{MAJ}^\omega + \Delta \vdash A(\underline{z}),$$

where Δ *is a collection of* Π_{br} *sentences, then there are closed monotone terms* $\underline{t}$ *of appropriate types of the language of* T^ω *such that*

$$\mathsf{T}^\omega + \Delta \vdash \tilde{\forall}\underline{a}\forall\underline{z} \leq^* \underline{a}\, A_{\mathrm{br}}(\underline{ta},\underline{z}).$$

Proof. It is enough to see that the arithmetical axioms are realized in T^ω. This is clearly the case for the arithmetical axioms that are universal statements, as well as for the quantifier-free induction axioms of the theories PRA_i^ω and $\mathsf{G}_n\mathsf{A}_i^\omega$ ($n \geq 2$). It remains to see that the scheme of unrestricted induction

$$A(0) \wedge \forall n(A(n) \to A(n+1)) \to \forall n A(n)$$

can be interpreted in HA^ω. An easy computation shows that one must present a monotone functionals $\underline{t}$ such that

$$\tilde{\forall}\underline{a},\underline{\Phi}(A_{\mathrm{br}}(\underline{a},0) \wedge \forall n\tilde{\forall}\underline{b}(A_{\mathrm{br}}(\underline{b},n) \to A_{\mathrm{br}}(\underline{\Phi b}n,n+1)) \to \forall n A_{\mathrm{br}}(\underline{ta\Phi}n,n))$$

Using the recursors of Gödel's T we define $\underline{t}$ such that $\underline{ta\Phi}0$ is $\underline{a}$ and $\underline{ta\Phi}(n+1)$ is $\max(\underline{\Phi}(\underline{ta\Phi}n)n,\underline{ta\Phi}n)$. This functional clearly does the job. $\square$

The theory $\mathsf{G}_2\mathsf{A}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\mathrm{free}} + \mathsf{MAJ}^\omega$ refutes Markov's principle and, therefore, it is *classically inconsistent*. In effect, suppose that

$$\forall x^1(\neg\neg\exists n^0(xn = 0) \to \exists n^0(xn = 0)).$$

By intuitionistic logic and $\mathsf{bIP}^\omega_{\tilde{\exists}\mathrm{free}}$, $\forall x^1\exists n^0(\neg\forall k^0(xk \neq 0) \to \exists i \leq n(xi = 0))$. By the bounded collection principle, one infers that there is a natural number l^0 such that $\forall x \leq_1 1(\neg\forall k^0(xk \neq 0) \to \exists n \leq l(xn = 0))$. This is a contradiction (just consider the number theoretic function that takes the value 1 for values less than or equal to l and is 0 afterwards).

Nevertheless, the theory $\mathsf{G}_2\mathsf{A}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\mathrm{free}} + \mathsf{MAJ}^\omega$ is *intuitionistic consistent* (relative to $\mathsf{G}_2\mathsf{A}^\omega$). This is a corollary of the Soundness Theorem:

Corollary 4. *Let* T^ω *be one of the theories* HA^ω*,* PRA^ω_i *or* $\mathsf{G}_n\mathsf{A}^\omega_i$ *(*$n \geq 2$*). The theory* $\mathsf{T}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\text{free}} + \mathsf{MAJ}^\omega$ *is consistent relative to* T^ω*.*

This corollary has an interesting interpretation (we are grateful to Reinhard Kahle for pointing this out to us). It can be viewed as the realization of a non-standard Hilbert program (*pace* the intuitionistic setting) for the "ideal" (and rather strange) world of $\mathsf{HA}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\text{free}} + \mathsf{MAJ}^\omega$. It is a consequence of the next corollary that this "ideal" world is even Π^0_2-conservative over the "real" world.

