Bounded Functional Interpretation

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Abstract

We present a new functional interpretation, based on a novel assignment of formulas. In contrast with Gödel's functional "Dialectica" interpretation, the new interpretation does not care for precise witnesses of existential statements, but only for bounds for them. New principles are supported by our interpretation, including (a version of) the FAN theorem, weak König's lemma and the lesser limited principle of omniscience. Conspicuous among these principles are also refutations of some laws of classical logic. Notwithstanding, we end up discussing some applications of the new interpretation to theories of classical arithmetic and analysis.

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1 Introduction

In 1958 Kurt Gödel presented an interpretation of Heyting Arithmetic HA into a quantifier-free theory T in all finite types. The interpretation hinges on

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a particular assignment of formulas of the language of first-order arithmetic to quantifier-free formulas of the language of T. Gödel's so-called functional interpretation (a.k.a. Gödel's *Dialectica* interpretation, after the journal where it was published [9]) interprets certain principles that are not intuitionistically acceptable showing, in effect, that these principles can be safely added to HA without thereby changing the provable Π^0_2-sentences. The particular assignment defined by Gödel cares for precise witnesses of existential statements (and decides disjunctions). For some years now, Ulrich Kohlenbach has been urging a shift of attention from the obtaining of precise witnesses to the obtaining of *bounds* for the witnesses. One of the main advantages of working with the extraction of bounds is that the non-computable mathematical objects whose existence is claimed by various ineffective principles can sometimes be bounded by computable ones, and this opens the way to obtaining effective bounds for $\forall\exists$ statements *as long as* these claims have the right logical form (see Subsection 7.1). The standard example is *weak König's lemma*. This principle states that every infinite binary tree has an infinite path, and it is ineffective in the sense that there are infinite recursive binary trees whose infinite branches are all non-recursive. It can be viewed as the logical counterpart of various ineffective analytical principles such as

- the attainment of the maximum by a continuous function on [0,1],
- every continuous function on [0,1] is uniformly continuous,
- Heine/Borel theorem (in the sequential form),

among many others (see [34] for a comprehensive list).

Another important benefit of extracting bounds (via the hereditary notion of bound known as *majorizability* [11]) is the uniformity obtained on parameters which are themselves also bounded. For instance, a witness for a theorem having the form $\forall f \forall g \leq_1 f \exists n A(f, g, n)$ is a functional ϕ, depending on f and g, producing a natural number n (the relation $\leq_1$ is the pointwise less than relation between functions of type 1, i.e. from $\mathbb{N}$ to $\mathbb{N}$). However, if a bound on n is all that is required, then this bound will only depend on bounds for f and g provided that the functional ϕ is majorizable. In this case, the bound can be given independently of g and depending only on a bound for f.

In at least two important situations a bound can be transformed back into an actual witness, namely

(I) when the range circumscribed by the bound is finite, and the relation under consideration is decidable. For instance, given a closed term t satisfying $\forall n \exists m \leq t(n)\, A(n, m)$, A being a decidable relation, one can obtain, by bounded search, a new closed term q such that $\forall n\, A(n, q(n))$.

(II) when the relation under consideration is monotone in the witnessing argument. For instance, if $A(f)$ is such that

$$A(f) \wedge f \leq_1 f' \rightarrow A(f')$$

then we easily get that $\exists f \leq_1 t\, A(f)$ implies $A(t)$, i.e. any bound is also a witness.

The latter observation – on the monotonicity of actual mathematical existence theorems – has paved the way to striking results in the analysis of mathematical proofs (e.g. [16,24,28]). In [18], Kohlenbach observed (see also the recent survey [27]) that various numerically interesting theorems in analysis can be written in the form

$$\forall x \in P\, \forall z \in K_x \exists n A_\exists(x, z, n), \tag{1}$$

where P is a Polish space (complete separable metric space), K_x is a family of compact Polish spaces, parametrized by elements of P, and $A_\exists$ is an existential formula. When formalized in the language of arithmetic in all finite types, via the standard representation of Polish spaces, sentences of the kind (1) have the logical form

$$\forall x^1 \forall z \leq_1 t(x) \exists n A_\exists(x, z, n), \tag{2}$$

where t is a type 1 closed term of the formal language. In very general situations (even ineffective) proofs of theorems having the logical form (2) are guaranteed to provide a bound on n depending only on x (independent of z). In more concrete terms, a systematic analysis of a proof of a theorem (2) is guaranteed to provide a closed term q and a proof of the *stronger theorem*

$$\forall x^1 \forall z \leq_1 t(x) \exists n \leq q(x)\, A_\exists(x, z, n).$$

If the formula $A_\exists$ is monotone on n (as it is the case for a wide range of statements in functional analysis, cf. [27]), one actually obtains a witness for n independent of z,

$$\forall x^1 \forall z \leq_1 t(x) A_\exists(x, z, q(x)). \tag{3}$$

The analysis of mathematical proofs with the help of proof theoretic techniques, in search for concrete new information, has been dubbed *proof mining*. The established technique in proof mining was introduced by Ulrich Kohlenbach in [14,18], and has been applied with increasing virtuosity and efficiency by Kohlenbach and his students ever since. The technique is called *monotone functional interpretation* (henceforth abbreviated by m.f.i.) and the proof of its soundness theorem juxtaposes the Gödelian argument (which yields precise witnesses) with a majorization argument.

This paper introduces a new functional interpretation, the *bounded functional interpretation* (abbreviated b.f.i.). Whereas m.f.i. uses majorizability techniques at the outer level *after* the passage from the given formula to its Gödelian assignment, our interpretation is new in the sense that it defines a *novel assignment of formulas* which, in effect, *always* disregards precise witnesses, caring only for bounds for them. The new interpretation does not rely on the decidability of prime formulas, not even for the verification of the interpretation (as m.f.i. does). It also interprets new classical principles, conspicuously weak König's lemma. This should be compared with m.f.i.'s treatment of weak König's lemma, according to which the lemma is eliminated at the end of the analysis, not by the interpretation itself. At the same time, a version of the (intuitionistically acceptable) FAN theorem is interpreted by the b.f.i.:

$$\forall g \leq_1 f \exists n A(g,n) \rightarrow \exists k \forall g \leq_1 f \exists n \leq k A(g,n),$$

where A is *any* formula, provided that we read the relation $\leq_1$ *intensionally* (more on this below). This is a blatantly false principle in classical mathematics. Intuitionistic mathematics accepts it due to reasons of continuity: If one warrants intuitionistically the antecedent $\forall g \leq_1 f \exists n A(g,n)$, then the existential witnesses n must depend solely on finite initial segments of g, yielding in fact a continuous dependence of n by g. Since the functionals g range below a given f, compactness reasons (that can be put in intuitionistic clothes) yield a bound for the n's. Notwithstanding, it is *not* continuity that is responsible for the elimination of the FAN theorem by b.f.i.: It is majorizability! This was first observed in [15] in connection with the FAN rule (see also [21] and [26], where closure under the FAN rule is obtained even for systems whose models must contain discontinuous functionals). Majorizability, as opposed to continuity, is also responsible for the elimination of the classically valid (but intuitionistically unacceptable) weak König's lemma. Finally, there is even a more radical departure of b.f.i. from Gödel's interpretation: The principles interpreted by Gödel's technique are all consistent with classical logic, whereas this is not the case for b.f.i. in the presence of a minimal amount of arithmetic (cf. Proposition 9).

Our treatment of the majorizability relation is a bit subtle. For reasons similar to the ones that prevent Gödel's *Dialectica* interpretation from interpreting full extensionality, we must not work with the extensional majorizability relation. Instead, we work with an *intensional* version thereof. With this intensional majorizability relation, the FAN theorem falls as a very particular case of an overarching *bounded collection principle*. The mark of our treatment of the intensional majorizability relation is the incorporation of a rule, instead of a corresponding axiom. The presence of this rule entails the failure of the deduction theorem which, according to received opinion (see, for instance, [36]), is not attractive. We beg to differ from this judgment. The failure of the deduction theorem allows the emergence of the distinction between *postulates* and *implicative assumptions*, the former ones being placed on the left-hand side of the provability sign, while the latter ones on the right-hand side (what can be proved with implicative assumptions can be proved with postulates, but not vice-versa). There are indications that this distinction plays an important role in the analysis of ordinary theorems of mathematics, being therefore imbued of relevant mathematical meaning (see Kohlenbach's recent work [13], particularly the *discussions* in section 3). These matters are far from being well understood, being in dire need of further study and clarification. Having said that, we leave them at this juncture.