Corollary 5. *Under the conditions of Theorem 4, if*

$$\mathsf{T}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\text{free}} + \mathsf{MAJ}^\omega + \Delta \vdash \forall x^\rho \exists y^\tau A(x, y),$$

where $\rho \in \{0, 1\}$*,* τ *is arbitrary,* A *is in* Γ_{br} *and the variables are as displayed, then there is a closed monotone term* $t^{\rho\to\tau}$ *of the language of* T^ω *such that*

$$\mathsf{T}^\omega + \Delta \vdash \forall x \exists y \leq^* tx\, A(x, y).$$

Proof. Clearly, $(\exists y^\tau A(x, y))^{\text{br}}$ is $\tilde{\exists} a^\tau, \underline{b} \exists y \leq^*_\tau a\, A_{\text{br}}(\underline{b}, x, y)$. Therefore, by Theorem 4, there are closed monotone terms t and $\underline{q}$ such that the theory $\mathsf{T}^\omega + \Delta$ proves $\exists y \leq^*_\tau tx^M A_{\text{br}}(\underline{q}x^M, x, y)$ (the case $\rho = 0$ is even simpler). The result follows from Proposition 4. □

As it is well-known, when $\rho = \tau = 0$, the functional t is of type 1 and corresponds to an $< \epsilon_0$-recursive function (respectively, primitive recursive, function in the n-level of the Grzegorczyk's hierarchy) when T^ω is HA^ω (respectively, PRA^ω_i, $\mathsf{G}_n\mathsf{A}^\omega_i$ with $n \geq 2$).

The following result is an adaptation of a result of Kohlenbach in [10].

Corollary 6. *Under the conditions of Theorem 4, if* t *is a closed term of the language,* C *is in* Π_{br}*,* D *is in* Γ_{br} *and*

$$\mathsf{T}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\text{free}} + \mathsf{MAJ}^\omega + \Delta \vdash \forall u^1 \forall v \leq_\gamma tu(C \to \exists w^2 D(w)),$$

with the free variables as displayed, then there is a closed monotone term Ψ *of the language of* T^ω *such that*

$$\mathsf{T}^\omega + \Delta \vdash \forall u^1 \forall v \leq_\gamma tu(C \to \exists w \leq_2 \Psi u\, D(w)).$$

Proof. Clearly, $\forall u \forall v \leq^*_\gamma \tilde{t}u^M(v \leq_\gamma tu \wedge C \to \exists w^2 D(w))$ is a consequence of the major theory above, where $\tilde{t}$ is a closed term such that $t \leq^* \tilde{t}$ (see Proposition 6). By $\mathsf{bIP}^\omega_{\Pi_{\text{br}}}$ and bBC^ω, we conclude that

$$\forall u \exists a^2 \forall v \leq^*_\gamma \tilde{t}u^M (v \leq_\gamma tu \wedge C \to \exists w \leq^*_2 a\, D(w)).$$

Since the formula that follows the quantifier $\exists a^2$ is in Γ_{br}, by the previous corollary there is a closed monotone term $r^{1\to 2}$ such that

$$\mathsf{T}^\omega + \Delta \vdash \forall u \forall v \leq^*_\gamma \tilde{t}u^M (v \leq_\gamma tu \wedge C \to \exists w \leq^*_2 ru\, D(w)).$$

Define Ψu^1 as $\lambda f^1.ruf^M$. By a simple computation and (iii) of Lemma 1, we get the desired result. □

4.2 About certain principles (Part I)

Principles that have the form Π_{br} (those which may occur in Δ above) are very convenient because they are self-realizable. However, principles that *follow* from the theory $\mathsf{T}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\mathrm{free}} + \mathsf{MAJ}^\omega$ are even better since they are not necessary for the verification of their own realizability. As a consequence, even if they are false principles certain *truthful* bounding information can be obtained from their use because they are not needed for the verification of the bounds. (In this paper, talk about 'true' and 'false' refers to the full set-theoretic structure associated with the language of arithmetic in all finite types, where the zero type ranges over the natural numbers.) This is the case with certain versions of the FAN theorem, under the heading of *uniform boundedness principles* (these were so baptized and studied by Kohlenbach in, e.g., [9] and [10]). Before we look at them, let us first briefly consider some forms of choice since, in general, choice principles do not come for free within the framework of the bounded modified realizability (contrary to Kohlenbach's monotone modified realizability).

It is an easy consequence of bAC^ω that

$$\forall x^\rho \exists y A(x,y) \to \tilde{\exists} f \forall x \exists y \leq^* fx\, A(x,y)$$

for $\rho \in \{0,1\}$, y of any type and A arbitrary. A simple trick shows that when the type of y is 0, 1 or 2 then the inequality $y \leq^* fx$ may be replaced by $y \leq fx$. Remark further that if A is quantifier-free and y is of type 0 then we can obtain the usual form of choice using a bounded search. In the next section we will say more about the usual form of choice.

Kohlenbach's *principle of uniform boundedness* UB_ρ for the type ρ is the following combination of the FAN theorem with choice:

$$\begin{cases} \forall y^{0\to\rho} (\forall k^0 \forall x \leq_\rho yk\, \exists z^0 A(x,y,k,z) \\ \qquad \to \exists \chi^1 \forall k^0 \forall x \leq_\rho yk\, \exists z \leq_0 \chi k\, A(x,y,k,z)), \end{cases}$$

for arbitrary A.

Proposition 7. *Let ρ be a given type. The principle of uniform boundedness* UB_ρ *is provable in* $\mathsf{T}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\text{free}} + \mathsf{MAJ}^\omega$.

Proof. Fix $y^{0\to\rho}$ and assume that $\forall k^0 \forall x \leq_\rho yk\, \exists z^0 A(x,y,k,z)$. By MAJ^ω, take $\tilde{y}$ such that $y \leq^* \tilde{y}$. Take k^0. Then, $\forall x \leq^*_\rho \tilde{y}k\, (x \leq_\rho yk \to \exists z^0 A(x,y,k,z))$. We can apply $\mathsf{bIP}^\omega_{\Pi_{\text{br}}}$ (see Corollary 3) and move the existential quantifier $\exists z^0$ to the front of the implication. Using bBC^ω and the arbitrariness of k we get $\forall k^0 \exists n^0 \forall x \leq^*_\rho \tilde{y}k\, \exists z \leq_0 n\, (x \leq_\rho yk \to A(x,y,k,z))$. By a form of choice discussed above and (iii) of Lemma 1, we get the desired conclusion. $\square$

This is perhaps a good place to say a few words concerning bounded realizability *vis-à-vis* Kohlenbach's monotone realizability. Kohlenbach's interpretation (cf. [10]) relies upon the transformation of formulas of Kreisel's modified realizability, differing from it in the statement of the soundness theorem. Let $(A)^{\text{mr}}$ and A_{mr} denote the formulas assigned to a given formula A by the modified realizability interpretation of Kreisel (here $(A)^{\text{mr}}$ is of the form $\exists \underline{x} A_{\text{mr}}(\underline{x})$, with $A_{\text{mr}}(\underline{x})$ a $\exists$-free formula). The soundness theorem for monotone realizability guarantees the existence of monotone closed terms $\underline{t}$ such that

$$\mathsf{T}^\omega \vdash \exists \underline{x} \leq^* \underline{t} \forall \underline{z} A_{\text{mr}}(\underline{x}(\underline{z}), \underline{z}),$$

whenever the theory $\mathsf{T}^\omega + \mathsf{AC}^\omega + \mathsf{IP}^\omega_{\exists\text{free}}$ proves $A(\underline{z})$. The principles AC^ω and $\mathsf{IP}^\omega_{\exists\text{free}}$ are (respectively) the axiom of choice for arbitrary matrices and the independence of premises principle for $\exists$-free antecendents (types are not restricted). The advantage of monotone realizability over Kreisel's realizability is due to the fact that a soundness theorem still holds good when one adjoins to the theory T^ω certain ineffective principles (see Theorem 3.10 of [10] for details).