Even though a detailed comparison between m.f.i. and b.f.i. is beyond the scope of the present paper, the use of b.f.i. in analyzing some theoretical applications arising from the work of Ulrich Kohlenbach has convinced us that b.f.i. *sheds light* on that work, explaining on principled reasons certain phenomena for which m.f.i. requires rather ad-hoc arguments. The theoretical applications of b.f.i. can be seen as vast generalizations of Parikh type results (cf. [32]), in part obtained because b.f.i. automatically removes "ideal elements" from ineffective proofs (in modern parlance, b.f.i. is specially suited for obtaining conservation results).

Finally, we should point out that b.f.i. was conceived so that it would leave (intensional) bounded formulas unaffected by the interpretation and, in particular, would leave first-order bounded formulas unaffected, even in feasible settings. Therefore, b.f.i. (as opposed to m.f.i.) is tailored for the elimination of ideal elements from weak theories of arithmetic and analysis (cf. [7,31]). Nevertheless, this issue is not dealt with in the present paper and will have to await for another work.

In the next section, we introduce the basic system over which we define the new interpretation (Section 3). In Section 4, we enumerate some principles that are not provable in the basic setting but which have a trivial interpretation. These turn out to be precisely the principles that are needed for proving a characterization theorem for the b.f.i. In Sections 5 and 6 we extend the interpretation to classical and arithmetical systems, respectively. In the final sections, we study some theoretical applications of b.f.i. by giving simple interpretations of (a uniform version of) weak König's lemma and of various (non-standard) boundedness principles in classical mathematics. These applications are versions of (or stem from) original and exciting results proved earlier by Kohlenbach using m.f.i. ([18,19,22]).

2 Basic Framework

Let $\mathcal{L}^{\omega}_{\leq}$ be a language in all finite types (based on a given ground type 0) with a distinguished binary relation symbol $\leq_0$ (infixing between terms of type 0) and distinguished constants m of type $0 \to (0 \to 0)$ and z of type 0 (the constant z is needed to ensure that each finite type is inhabited by at least one closed term). The theory $\mathsf{IL}^{\omega}_{\leq}$ is intuitionistic logic in all finite types (see [1] for the formalization we will be using) with axioms stating that $\leq_0$ is reflexive, transitive, and with the axioms

$$\begin{aligned} \mathsf{A}_1 &: x \leq_0 \mathrm{m}(x, y) \wedge y \leq_0 \mathrm{m}(x, y) \\ \mathsf{A}_2 &: x \leq_0 x' \wedge y \leq_0 y' \to \mathrm{m}(x, y) \leq_0 \mathrm{m}(x', y') \end{aligned}$$

Our treatment of equality is based on the minimal alternative described by Troelstra in the end of section 3.1 and the beginning of section 3.3 of [36]. There is a symbol of equality only for terms of type 0. Its axioms are

$$\begin{aligned} \mathsf{E}_1 &: x =_0 x \\ \mathsf{E}_2 &: x =_0 y \wedge \phi[x/w] \to \phi[y/w] \end{aligned}$$

where ϕ is an atomic formula with a distinguished type 0 variable w. In order to characterize the behaviour of the logical constants (combinators) Π and Σ, we must also add

$$\mathsf{E}_{\Pi,\Sigma} \ : \ \phi[\Pi(x, y)/w] \leftrightarrow \phi[x/w], \quad \phi[\Sigma(x, y, z)/w] \leftrightarrow \phi[xy(xz)/w],$$

where ϕ is a an atomic formula with a distinguished variable w, and x, y and z are variables of appropriate type.

In the language $\mathcal{L}^{\omega}_{\leq}$ we can define Bezem's strong majorizability relation [2] (a modification of Howard's hereditary majorizability relation [11] that, as opposed to Howard's, is provably *transitive* – a necessity for our interpretation) and prove its main properties. We write $\leq^*_\rho$ for Bezem's strong majorizability relation for type ρ. This relation is defined by induction on the types:

(a) $x \leq^*_0 y := x \leq_0 y$
(b) $x \leq^*_{\rho\to\sigma} y := \forall u^\rho, v^\rho (u \leq^*_\rho v \to xu \leq^*_\sigma yv \wedge yu \leq^*_\sigma yv))$

The following is a result of [2]:

Lemma 1 $\mathsf{IL}^{\omega}_{\leq}$ *proves*

(i) $x \leq^* y \to y \leq^* y$.
(ii) $x \leq^* y \wedge y \leq^* z \to x \leq^* z$.

Proof. The type 0 case for (i) is due to the reflexivity of $\leq_0$, while the type non-zero cases follow directly from the definition. Property (ii) is proved by induction on the type. The type 0 case is given. We now must argue for $xu \leq^*_\sigma zv$ and $zu \leq^*_\sigma zv$ under the hypothesis that $x \leq^*_{\rho\to\sigma} y$, $y \leq^*_{\rho\to\sigma} z$ and $u \leq^*_\rho v$. We get immediately that $xu \leq^*_\sigma yv$. By (i), $v \leq^*_\rho v$. Therefore, $yv \leq^*_\sigma zv$. By the induction hypothesis, $xu \leq^*_\sigma zv$. The other property follows from the fact that, according to (i), $z \leq^*_{\rho\to\sigma} z$. □

In order to formulate the new functional interpretation, we introduce an extension $\mathcal{L}^\omega_\trianglelefteq$ of the language $\mathcal{L}^\omega_\leq$, obtained from the latter by the adjunction of new primitive binary relation symbols $\trianglelefteq_\rho$, one for each type ρ (we use infix notation for these symbols). The relation $\trianglelefteq_\rho$ is the intensional counterpart of the extensional relation $\leq^*_\rho$. The terms of $\mathcal{L}^\omega_\trianglelefteq$ are the same as the terms of the original language $\mathcal{L}^\omega_\leq$. Formulas of the form $s \trianglelefteq_\rho t$, where s and t are terms of type ρ, are the new atomic formulas of the language. We also add, as a new syntactic device, *bounded quantifiers*, i.e. quantifications of the form $\forall x \trianglelefteq t A(x)$ and $\exists x \trianglelefteq t A(x)$, for terms t not containing x. *Bounded formulas* are those formulas in which every quantifier is bounded.

Definition 1 *The theory* $\mathsf{IL}^\omega_\trianglelefteq$ *is an extension of* $\mathsf{IL}^\omega_\leq$ *with the axiom schemae:*

$$\mathsf{B}_\forall : \forall x \trianglelefteq t A(x) \leftrightarrow \forall x(x \trianglelefteq t \to A(x))$$
$$\mathsf{B}_\exists : \exists x \trianglelefteq t A(x) \leftrightarrow \exists x(x \trianglelefteq t \wedge A(x)),$$

with the restriction that x does not occur in t. There are also two further axioms

$$\mathsf{M}_1 : x \trianglelefteq_0 y \leftrightarrow x \leq_0 y$$
$$\mathsf{M}_2 : x \trianglelefteq_{\rho\to\sigma} y \to \forall u \trianglelefteq_\rho v(xu \trianglelefteq_\sigma yv \wedge yu \trianglelefteq_\sigma yv)$$

and a rule $\mathsf{RL}_\trianglelefteq$

$$\frac{A_{\mathrm{b}} \wedge u \trianglelefteq v \to su \trianglelefteq tv \wedge tu \trianglelefteq tv}{A_{\mathrm{b}} \to s \trianglelefteq t}$$

where s and t are terms of $\mathsf{IL}^\omega_\trianglelefteq$, *$A_{\mathrm{b}}$ is a bounded formula and u and v are variables that do not occur free in the conclusion.*

Warning 1 *As we will show in Proposition 8, the presence of the rule* $\mathsf{RL}_\trianglelefteq$ *entails the failure of the Deduction Theorem for the arithmetical theories (here considered). We must use the rule instead of the corresponding implication (the converse of* M_2*) due to the fact that the implication does not have a bounded functional interpretation. A similar problem occurs with the treatment of full extensionality by the usual Gödel's functional interpretation, in which case the axiom must be replaced by a rule of extensionality (cf. [23] and [35]).*

The proof that we gave of Lemma 1 does not use the converse of the implication M_2, only its weakened version given by the rule $\mathsf{RL}_{\trianglelefteq}$. This observation justifies the first two claims of the lemma below. The third claim is immediate.

Lemma 2 $\mathsf{IL}^{\omega}_{\trianglelefteq}$ *proves*

(i) $x \trianglelefteq y \to y \trianglelefteq y$.
(ii) $x \trianglelefteq y \wedge y \trianglelefteq z \to x \trianglelefteq z$.
(iii) $x \trianglelefteq_1 y \to x \leq^*_1 y$. *Hence,* $x \trianglelefteq_1 y \to x \leq_1 y$.

Observation 1 *The relation* $\leq_\sigma$ *is the usual pointwise "less than or equal to" relation. It is the relation* $\leq_0$ *for type 0, and* $x \leq_{\rho\to\sigma} y$ *is defined recursively by* $\forall u^\rho (xu \leq_\sigma yu)$.

Since the extended language $\mathcal{L}^{\omega}_{\trianglelefteq}$ has new atomic formulas, we must check whether our axioms ensure that we have a decent theory of identity for all types. The following two propositions guarantee just that.

Proposition 1 *Let* ϕ *be any formula of* $\mathcal{L}^{\omega}_{\trianglelefteq}$ *with a distinguished type 0 free variable* w. *The theory* $\mathsf{IL}^{\omega}_{\trianglelefteq}$ *proves*

$$x =_0 y \wedge \phi[x/w] \to \phi[y/w]$$

where x *and* y *are free for* w *in* ϕ.

Proof. It is enough to prove the above for atomic formulas ϕ. If the atomic formula is in the original language $\mathcal{L}^{\omega}_{\leq}$, the result follows from E_2. Let us now deal with the atomic formulas originating from the new relational symbols $\trianglelefteq_\sigma$. We must show that, for every type σ, if $r[w]$ and $q[w]$ are terms of type σ with a distinguished type 0 variable w, then

$$x =_0 y \wedge r[x/w] \trianglelefteq_\sigma q[x/w] \to r[y/w] \trianglelefteq_\sigma q[y/w].$$

We prove this by induction on the type σ. For the base type, use the axiom M_1 to reduce the $\trianglelefteq_0$-inequation to an inequation with relation symbol $\leq_0$. By rule $\mathsf{RL}_{\trianglelefteq}$, in order to prove that

$$x =_0 y \wedge r[x/w] \trianglelefteq_{\sigma\to\tau} q[x/w] \to r[y/w] \trianglelefteq_{\sigma\to\tau} q[y/w].$$

it is enough to prove the implication

$$\begin{cases} x =_0 y \wedge r[x/w] \trianglelefteq_{\sigma\to\tau} q[x/w] \wedge u \trianglelefteq_\sigma v \to \\ \qquad r[y/w]u \trianglelefteq_\tau q[y/w]v \wedge q[y/w]u \trianglelefteq_\tau q[y/w]v. \end{cases}$$

By the axiom M_2, the last two conjuncts of the antecedent of the implication entail $r[x/w]u \trianglelefteq_\tau q[x/w]v$ and $q[x/w]u \trianglelefteq_\tau q[x/w]v$. By induction hypothesis applied to the type τ term couples $r[w]u$, $q[w]v$ and $q[w]u$, $q[w]v$, we obtain the consequent of the implication. $\square$

Proposition 2 *Let ϕ be any formula of $\mathcal{L}^{\omega}_{\trianglelefteq}$ with a distinguished free variable w. The theory $\mathsf{IL}^{\omega}_{\trianglelefteq}$ proves the equivalences*

$$\phi[\Pi(x,y)/w] \leftrightarrow \phi[x/w], \quad \phi[\Sigma(x,y,z)/w] \leftrightarrow \phi[xy(xz)/w]$$

where x, y and z are free for w in ϕ.

Proof. It is enough to prove the above for atomic formulas ϕ. If ϕ is in the original language $\mathcal{L}^{\omega}_{\leq}$, the result follows from $\mathsf{E}_{\Pi,\Sigma}$. The atomic inequations with relation symbol $\trianglelefteq_{\sigma}$ are dealt with by induction on the type σ, similarly to the argument of the previous proposition. $\square$

Lemma 3 $\mathsf{IL}^{\omega}_{\trianglelefteq}$ *proves that $\Pi \trianglelefteq \Pi$ and $\Sigma \trianglelefteq \Sigma$.*

Proof. The usual proof of these facts for the $\leq^*$ relation only uses the rule $\mathsf{RL}_{\trianglelefteq}$, not the unwarranted implication. For instance, to check that $\Sigma \trianglelefteq \Sigma$ it is enough to prove (by several applications of rule $\mathsf{RL}_{\trianglelefteq}$) the implication

$$x \trianglelefteq x' \wedge y \trianglelefteq y' \wedge z \trianglelefteq z' \rightarrow \Sigma xyz \trianglelefteq \Sigma x'y'z'$$

By the above proposition, the consequent of the implication is equivalent to $xy(xz) \trianglelefteq x'y'(x'z')$. This, in turn, is (under the antecedent of the implication) an easy consequence of M_2. $\square$

2.1 Majorizability Theories

In the following the reader should observe that the language $\mathcal{L}^{\omega}_{\trianglelefteq}$ is allowed to include relational and constant symbols besides $\leq_0$, $\trianglelefteq_{\sigma}$ (σ a finite type), m and z.

Definition 2 *Consider a fixed language $\mathcal{L}^{\omega}_{\trianglelefteq}$. A theory $\mathsf{T}^{\omega}_{\trianglelefteq}$ in $\mathcal{L}^{\omega}_{\trianglelefteq}$ is called a* majorizability theory *for $\mathcal{L}^{\omega}_{\trianglelefteq}$ if it extends $\mathsf{IL}^{\omega}_{\trianglelefteq}$ and, for every constant c^{ρ}, there is a closed term t^{ρ} such that $\mathsf{T}^{\omega}_{\trianglelefteq} \vdash c \trianglelefteq_{\rho} t$.*

If the constants of the language $\mathcal{L}^{\omega}_{\trianglelefteq}$ are just m, z and the combinators Π and Σ, then Lemma 3 guarantees that $\mathsf{IL}^{\omega}_{\trianglelefteq}$ is a majorizability theory. Later in the paper, we will associate majorizability theories to the theories of arithmetic $\mathsf{HA}^{\omega}, \mathsf{PRA}^{\omega}_{i}$ and $\mathsf{G}_n\mathsf{A}^{\omega}_{i}$ ($n \geq 2$).

In a majorizability theory we define by induction on the type, a binary relation m_{ρ} of type $\rho \rightarrow (\rho \rightarrow \rho)$ according to the following clauses:

(a) m_0 is m
(b) $\mathrm{m}_{\rho\to\sigma}(x,y) := \lambda u^\rho.\mathrm{m}_\sigma(xu, yu)$

It is well known that the combinators Π and Σ enjoy the property of *combinatorial completeness* whereby, given any term t with a distinguished variable u, there is a term $\lambda u.t$ whose free variables are those of t except for u, such that $(\lambda u.t)(q)$ is (in the sense of allowing the pertinent substitutions of one term for the other) $t[q/u]$. We are using this fact in the above definition.

Lemma 4 *Let* $\mathsf{T}^\omega_\trianglelefteq$ *be a majorizability theory.* $\mathsf{T}^\omega_\trianglelefteq$ *proves*

(i) $x \trianglelefteq x \wedge y \trianglelefteq y \to x \trianglelefteq \mathrm{m}(x,y) \wedge y \trianglelefteq \mathrm{m}(x,y)$
(ii) $\mathrm{m}_\rho \trianglelefteq \mathrm{m}_\rho$.

Proof. Firstly, we prove *(ii)* by induction on the type ρ. The base case is trivial. By rule $\mathsf{RL}_\trianglelefteq$ (twice), in order to prove that $\mathrm{m}_{\rho\to\sigma} \trianglelefteq \mathrm{m}_{\rho\to\sigma}$ it is enough to prove

$$x \trianglelefteq_{\rho\to\sigma} x' \wedge y \trianglelefteq_{\rho\to\sigma} y' \to \mathrm{m}(x,y) \trianglelefteq_{\rho\to\sigma} \mathrm{m}(x',y').$$

In order to prove *this*, by rule $\mathsf{RL}_\trianglelefteq$ (twice again), it is sufficient to prove

$$x \trianglelefteq_{\rho\to\sigma} x' \wedge y \trianglelefteq_{\rho\to\sigma} y' \wedge u \trianglelefteq_\rho v \to \mathrm{m}(xu,yu) \trianglelefteq_\sigma \mathrm{m}(x'v,y'v).$$

This follows from the induction hypothesis. Claim *(i)* is also proved by induction on the type. The base case is clear. We must now prove the implication whose antecedent is $x \trianglelefteq_{\rho\to\sigma} x \wedge y \trianglelefteq_{\rho\to\sigma} y$ and whose consequent is $x \trianglelefteq_{\rho\to\sigma} \mathrm{m}_{\rho\to\sigma}(x,y)$ (the other conjunct is similar). By rule $\mathsf{RL}_\trianglelefteq$, it is sufficient to prove

$$x \trianglelefteq x \wedge y \trianglelefteq y \wedge u \trianglelefteq_\rho v \to xu \trianglelefteq_\sigma \mathrm{m}(xv,yv) \wedge \mathrm{m}(xu,yu) \trianglelefteq_\sigma \mathrm{m}(xv,yv).$$

The second conjunct of the consequent follows from part *(ii)* of the lemma. For the first conjunct, observe that $xu \trianglelefteq_\sigma xv$ and that, by induction hypothesis, $xv \trianglelefteq_\sigma \mathrm{m}_\sigma(xv,yv)$. Now use the transitivity of $\trianglelefteq_\sigma$. $\square$

The following result can be proven by an easy induction on the structure of terms.