The most conspicuous difference between monotone realizability and bounded realizability is the fact that the first is based on a transformation of formulas which is classically correct whereas the second is *not*. Due to this correctness, monotone realizability is able to extract classically correct bounds from consequences of the form $\forall\exists F$, where F is an *arbitrary* matrix. These bounds can be obtained in the presence of certain classically correct ineffective principles and are obtained within a formal framework where FAN-like *rules* (with arbitrary matrix) are admissible. On the other hand, the classically incorrect transformation of formulas upon which bounded realizability is based hinders the extraction of classically correct bounds from arbitrary $\forall\exists$ consequences. However, bounded realizability is specially taylored for obtaining new *conservation results* (see Corollary 5 above) regarding certain special kinds of sentences (of the form $\forall\exists$, followed by a Γ_{br} matrix) and, also, for extracting classically correct bounds for consequences of this kind. These results can be obtained in the presence of *very general* FAN-like *principles* (as opposed to rules), many of which false, and also of certain ineffective principles (see the next section).

4.3 About certain principles (Part II)

We present a list of principles which follow from $\mathsf{T}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\mathrm{free}} + \mathsf{MAJ}^\omega$ with the aid of *true* assertions that have the form Π_{br} (over the base theory T^ω). According to Theorem 4 and related results, the bounds resulting from the application of these results to the principles are *truthful*, i.e., can be verified in a true theory. We call such principles *benign*. Note that benign principles can be false!

1. Extensional equality in higher types is defined by induction on the type. For type 0, it is the given equality at type 0. The equality $x =_{\rho\to\tau} y$ is defined by $\forall u^\rho (xu =_\tau yu)$. *Full extensionality* is the collection of axioms of the form $\forall \phi^{\rho\to\tau} \forall x^\rho, y^\rho (x =_\rho y \to \phi x =_\tau \phi y)$. The scheme of full extensionality does not seem to be self-realizable because of the relativization to majorizable functionals in bounded modified realizability. However, the axioms of extensionality for types $\rho = 0, 1, 2$ can easily be put in the form Π_{br}. Hence, these forms of extensionality are benign.

2. The classical, but not intuitionistic, truth

$$(\S) \quad \forall x \forall y (A(x) \vee B(y)) \to \forall x A(x) \vee \forall y B(y),$$

where A and B are Π_{br}-formulas modulo $\mathsf{T}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\mathrm{free}} + \mathsf{MAJ}^\omega$, and the types of x and y are unrestricted, is benign. Note that when both types are 0 and A and B are quantifier-free formulas, we have the well-known *lesser limited principle of omniscience* LLPO (cf. [4]). Also, by the argument after Corollary 3, it follows from (§) that the scheme

$$\forall x \forall y (\neg A(x) \vee \neg B(y)) \to \forall x \neg A(x) \vee \forall y \neg B(y),$$

where A and B are arbitrary, is benign. In order to see that (§) is benign, observe that this principle follows (over $\mathsf{T}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\mathrm{free}} + \mathsf{MAJ}^\omega$) from:

$$\tilde{\forall} a \forall x \leq^* a \tilde{\forall} b \forall y \leq^* b\, (A(x) \vee B(y)) \to \tilde{\forall} a \forall x \leq^* a\, A(x) \vee \tilde{\forall} b \forall y \leq^* b\, B(y),$$

where A and B are $\tilde{\exists}$-free formulas (use Proposition 5). Since this restricted implication is in Π_{br} and it is true, we are done.

3. The law of excluded middle $A \vee \neg A$, for $A \in \Pi_{\mathrm{br}}$, is benign because (over $\mathsf{T}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\mathrm{free}} + \mathsf{MAJ}^\omega$) it follows from the Π_{br}-scheme $A \vee \neg A$, with A in the class of $\tilde{\exists}$-free formulas. Note that this form of excluded middle includes $\Pi^0_1 - \mathsf{LEM}$, i.e., $\forall n^0 A_0(n) \vee \neg \forall n^0 A_0(n)$ for A_0 a first-order bounded formula. In view of the fact that $\mathsf{T}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\mathrm{free}} + \mathsf{MAJ}^\omega$ refutes Markov's principle, one is drawn to the conclusion that $\Pi^0_1 - \mathsf{LEM}$ does not prove Markov's principle. This result first appeared in [1].