Lemma 5 *Let* $\mathsf{T}^\omega_\trianglelefteq$ *be a majorizability theory. For every closed term* t^ρ *of* $\mathcal{L}^\omega_\trianglelefteq$ *there exists another closed term* $\tilde{t}^\rho$ *of* $\mathcal{L}^\omega_\trianglelefteq$*, such that*

$$\mathsf{T}^\omega_\trianglelefteq \vdash t \trianglelefteq \tilde{t}.$$

Notation 1 *An underlined term* $\underline{t}$ *is an abbreviation of a (possibly empty) tuple of terms* $t_1, t_2, \ldots, t_k$*. We use this notation for tuples in several contexts, and it should be evident what it means. For instance,* $\forall \underline{x} \trianglelefteq \underline{t}$ *abbreviates the* *sequence of quantifications* $\forall x_1 \trianglelefteq t_1 \forall x_2 \trianglelefteq t_2 \ldots \forall x_k \trianglelefteq t_k$*, where* $x_1, x_2, \ldots, x_k$ *and* $t_1, t_2, \ldots, t_k$ *are sequences of variables, respectively terms, with matching types. After a while, we will no longer underline terms. It will be clear from the context when we are referring to a tuple of terms, instead of a single term only.*

Definition 3 *Within the context of a language* $\mathcal{L}^\omega_\trianglelefteq$*, by a* majorant *of an (open) term* t *with (free) variables* $\underline{w}$ *we mean a term* $\tilde{t}$ *with the same (free) variables such that* $\mathsf{T}^\omega_\trianglelefteq \vdash \lambda \underline{w}.t \trianglelefteq \lambda \underline{w}.\tilde{t}$*. A term* t *is called* monotone *if it is self-majorizing. To say that a functional* f *is* monotone *is to assume that* $f \trianglelefteq f$*.*

It is an easy consequence of Lemma 5 that every (open) term has a majorant. In the sequel, we shall often quantify over monotone functionals. We abbreviate the quantifications $\forall f(f \trianglelefteq f \to A(f))$ and $\exists f(f \trianglelefteq f \wedge A(f))$ by $\tilde{\forall} f A(f)$ and $\tilde{\exists} f A(f)$, respectively.

3 The Bounded Functional Interpretation

In this section, we define a new functional interpretation (the *Bounded Functional Interpretation*) within $\mathsf{IL}^\omega_\trianglelefteq$ and prove a corresponding soundness theorem.

Definition 4 *To each formula* A *of the language* $\mathcal{L}^\omega_\trianglelefteq$ *we associate formulas* $(A)^{\mathrm{B}}$ *and* A_{B} *of the same language so that* $(A)^{\mathrm{B}}$ *is of the form* $\tilde{\exists} \underline{b} \tilde{\forall} \underline{c} A_{\mathrm{B}}(\underline{b}, \underline{c})$*, with* $A_{\mathrm{B}}(\underline{b}, \underline{c})$ *a bounded formula.*

1. $(A_{\mathrm{b}})^{\mathrm{B}}$ *and* $(A_{\mathrm{b}})_{\mathrm{B}}$ *are simply* A_{b}*, for bounded formulas* A_{b}*.*

If we have already interpretations for A *and* B *given by* $\tilde{\exists} \underline{b} \tilde{\forall} \underline{c} A_{\mathrm{B}}(\underline{b}, \underline{c})$ *and* $\tilde{\exists} \underline{d} \tilde{\forall} \underline{e} B_{\mathrm{B}}(\underline{d}, \underline{e})$ *(respectively) then, we define*

2. $(A \wedge B)^{\mathrm{B}}$ *is* $\tilde{\exists} \underline{b}, \underline{d} \tilde{\forall} \underline{c}, \underline{e} (A_{\mathrm{B}}(\underline{b}, \underline{c}) \wedge B_{\mathrm{B}}(\underline{d}, \underline{e}))$*,*
3. $(A \vee B)^{\mathrm{B}}$ *is* $\tilde{\exists} \underline{b}, \underline{d} \tilde{\forall} \underline{c}, \underline{e} (\tilde{\forall} \underline{c}' \trianglelefteq \underline{c} A_{\mathrm{B}}(\underline{b}, \underline{c}') \vee \tilde{\forall} \underline{e}' \trianglelefteq \underline{e} B_{\mathrm{B}}(\underline{d}, \underline{e}'))$*,*
4. $(A \to B)^{\mathrm{B}}$ *is* $\tilde{\exists} \underline{f}, \underline{g} \tilde{\forall} \underline{b}, \underline{e} (\tilde{\forall} \underline{c} \trianglelefteq \underline{g} \underline{b} \underline{e} A_{\mathrm{B}}(\underline{b}, \underline{c}) \to B_{\mathrm{B}}(\underline{f} \underline{b}, \underline{e}))$*.*

For bounded quantifiers we have:

5. $(\forall x \trianglelefteq t A(x))^{\mathrm{B}}$ *is* $\tilde{\exists} \underline{b} \tilde{\forall} \underline{c} \forall x \trianglelefteq t\, A_{\mathrm{B}}(\underline{b}, \underline{c}, x)$*,*
6. $(\exists x \trianglelefteq t A(x))^{\mathrm{B}}$ *is* $\tilde{\exists} \underline{b} \tilde{\forall} \underline{c} \exists x \trianglelefteq t \tilde{\forall} \underline{c}' \trianglelefteq \underline{c}\, A_{\mathrm{B}}(\underline{b}, \underline{c}', x)$*.*

And for unbounded quantifiers we define

7. $(\forall x A(x))^{\mathrm{B}}$ *is* $\tilde{\exists} \underline{f} \tilde{\forall} a, \underline{c} \forall x \trianglelefteq a A_{\mathrm{B}}(\underline{f} a, \underline{c}, x)$*.*

8. $(\exists x A(x))^{\mathrm{B}}$ *is* $\tilde{\exists} a, \underline{b} \tilde{\forall} \underline{c} \exists x \trianglelefteq a \tilde{\forall} \underline{c}' \trianglelefteq \underline{c} A_{\mathrm{B}}(\underline{b}, \underline{c}', x)$.

In the above, it is understood that $(\exists x A)_{\mathrm{B}}$ is $\exists x \trianglelefteq a \tilde{\forall} \underline{c}' \trianglelefteq \underline{c} A_{\mathrm{B}}(\underline{b}, \underline{c}', x)$. Similarly for the other clauses. Note that the *universal* bounded quantifiers that occur in the clauses *3*, *4*, *6* and *8* are (as opposed to the others) restricted to monotone variables. The case of negation is a particular case of the implication. We get,

9. $(\neg A)^{\mathrm{B}}$ *is* $\tilde{\exists} \underline{g} \tilde{\forall} \underline{b} \neg \tilde{\forall} \underline{c} \trianglelefteq \underline{g} \underline{b} A_{\mathrm{B}}(\underline{b}, \underline{c})$.

An inspection of the clauses of the definition of the bounded functional interpretation easily shows that

Lemma 6 (Monotonicity Lemma) *Let* $(A)^{\mathrm{B}}$ *be* $\tilde{\exists} \underline{b} \tilde{\forall} \underline{c} A_{\mathrm{B}}(\underline{b}, \underline{c}, \underline{x})$. *The following* monotonicity property *holds:*

$$\mathsf{T}^{\omega}_{\trianglelefteq} \vdash \underline{b} \trianglelefteq \underline{b}' \wedge \underline{c} \trianglelefteq \underline{c} \wedge A_{\mathrm{B}}(\underline{b}, \underline{c}, \underline{x}) \rightarrow A_{\mathrm{B}}(\underline{b}', \underline{c}, \underline{x}).$$

We are now ready to formulate and prove a soundness theorem for the bounded functional interpretation:

Theorem 1 (Soundness) *Consider a fixed language* $\mathcal{L}^{\omega}_{\trianglelefteq}$. *Let* $\mathsf{T}^{\omega}_{\trianglelefteq}$ *be a majorizability theory for* $\mathcal{L}^{\omega}_{\trianglelefteq}$ *and assume that* $(A(\underline{z}))^{\mathrm{B}}$ *is* $\tilde{\exists} \underline{b} \tilde{\forall} \underline{c} A_{\mathrm{B}}(\underline{b}, \underline{c}, \underline{z})$, *where* $A(\underline{z})$ *is an arbitrary formula of* $\mathcal{L}^{\omega}_{\trianglelefteq}$ *with its free variables as displayed. If*

$$\mathsf{IL}^{\omega}_{\trianglelefteq} \vdash A(\underline{z}),$$

then there are closed monotone terms $\underline{t}$ *of appropriate types such that*

$$\mathsf{T}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall} \underline{a} \forall \underline{z} \trianglelefteq \underline{a} \tilde{\forall} \underline{c}\, A_{\mathrm{B}}(\underline{t}\underline{a}, \underline{c}, \underline{z}).$$

Proof. The proof proceeds by induction on the length of the derivation of $A(z)$. Wherever convenient we shall build a term t containing also the variables a (the majorants of z), instead of a closed term which only at the end gets applied to a.

The axioms M_1, M_2 and the equations characterizing the behaviour of the combinators are universal statements and it is to check that their interpretation is sound. Let us now consider an instance of the rule $\mathsf{RL}_{\trianglelefteq}$. Assume that the premise of the instance

$$A_{\mathrm{b}}[z] \wedge u \trianglelefteq v \rightarrow s[z]u \trianglelefteq t[z]v \wedge t[z]u \trianglelefteq t[z]v.$$

has been derived (for the sake of simplicity, let z be the only parameter). By induction hypothesis, $\mathsf{T}^{\omega}_{\trianglelefteq}$ proves that for all monotone a, c and d

$$\forall z \trianglelefteq a \forall u \trianglelefteq c \forall v \trianglelefteq d (A_{\mathrm{b}}[z] \wedge u \trianglelefteq v \rightarrow s[z]u \trianglelefteq t[z]v \wedge t[z]u \trianglelefteq t[z]v).$$

By *(i)* of Lemma 2 (with $c = d = v$), the above entails that $\mathsf{T}^\omega_\trianglelefteq$ proves

$$\tilde{\forall} a \forall z \trianglelefteq a (A_{\mathrm{b}}[z] \wedge u \trianglelefteq v \to s[z]u \trianglelefteq t[z]v \wedge t[z]u \trianglelefteq t[z]v),$$

which is equivalent to

$$\forall a \forall z (A_{\mathrm{b}}[z] \wedge z \trianglelefteq a \wedge a \trianglelefteq a \wedge u \trianglelefteq v \to s[z]u \trianglelefteq t[z]v \wedge t[z]u \trianglelefteq t[z]v).$$

By the rule $\mathsf{RL}_\trianglelefteq$ we conclude

$$\forall a \forall z (A_{\mathrm{b}}[z] \wedge z \trianglelefteq a \wedge a \trianglelefteq a \to s[z] \trianglelefteq t[z]).$$

which is equivalent to $\tilde{\forall} a \forall z \trianglelefteq a (A_{\mathrm{b}}[z] \to s[z] \trianglelefteq t[z])$, as desired.

Let us now consider the axioms for bounded quantifiers. We assume that each of the bounded principles $\mathsf{B}_\forall$ and $\mathsf{B}_\exists$ is a shorthand for two separate principles: the left-to-right and the right-to-left implications.

$\mathsf{B}_\forall$. $\forall x \trianglelefteq t A(x) \leftrightarrow \forall x (x \trianglelefteq t \to A(x))$. Assume $(A(x))^{\mathrm{B}}$ is $\tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b, c, x)$ and that z includes all the free variables of t (as mentioned in Notation 1 we will omit the underlining of tuples). Then

$$(\forall x \trianglelefteq t A(x))^{\mathrm{B}} := \tilde{\exists} b \tilde{\forall} c \forall x \trianglelefteq t A_{\mathrm{B}}(b, c, x),$$

while the interpretation of the right hand side gives

$$(\forall x (x \trianglelefteq t \to A(x)))^{\mathrm{B}} := \tilde{\exists} f \tilde{\forall} c, d \forall x \trianglelefteq d (x \trianglelefteq t \to A_{\mathrm{B}}(fd, c, x).$$

The left-to-right implication asks for monotone terms q and r such that

$$\tilde{\forall} b, c, d (\tilde{\forall} c' \trianglelefteq qbcd \forall x \trianglelefteq t A_{\mathrm{B}}(b, c', x) \to \forall x \trianglelefteq d (x \trianglelefteq t \to A_{\mathrm{B}}(rbd, c, x))).$$

Clearly, $q(b, c, d) := c$ and $r(b, d) := b$ do the job. For the right-to-left implication, we must find monotone terms q, s and r such that

$$\tilde{\forall} f, c (\tilde{\forall} d \trianglelefteq qfc \tilde{\forall} c' \trianglelefteq sfc \forall x \trianglelefteq d (x \trianglelefteq t \to A_{\mathrm{B}}(fd, c', x)) \to \forall x \trianglelefteq t A_{\mathrm{B}}(rf, c, x)).$$

It is easy to see that $q(f, c) := \tilde{t}[a/z]$, $s(f, c) := c$ and $r(f) = f(\tilde{t}[a/z])$, where $\tilde{t}$ and a are majorants for t and z respectively, do the job.

$\mathsf{B}_\exists$. $\exists x \trianglelefteq t A(x) \leftrightarrow \exists x (x \trianglelefteq t \wedge A(x))$. Assume $(A(x))^{\mathrm{B}}$ is $\tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b, c, x)$. Then

$$(\exists x \trianglelefteq t A(x))^{\mathrm{B}} := \tilde{\exists} b \tilde{\forall} c \exists x \trianglelefteq t \tilde{\forall} c' \trianglelefteq c A_{\mathrm{B}}(b, c', x),$$

while the interpretation of the right hand side gives

$$(\exists x (x \trianglelefteq t \wedge A(x)))^{\mathrm{B}} := \tilde{\exists} b, d \tilde{\forall} c \exists x \trianglelefteq d \tilde{\forall} c' \trianglelefteq c (x \trianglelefteq t \wedge A_{\mathrm{B}}(b, c', x)).$$

The left-to-right implication asks for monotone terms q, r and s such that

$$\begin{cases} \tilde{\forall} b, c(\tilde{\forall} c'' \trianglelefteq qbc \exists x \trianglelefteq t \tilde{\forall} c' \trianglelefteq c'' A_{\mathrm{B}}(b, c', x) \to \\ \qquad \exists x \trianglelefteq rb \tilde{\forall} c' \trianglelefteq c(x \trianglelefteq t \wedge A_{\mathrm{B}}(sb, c', x))). \end{cases}$$

We just take $q(b,c) := c$, $r(b) := \tilde{t}[a/z]$ and $s(b) := b$, where $\tilde{t}$ and a are majorants for t and z respectively. For the right-to-left implication, we must find monotone terms q and r such that

$$\begin{cases} \tilde{\forall} b, c, d(\tilde{\forall} c'' \trianglelefteq qbcd \exists x \trianglelefteq d \tilde{\forall} c' \trianglelefteq c''(x \trianglelefteq t \wedge A_{\mathrm{B}}(b, c', x)) \to \\ \qquad \exists x \trianglelefteq t \tilde{\forall} c' \trianglelefteq c A_{\mathrm{B}}(rbd, c', x)). \end{cases}$$

It is easy to see that $q(b,c,d) := c$ and $r(b,d) := b$ do the job.

For the induction steps assume that $(A)^{\mathrm{B}}$ is $\tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b,c)$ and $(B)^{\mathrm{B}}$ is $\tilde{\exists} d \tilde{\forall} e B_{\mathrm{B}}(d,e)$ (we omit the free variables of A and B whenever not relevant).

1. $A, A \to B \Rightarrow B$. By induction hypothesis we have monotone terms t, s and r such that

(i) $\tilde{\forall} c A_{\mathrm{B}}(t,c)$
(ii) $\tilde{\forall} b, e(\tilde{\forall} c \trianglelefteq r(b,e) A_{\mathrm{B}}(b,c) \to B_{\mathrm{B}}(s(b), e))$.

Let $q := s(t)$. It is easy to see that $\tilde{\forall} e B_{\mathrm{B}}(q,e)$ follows from (i) and (ii), since, for a fixed e, (i) implies $\tilde{\forall} c \trianglelefteq r(t,e) A_{\mathrm{B}}(t,c)$ and (ii) implies $\tilde{\forall} c \trianglelefteq r(t,e) A_{\mathrm{B}}(t,c) \to B_{\mathrm{B}}(s(t), e)$. Moreover, if s and t are monotone, then q is also monotone.

2. $A \to B, B \to C \Rightarrow A \to C$. Assume $(C)^{\mathrm{B}} := \tilde{\exists} u \tilde{\forall} v C_{\mathrm{B}}(u,v)$. By induction hypothesis we have monotone terms s and t such that

(i) $\tilde{\forall} b, e(\tilde{\forall} c \trianglelefteq sbe A_{\mathrm{B}}(b,c) \to B_{\mathrm{B}}(tb, e))$

and monotone terms r, q such that

(ii) $\tilde{\forall} d, v(\tilde{\forall} e \trianglelefteq rdv B_{\mathrm{B}}(d,e) \to C_{\mathrm{B}}(qd, v))$.