As in the discussion of the previous point, by the argument following Corollary 3, we get that the scheme $\neg A \vee \neg\neg A$ is benign, where A is arbitrary.

4. The following principle can easily be put in Π_{br}-form:

$$(\$) \quad \tilde{\forall} h^{1\to\rho} (\forall x^1 \exists y \leq^*_\rho \hat{h}x\, A(x,y) \to \exists f \leq^*_{1\to\rho} \hat{h}\, \forall x A(x, fx)),$$

where $\hat{h}$ is the functional of type $1 \to \rho$ defined by $\hat{h} := \lambda x^1 . h x^M$, and A is a $\tilde{\exists}$-free formula. This is clearly a true principle. Note that a tuple of h's would also be alright. The choice principle $\mathsf{AC}^{1,\rho}$ is

$$\forall x^1 \exists y^\rho A(x,y) \to \exists f^{1\to\rho} \forall x A(x, fx),$$

where A is an *arbitrary* formula. We show that this choice principle is provable in $\mathsf{T}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\mathrm{free}} + \mathsf{MAJ}^\omega$ together with the tuple version of the above Π_{br}-principle. As a consequence, $\mathsf{AC}^{1,\rho}$ is benign ($\mathsf{AC}^{0,\rho}$ is also benign). Suppose that $\forall x^1 \exists y^\rho A(x,y)$. By the Characterization Theorem 2, we deduce that $\forall x^1 \exists y^\rho \tilde{\exists} \underline{b} A_{\mathrm{br}}(\underline{b}, x, y)$. By bAC^ω there are monotone functionals $h^{1\to\rho}$ and $\underline{\phi}$ such that

$$\forall x^1 \exists y \leq^*_\rho \hat{h}x\, \exists \underline{b} \leq^* \hat{\underline{\phi}}x\, (\underline{b} \leq^* \underline{b} \wedge A_{\mathrm{br}}(\underline{b}, x, y)).$$

We deduce $\exists f \leq^*_{1\to\rho} \hat{h}\, \exists \underline{\psi} \leq^* \hat{\underline{\phi}}\, \forall x^1 (\underline{\psi}x \leq^* \underline{\psi}x \wedge A_{\mathrm{br}}(\underline{\psi}x, x, fx))$ by the tuple version of (\$). Hence, $\exists f^{1\to\rho} \forall x^1 \tilde{\exists} \underline{b}\, A_{\mathrm{br}}(\underline{b}, x, fx)$. A new application of the Characterization Theorem yields the result.

5. The discussion above does not generalize to choice principles in which x is of type greater than 1 because, for such types, (\$) cannot be put in the form Π_{br}. However, there is a benign version of choice with no restrictions on the types:

$$\forall x \leq^*_\tau a\, \exists y^\rho A(x,y) \to \exists f^{\tau\to\rho} \forall x \leq^*_\tau a\, A(x, fx),$$

where A is an arbitrary formula and a is monotone. We reason within the theory $\mathsf{T}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\text{free}} + \mathsf{MAJ}^\omega$. Assume that $\forall x \leq^*_\tau a\, \exists y^\rho A(x,y)$. By the Characterization Theorem, $\forall x \leq^*_\tau a\, \exists y^\rho \tilde{\exists} \underline{b} A_{\text{br}}(\underline{b}, x, y)$. By bBC^ω, there are monotone c and $\underline{b}'$ such that $\forall x \leq^*_\tau a\, \exists y \leq^*_\rho c\, \tilde{\exists} \underline{b} \leq^* \underline{b}'\, A_{\text{br}}(\underline{b}, x, y)$. Consider the conditional:

$$\begin{cases} \forall x \leq^*_\tau a\, \exists y \leq^*_\rho c\, \tilde{\exists} \underline{b} \leq^* \underline{b}'\, A_{\text{br}}(\underline{b}, x, y) \to \\ \qquad \exists f \leq^*_{\tau\to\rho} \lambda x^\tau . c^\rho\, \forall x \leq^*_\tau a\, \tilde{\exists} \underline{b} \leq^* \underline{b}' A_{\text{br}}(\underline{b}, x, fx). \end{cases}$$

This conditional is of $\tilde{\exists}$-form and it is true. From it, by Modus Ponens, it follows that $\exists f^{\tau\to\rho} \forall x \leq^*_\tau a\, \tilde{\exists} \underline{b}\, A_{\text{br}}(\underline{b}, x, fx)$. Again by the Characterization Theorem, we get $\exists f^{\tau\to\rho} \forall x \leq^*_\tau a\, A(x, fx)$.