We have to produce monotone terms p and l satisfying

$$\tilde{\forall} b, v(\tilde{\forall} c \trianglelefteq lbv A_{\mathrm{B}}(b,c) \to C_{\mathrm{B}}(pb, v)).$$

Let $l(b,v) := s(b, r(tb, v))$ and $p(b) := q(tb)$. Fix monotone b, v and assume that (iii) $\tilde{\forall} c \trianglelefteq s(b, r(tb,v)) A_{\mathrm{B}}(b,c)$. By (i) we get

(iv) $\tilde{\forall} e \trianglelefteq r(tb,v)(\tilde{\forall} c \trianglelefteq sbe A_{\mathrm{B}}(b,c) \to B_{\mathrm{B}}(tb,e))$.

From (iii) and (iv) we have (v) $\tilde{\forall} e \trianglelefteq r(tb,v) B_{\mathrm{B}}(tb,e)$. From (ii) and (v) we get (by taking $d := tb$) $C_{\mathrm{B}}(q(tb), v)$.

3a. $A \vee A \to A$. To interpret this axiom we must find monotone terms t, q_1 and q_2 such that

$$\tilde{\forall} b_1, b_2, c(\tilde{\forall} c'_1 \trianglelefteq q_1 b_1 b_2 c \tilde{\forall} c'_2 \trianglelefteq q_2 b_1 b_2 c \bigvee_{i=1}^{2} \tilde{\forall} c''_i \trianglelefteq c'_i A_{\mathrm{B}}(b_i, c''_i) \to A_{\mathrm{B}}(t b_1 b_2, c)).$$

Let $t(b_1, b_2) := \mathrm{m}(b_1, b_2)$ and $q_i(b_1, b_2, c) := c$. Clearly, the terms t and q_i are monotone (see Lemma 4). By the Monotonicity Property and Lemma 4, these terms do the job.

3b. $A \to A \wedge A$. To interpret this axiom we must find monotone terms t_1, t_2 and q such that

$$\tilde{\forall} b, c_1, c_2(\tilde{\forall} c \trianglelefteq q(b, c_1, c_2) A_{\mathrm{B}}(b, c) \to \bigwedge_{i=1}^{2} A_{\mathrm{B}}(t_i(b), c_i)).$$

Let $t_i(b) := b$ and $q(b, c_1, c_2) := \mathrm{m}(c_1, c_2)$. These terms will do. Note that the construction of the term q is *canonical* at this step, contrary to Gödel's interpretation where one is faced with a choice of terms.

4. $A \to A \vee B$ and $A \wedge B \to A$. Trivial.

5. $A \vee B \to B \vee A$ and $A \wedge B \to B \wedge A$. Trivial.

6. $A \to B \Rightarrow C \vee A \to C \vee B$. Assume $(C)^{\mathrm{B}} := \tilde{\exists} u \tilde{\forall} v C_{\mathrm{B}}(u, v)$. By induction hypothesis, there are monotone terms t and q such that

(i) $\tilde{\forall} b \tilde{\forall} e(\tilde{\forall} c \trianglelefteq tbe A_{\mathrm{B}}(b, c) \to B_{\mathrm{B}}(qb, e))$.

Let us compute the bounded functional interpretation of the conclusion of the rule. At first,

$$\begin{cases} \tilde{\exists} u, b \tilde{\forall} v, c(\tilde{\forall} v' \trianglelefteq v C_{\mathrm{B}}(u, v') \vee \tilde{\forall} c' \trianglelefteq c A_{\mathrm{B}}(b, c') \to \\ \qquad \tilde{\exists} u, d \tilde{\forall} v, e(\tilde{\forall} v' \trianglelefteq v C_{\mathrm{B}}(u, v') \vee \tilde{\forall} e' \trianglelefteq e B_{\mathrm{B}}(b, e')). \end{cases}$$

We must find monotone terms r, s, p and l such that for all monotone u, b, v and e,

$$\begin{cases} \forall v'' \trianglelefteq rubve \tilde{\forall} c \trianglelefteq subve(\tilde{\forall} v' \trianglelefteq v'' C_{\mathrm{B}}(u, v') \vee \tilde{\forall} c' \trianglelefteq c A_{\mathrm{B}}(b, c')) \to \\ \qquad \forall v' \trianglelefteq v C_{\mathrm{B}}(pub, v') \vee \tilde{\forall} e' \trianglelefteq e B_{\mathrm{B}}(lub, e'). \end{cases}$$

We take $r(u, b, v, e) := v$, $s(u, b, v, e) := t(b, e)$, $p(u, b) := u$ and $l(u, b) = q(b)$. These are clearly monotone terms. Let us verify that they do the job. Fix monotone u, b, v and e, and suppose that

$$\tilde{\forall} v'' \trianglelefteq v \tilde{\forall} c \trianglelefteq tbe(\tilde{\forall} v' \trianglelefteq v'' C_{\mathrm{B}}(u, v') \vee \tilde{\forall} c' \trianglelefteq c A_{\mathrm{B}}(b, c')).$$

In particular,

$$\tilde{\forall} v' \trianglelefteq v C_{\mathrm{B}}(u, v') \vee \forall c' \trianglelefteq tbe A_{\mathrm{B}}(b, c').$$

If the first disjunct holds, we are done. Suppose that the second disjunct is the case. Take any monotone $e' \trianglelefteq e$. Clearly, if $c' \trianglelefteq tbe'$ then $c' \trianglelefteq tbe$. Hence, $A_{\mathrm{B}}(b, c')$. We have showed that, $\tilde{\forall} c' \trianglelefteq tbe' A_{\mathrm{B}}(b, c')$. By (i), we get $B_{\mathrm{B}}(qb, e')$. By the arbitrariness of e', we are done.

7a. $A \wedge B \to C \Rightarrow A \to (B \to C)$. Assume $(C)^{\mathrm{B}} := \tilde{\exists} u \tilde{\forall} v C_{\mathrm{B}}(u, v)$. Just observe that the interpretation of $A \wedge B \to C$ asks for monotone terms t_1, t_2, s such that

$$\tilde{\forall} b, d, v(\tilde{\forall} c \trianglelefteq t_1 bdv A_{\mathrm{B}}(b, c) \wedge \tilde{\forall} e \trianglelefteq t_2 bdv B_{\mathrm{B}}(d, e) \to C_{\mathrm{B}}(sbd, v)),$$

while the conclusion of the rule asks for terms t_1, t_2, s such that

$$\tilde{\forall} b, d, v(\tilde{\forall} c \trianglelefteq t_1 bdv A_{\mathrm{B}}(b, c) \to (\tilde{\forall} e \trianglelefteq t_2 bdv B_{\mathrm{B}}(d, e) \to C_{\mathrm{B}}(sbd, v))).$$

7b. $A \to (B \to C) \Rightarrow A \wedge B \to C$. Similar to 7a.

8. $\perp \to A$. Trivial.

9. $A \to B(z) \Rightarrow A \to \forall z B(z)$. The interpretation of $A \to B(z)$ is

$$\tilde{\exists} f, g \tilde{\forall} b, e(\tilde{\forall} c \trianglelefteq gbe A_{\mathrm{B}}(b, c) \to B_{\mathrm{B}}(fb, e, z)).$$

By induction hypothesis, we have monotone terms r, s such that

$$\tilde{\forall} a, b, e \forall z \trianglelefteq a(\tilde{\forall} c \trianglelefteq rabe A_{\mathrm{B}}(b, c) \to B_{\mathrm{B}}(sab, e, z)).$$

The interpretation of the conclusion of the rule asks for monotone terms t and q satisfying,

$$\tilde{\forall} a, b, e(\tilde{\forall} c \trianglelefteq tabe A_{\mathrm{B}}(b, c) \to \forall z \trianglelefteq a B_{\mathrm{B}}(qab, e, z)).$$

It is clear that we can take $t := r$ and $q := s$.

10. $\forall x A(x) \to A(t)$. For this axiom it is important to show all the free variables. The interpretation of $\forall x A(x, z) \to A(t[z], z)$ is

$$\tilde{\exists} \phi, \psi, \theta \tilde{\forall} g, c(\tilde{\forall} d \trianglelefteq \phi gc \tilde{\forall} c' \trianglelefteq \psi gc \forall x \trianglelefteq d A_{\mathrm{B}}(gd, c', x, z) \to A_{\mathrm{B}}(\theta g, c, t[z], z)).$$

The soundness asks for monotone terms t, q, r such that for all g, c, a and $z \trianglelefteq a$

$$\tilde{\forall} d \trianglelefteq tgca \tilde{\forall} c' \trianglelefteq qgca \forall x \trianglelefteq d A_{\mathrm{B}}(gd, c', x, z) \to A_{\mathrm{B}}(rga, c, t[z], z).$$

Let $t(g, c, a) := \tilde{t}[a/z]$, $q(g, c, a) := c$ and $r(g, a) := g(\tilde{t}[a/z])$, where $\tilde{t}$ is a term that majorizes t. It is easy to see that for all g, c, a and $z \trianglelefteq a$

$$\tilde{\forall} d \trianglelefteq \tilde{t}[a/z] \tilde{\forall} c' \trianglelefteq c \forall x \trianglelefteq d A_{\mathrm{B}}(gd, c, x, z) \to A_{\mathrm{B}}(g(\tilde{t}[a/z]), c, t[z], z),$$

since $t[z] \trianglelefteq \tilde{t}[a/z]$. It is clear that t, q and r are monotone.

11. $A(t) \to \exists x A(x)$. Here, it is also important to make explicit all the free variables. The axiom $A(t[z], z) \to \exists x A(x, z)$ has (partial) interpretation

$$\tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b, c, t[z], z) \to \tilde{\exists} d, b \tilde{\forall} c \exists x \trianglelefteq d \tilde{\forall} c' \trianglelefteq c A_{\mathrm{B}}(b, c', x, z)$$

and the soundness asks for monotone terms q, r and s such that for all b, c, a and $z \trianglelefteq a$

$$\tilde{\forall} c' \trianglelefteq qbca A_{\mathrm{B}}(b, c', t[z], z) \to \exists x \trianglelefteq rba \tilde{\forall} c' \trianglelefteq c A_{\mathrm{B}}(sba, c', x, z).$$

Let $q(b, c, a) := c$, $r(b, a) := t^*[a/z]$ and $s(b, a) := b$. Then, for all b, e, a and $z \trianglelefteq a$

$$\tilde{\forall} c' \trianglelefteq c A_{\mathrm{B}}(b, c', t[z], z) \to \exists x \trianglelefteq t^*[a/z] \tilde{\forall} c' \trianglelefteq c A_{\mathrm{B}}(b, c', x, z),$$

follows since $t[z] \trianglelefteq t^*[a/z]$. It is also clear that q, r and s are monotone.

12. $A(z) \to B \Rightarrow \exists z A(z) \to B$. The interpretation of $A(z) \to B$ is

$$\tilde{\exists} f, g \tilde{\forall} b, e (\tilde{\forall} c \trianglelefteq gbe A_{\mathrm{B}}(b, c, z) \to B_{\mathrm{B}}(fb, e))$$

By induction hypothesis we have monotone terms r, s such that for all a, b, e and $z \trianglelefteq a$

$$\tilde{\forall} c \trianglelefteq rabe A_{\mathrm{B}}(b, c, z) \to B_{\mathrm{B}}(sab, e).$$

The interpretation of the conclusion of the rule asks for monotone terms t and q satisfying for all a, b, e

$$\tilde{\forall} c \trianglelefteq tabe \exists z \trianglelefteq a \tilde{\forall} c' \trianglelefteq c A_{\mathrm{B}}(b, c', z) \to B_{\mathrm{B}}(qab, e).$$

It's clear that we can take $t := r$ and $q := s$. □

4 The Interpretation at Work

Gödel's original interpretation [9] interprets certain principles, whose status goes beyond the intuitionistically acceptable (this was studied by M. Yasugi in [38]; see sections 3.5.7-3.5.11 of [37] for an exposition of these matters). The Bounded Functional Interpretation also interprets certain principles beyond those provable in $\mathsf{IL}^{\omega}_{\trianglelefteq}$. While some of these principles are *related* to the principles that are vindicated by Gödel's interpretation, others are completely new.

We finish the section with a Characterization Theorem for the bounded functional interpretation.

4.1 Interpretable Principles

As we will show, the following principles have a bounded functional interpretation:

1. The *Bounded Choice Principle*

$$\mathsf{bAC}^{\rho,\tau}[\trianglelefteq]:\ \forall x^\rho \exists y^\tau A(x,y) \to \tilde{\exists} f \tilde{\forall} b \forall x \trianglelefteq b \exists y \trianglelefteq fb A(x,y),$$

where A is an arbitrary formula of the language $\mathcal{L}^\omega_\trianglelefteq$. The standard Axiom of Choice does not seem to be interpretable in general. Still, in Subsection 4.2 we will see that a monotone version of the Axiom of Choice is interpreted by the Bounded Functional Interpretation.

2. The *Bounded Independence of Premises Principle*

$$\mathsf{bIP}^\rho_{\forall\mathrm{bd}}[\trianglelefteq]:\ (\forall \underline{x} A_\mathrm{b}(\underline{x}) \to \exists y^\rho B(y)) \to \tilde{\exists} b (\forall \underline{x} A_\mathrm{b}(\underline{x}) \to \exists y \trianglelefteq b B(y)),$$

where A_b is a bounded formula and B is an arbitrary formula.

3. The *Bounded Markov's Principle*

$$\mathsf{bMP}^\rho_{\mathrm{bd}}[\trianglelefteq]:\ (\forall y^\rho \forall \underline{x} A_\mathrm{b}(\underline{x}, y) \to B_\mathrm{b}) \to \tilde{\exists} b (\forall y \trianglelefteq b \forall \underline{x} A_\mathrm{b}(\underline{x}, y) \to B_\mathrm{b}),$$

where A_b and B_b are bounded formulas. When B_b is $\perp$, with the help of some intuitionistic logic, we get the following useful version of the above principle: $\neg\neg\exists y^\rho A_\mathrm{b}(y) \to \tilde{\exists} b \neg\neg\exists y \trianglelefteq b A_\mathrm{b}(y)$, where A_b is a bounded formula.

4. The *Bounded Universal Disjunction Principle*

$$\mathsf{bUD}^{\underline{\rho},\underline{\tau}}_{\forall\mathrm{bd}}[\trianglelefteq]:\ \tilde{\forall} \underline{b}^{\underline{\rho}} \tilde{\forall} \underline{c}^{\underline{\tau}} (\forall \underline{x} \trianglelefteq \underline{b} A_\mathrm{b}(\underline{x}) \vee \forall \underline{y} \trianglelefteq \underline{c} B_\mathrm{b}(\underline{y})) \to \forall \underline{x} A_\mathrm{b}(\underline{x}) \vee \forall \underline{y} B_\mathrm{b}(\underline{y}),$$

where A_b and B_b are bounded formulas. There is no analogue of this principle in Gödel's functional interpretation. In a setting where bounded first-order formulas are decidable, this principle generalizes Bishop's lesser limited principle of omniscience LLPO (cf. [3], but also [4]) viz. that $\forall x^0, y^0 (A(x) \vee B(y)) \to \forall x A(x) \vee \forall y B(y)$, where $A(x)$ and $B(y)$ are bounded first-order formulas.

5. The *Bounded Contra Collection Principle*

$$\mathsf{bBCC}^{\rho,\underline{\tau}}_{\mathrm{bd}}[\trianglelefteq]:\ \tilde{\forall} c^\rho (\tilde{\forall} \underline{b}^{\underline{\tau}} \exists z \trianglelefteq c \forall \underline{y} \trianglelefteq \underline{b} A_\mathrm{b}(\underline{y}, z) \to \exists z \trianglelefteq c \forall \underline{y} A_\mathrm{b}(\underline{y}, z)),$$

where A_b is a bounded formula. This principle allows the conclusion of certain existentially bounded statements from the assumption of weakenings thereof (so-called ϵ-*versions* or ϵ-*weakenings*, in a terminology that Kohlenbach introduced in [14] regarding a more concrete situation – see also Section 7.1 below). As we shall discuss in Section 7, the Bounded Contra Collection Principle entails certain classical (non-constructive) principles related to weak König's lemma.

6. And finally, the *Majorizability Axioms*

$$\mathsf{MAJ}^\rho[\trianglelefteq] : \ \forall x^\rho \exists y(x \trianglelefteq y).$$

We use $\mathsf{bAC}^\omega[\trianglelefteq]$, $\mathsf{bIP}^\omega_{\forall\mathrm{bd}}[\trianglelefteq]$, $\mathsf{bMP}^\omega_{\mathrm{bd}}[\trianglelefteq]$, $\mathsf{bUD}^\omega_{\forall\mathrm{bd}}[\trianglelefteq]$, $\mathsf{bBCC}^\omega_{\mathrm{bd}}[\trianglelefteq]$ and $\mathsf{MAJ}^\omega[\trianglelefteq]$, respectively, for the aggregate of each of the above principles over all types. We denote by $\mathsf{P}[\trianglelefteq]$ the sum total of all these principles.