6. It is well known that the FAN theorem is *true* (and intuitionistically acceptable) for quantifier-free matrices *without* parameters of type greater than 1. A modification of its contrapositive, which is equivalent to *weak König's lemma* WKL, is rejected intuitionistically. Nevertheless, it is clear that this modified contrapositive:

$$\forall u^1 (\forall n^0 \exists x \leq_1 u\, \forall k \leq_0 n\, A(x,k) \to \exists x \leq_1 u\, \forall k\, A(x,k)),$$

can be put in the form Π_{br} when A is a quantifier-free formula. Therefore, it is a benign principle. A related principle is *uniform weak König's lemma* (first considered in [11]), UWKL for short:

$$\exists \Phi^{1\to1} \forall f^1 (\forall k \exists s \in \{0,1\}^k (s \in f^{\text{tree}}(s)) \to \forall k\, (\overline{\Phi f}(k) \in f^{\text{tree}})),$$

where we are using the notation of [2]. Remark that the functional $\Phi^{1\to1}$ can be bounded (in the sense of $\leq^*_{1\to1}$) by the functional $\lambda f^1, k^0 . 1^0$. Hence, UWKL can be put in a Π_{br}-form. Alternatively, we can analyze UWKL by noting that this principle is provable in $\mathsf{T}^\omega + \mathsf{AC}^{1,1}$ with the aid of the above contrapositive. Note that, according to 3 above, $\mathsf{AC}^{1,1}$ is a benign principle.

7. The forms of comprehension $\mathsf{CA}^\rho_{\Pi_{\text{br}}}$:

$$\exists \Phi \leq^*_{\rho\to0} \lambda x^\rho . 1^0\, \forall y^\rho (\Phi y =_0 0 \leftrightarrow A(y)),$$

for A a Π_{br}-formula, are benign. They follow (over $\mathsf{T}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\text{free}} + \mathsf{MAJ}^\omega$) from their restrictions to formulas A that are $\tilde{\exists}$-free (use Proposition 5). These restrictions are true and have the form Π_{br}. By previous discussions, it is clear that comprehension $\mathsf{CA}^\rho_\neg$ for negated formulas is also a benign principle:

$$\exists\Phi \leq^*_{\rho\to 0} \lambda x^\rho.1^0\, \forall y^\rho(\Phi y =_0 0 \leftrightarrow \neg A(y)).$$

8. By a similar argument, the scheme of induction

$$A(0) \wedge \forall n(A(n) \to A(n+1)) \to \forall n A(n),$$

is benign for A a Π_{br}-formula or a negated formula.

9. In [10], Kohlenbach considered the principles F_ρ:

$$\forall \Phi^{0\to(\rho\to 0)} \forall y^{0\to\rho} \exists y_0 \leq_{0\to\rho} y\, \forall k^0 \forall z \leq_\rho yk\, (\Phi kz \leq_0 \Phi k(y_0 k)).$$

These principles are false for $\rho \neq 0$. They are, nevertheless, benign. We make some preliminary remarks. Let $(x \leq_\rho y)^{\mathcal{M}}$ be the formula obtained from $x \leq_\rho y$ by substituting every universal quantifier $\forall u$ by the complex $\tilde{\forall} v \forall u \leq^* v$. The new relation $(x \leq_\rho y)^{\mathcal{M}}$ is the old relation $x \leq_\rho y$ *relativized* to the class of $\mathcal{M} := \{u : \tilde{\exists} v(u \leq^* v)\}$ of majorizable functionals. Consider:

$$(\star) \left\{ \begin{array}{l} \forall \Psi^{\rho\to 0} \tilde{\forall} v \forall u \leq^*_\rho v \forall n^0 (\forall z \leq^*_\rho v((z \leq_\rho u)^{\mathcal{M}} \to \Psi z \leq_0 n) \to \\ \qquad \exists x_0 \leq^*_\rho v((x_0 \leq_\rho u)^{\mathcal{M}} \wedge \forall z \leq^*_\rho v((z \leq_\rho u)^{\mathcal{M}} \to \Psi z \leq_0 \Psi x_0))). \end{array} \right.$$

This principle is *true* and has the form Π_{br}. We argue that over the theory $\mathsf{T}^\omega + \mathsf{bAC}^\omega + \mathsf{bIP}^\omega_{\tilde{\exists}\mathrm{free}} + \mathsf{MAJ}^\omega$ the principle $(\star)$ and $\mathsf{AC}^{0,\rho}$ entails F_ρ. This shows that F_ρ is benign.

Due to the presence of MAJ^ω, the formulas $x \leq_\rho y$ and $(x \leq_\rho y)^{\mathcal{M}}$ are equivalent. We will use this fact systematically. Take $\Phi^{0\to(\rho\to 0)}$ and $y^{0\to\rho}$. Let k^0 be given. Put $\Psi := \Phi k$ and $u := yk$. Let $\tilde{\Psi}$ and v be such that $\Psi \leq^*_{\rho\to 0} \tilde{\Psi}$ and $u \leq^*_\rho v$. For $z \leq^*_\rho v$ we get $\Psi z \leq_0 \tilde{\Psi} v$. In particular, $\forall z \leq^*_\rho v(z \leq_\rho u \to \Psi z \leq_0 n)$, where n is $\tilde{\Psi} v$. By $(\star)$ and (iii) of Lemma 1, $\exists x_0(x_0 \leq_\rho u \wedge \forall z(z \leq_\rho u \to \Psi z \leq_0 \Psi x_0))$. By the arbitrariness of k^0 we have

$$\forall k^0 \exists x_0(x_0 \leq_\rho yk \wedge \forall z(z \leq_\rho yk \to \Phi kz \leq_0 \Phi k x_0)).$$

An application of $\mathsf{AC}^{0,\rho}$ yields F_ρ.

5 Closing

For some years now, Ulrich Kohlenbach and his students have been showing the *practical* use of Proof Theory in obtaining numerical bounds from *classical* proofs of analysis (see [12] for a recent survey). Kohlenbach's methods are *not* based on realizability because realizability notions (including bounded realizability) are *not* taylored for the analysis of *classical* proofs. In effect, even though a classical proof can be translated into an intuitionistic proof via (e.g.) the Gödel-Gentzen negative translation, the translation destroys existential statements – replacing them by negated universal statements – with the consequence that realizers yield no computational information. Of course, this shortcoming is related with the fact that Markov's principle is *not* benign. That notwithstanding, bounded modified realizability (and monotone modified realizability [10]) supports many classical principles that go beyond intuitionistic logic (see Subsections 4.2 and 4.3).

In order to deal with full classical reasoning, one must use the more sophisticated tool of the functional interpretation, introduced by Kurt Gödel in [6] (one should add, for quite different reasons – see the discussion in [14]). Contrary to realizability, Gödel's interpretation is efficient in analyzing classical proofs because it supports Markov's principle. The proof-theoretical tool used by Ulrich Kohlenbach in his program of searching for numerical information in classical proofs of analysis (so-called Proof Mining) is a modification of Gödel's interpretation, still based on Gödel's assignment of formulas: *monotone functional interpretation*. As we have said in the introduction, the realizability notion studied in this paper stems from work on a newly found functional interpretation, based on a novel assignment of formulas: *bounded functional interpretation*. The new functional interpretation supports (a variation of) Markov's principle, being also efficient for the analysis of *classical* proofs. Moreover, it has the peculiarity of using *intensional* majorizability relations (i.e., regulated by rules, instead of axioms). The curious reader can learn about the new interpretation in [5].

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