The next principle is a vast generalization of a version of Brouwer's FAN theorem (Brouwer's theorem is the case $\rho = 1$, $\tau = 0$).[3] Under the label of *uniform boundedness* principles UB_ρ, Kohlenbach already considered in [21] a related generalization of the FAN theorem.

Proposition 3 *The theory* $\mathsf{IL}^\omega_\trianglelefteq + \mathsf{P}[\trianglelefteq]$ *proves the* Bounded Collection Principle

$$\mathsf{bBC}^{\rho,\tau}[\trianglelefteq] : \ \tilde{\forall} c(\forall z \trianglelefteq c^\rho \exists y^\tau A(y,z) \to \tilde{\exists} b \forall z \trianglelefteq c \exists y \trianglelefteq b A(y,z)),$$

where A is an arbitrary formula. (We use the acronym $\mathsf{bBC}^\omega[\trianglelefteq]$ for the aggregate of this principle over all types.)

Observation 2 *In the above, the formula A is arbitrary (in consonance with the FAN theorem). The Bounded Contra Collection Principle is (classically) the contrapositive of the Bounded Collection Principle,* restricted *to bounded matrices only. Note that the Bounded Contra Collection Principle is not intuitionistically acceptable, even for bounded matrices, $\rho = 1$ and $\tau = 0$ (it entails weak König's lemma, as we shall see).*

Proof. Let c monotone be fixed. Assume that

$$\forall z(z \trianglelefteq c^\rho \to \exists y^\tau A(y,z)).$$

By $\mathsf{bIP}^\omega_{\forall\mathrm{bd}}[\trianglelefteq]$ we get

$$\forall z \tilde{\exists} b(z \trianglelefteq c^\rho \to \exists y \trianglelefteq b A(y,z)),$$

which by $\mathsf{bAC}^\omega[\trianglelefteq]$ gives

[3] The usual formulation of Brouwer's FAN theorem in, for instance, section 1.9.24 of [37], differs from the statement herein (seen in the arithmetical setting) in that it concerns continuity, as opposed to majorizability.

$$\tilde{\exists} f \tilde{\forall} a \forall z \trianglelefteq a \tilde{\exists} b \trianglelefteq fa(z \trianglelefteq c^\rho \to \exists y \trianglelefteq bA(y,z)),$$

which implies

$$\tilde{\exists} f \tilde{\forall} a \forall z \trianglelefteq a(z \trianglelefteq c^\rho \to \exists y \trianglelefteq faA(y,z)).$$

Taking $a := c$, we get $\tilde{\exists} b \forall z \trianglelefteq c \exists y \trianglelefteq bA(y,z)$. □

Theorem 2 (Soundness, General Extension) *Consider a fixed language* $\mathcal{L}^\omega_\trianglelefteq$. *Let* $\mathsf{T}^\omega_\trianglelefteq$ *be a majorizability theory for* $\mathcal{L}^\omega_\trianglelefteq$ *and assume that* $(A(\underline{z}))^{\mathrm{B}}$ *is* $\tilde{\exists} \underline{b} \tilde{\forall} \underline{c} A_{\mathrm{B}}(\underline{b}, \underline{c}, \underline{z})$, *where* $A(\underline{z})$ *is an arbitrary formula of* $\mathcal{L}^\omega_\trianglelefteq$ *with its free variables as displayed. If*

$$\mathsf{IL}^\omega_\trianglelefteq + \mathsf{P}[\trianglelefteq] \vdash A(\underline{z}),$$

then there are closed monotone terms $\underline{t}$ *of appropriate types such that*

$$\mathsf{T}^\omega_\trianglelefteq \vdash \tilde{\forall} \underline{a} \forall \underline{z} \trianglelefteq \underline{a}\, \tilde{\forall} \underline{c} A_{\mathrm{B}}(\underline{ta}, \underline{c}, \underline{z}).$$

Proof. Given any principle $P \in \mathsf{P}[\trianglelefteq]$ (where $(P)^{\mathrm{B}}$ is $\tilde{\exists} b \tilde{\forall} c P_{\mathrm{B}}(b,c)$), we must argue that there are monotone closed terms s such that $\mathsf{T}^\omega_\trianglelefteq \vdash \tilde{\forall} c P_{\mathrm{B}}(s,c)$. Let us first look at the interpretation of $\mathsf{bAC}^\omega[\trianglelefteq]$. Assume that $(A(x,y))^{\mathrm{B}}$ is $\tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b,c,x,y)$. The interpretation of $\forall x \exists y A(x,y)$ is

$$\tilde{\exists} f, g \tilde{\forall} a, c B(a, ga, c, f),$$

where $B(a,b,c,f)$ is $\forall x \trianglelefteq a \exists y \trianglelefteq fa \tilde{\forall} c' \trianglelefteq c A_{\mathrm{B}}(b, c', x, y)$. The conclusion of $\mathsf{bAC}^\omega[\trianglelefteq]$ has interpretation

$$\tilde{\exists} f, g \tilde{\forall} a, c \tilde{\exists} f' \trianglelefteq f \tilde{\forall} a'' \trianglelefteq a \tilde{\forall} c'' \trianglelefteq c \tilde{\forall} a' \trianglelefteq a'' B(a', ga'', c'', f').$$

We have to show that there are monotone terms t and q, r_1 and r_2 such that, for all monotone f, g, a and c,

$$\begin{cases} \tilde{\forall} a' \trianglelefteq r_1 fgac \tilde{\forall} c'' \trianglelefteq r_2 fgac\, B(a', ga', c'', f) \to \\ \qquad \tilde{\exists} f' \trianglelefteq tfg \tilde{\forall} a'' \trianglelefteq a \tilde{\forall} c'' \trianglelefteq c \tilde{\forall} a' \trianglelefteq a'' B(a', qfga'', c'', f'). \end{cases}$$

Let $t(f,g) := f$, $q(f,g) := g$, $r_1(f,g,a,c) = a$, $r_2(f,g,a,c) := c$ and take f' as f. The implication

$$\begin{cases} \tilde{\forall} a' \trianglelefteq a \tilde{\forall} c' \trianglelefteq c B(a', ga', c', f) \to \\ \qquad \tilde{\forall} a'' \trianglelefteq a \tilde{\forall} c'' \trianglelefteq c \tilde{\forall} a' \trianglelefteq a'' B(a', ga'', c'', f), \end{cases}$$

follows due to the monotonicity property of the second entry of B, the monotonicity of g and the transitivity of $\trianglelefteq$.

Let us now look at the interpretation of $\mathsf{bIP}^{\omega}_{\forall \mathrm{bd}}[\trianglelefteq]$. Its premise has interpretation

$$\tilde{\exists} b, d, g \tilde{\forall} e (\tilde{\forall} a \trianglelefteq ge \forall x \trianglelefteq a A_{\mathrm{b}}(x) \to \exists y \trianglelefteq b \tilde{\forall} e' \trianglelefteq e B_{\mathrm{B}}(d, e', y))$$

while the interpretation of the conclusion is

$$\begin{cases} \tilde{\exists} b, d, g \tilde{\forall} e \exists b' \trianglelefteq b \tilde{\forall} e'' \trianglelefteq e \Big(b' \trianglelefteq b' \wedge \\ \qquad (\tilde{\forall} a \trianglelefteq g e'' \forall x \trianglelefteq a A_{\mathrm{b}}(x) \to \exists y \trianglelefteq b' \tilde{\forall} e' \trianglelefteq e'' B_{\mathrm{B}}(d, e', y)) \Big). \end{cases}$$

It is now easy to check that there are straightforward monotone terms (projections) that interpret the above principle.

We now study the interpretation of $\mathsf{bMP}^{\omega}_{\mathrm{bd}}[\trianglelefteq]$. Its antecedent has interpretation

$$\tilde{\exists} b, a (\tilde{\forall} b' \trianglelefteq b \tilde{\forall} a' \trianglelefteq a \forall y \trianglelefteq b' \forall x \trianglelefteq a' A_{\mathrm{b}}(x, y) \to B_{\mathrm{b}})$$

while the interpretation of the conclusion is

$$\tilde{\exists} b, a \tilde{\exists} b' \trianglelefteq b (\tilde{\forall} a' \trianglelefteq a \forall y \trianglelefteq b' \forall x \trianglelefteq a' A_{\mathrm{b}}(x, y) \to B_{\mathrm{b}}).$$

Again, it is easy to check that there are suitable terms that interpret $\mathsf{bMP}^{\omega}_{\mathrm{bd}}[\trianglelefteq]$.

The interpretations of $\mathsf{bUD}^{\omega}_{\forall \mathrm{bd}}[\trianglelefteq]$ and $\mathsf{bBCC}^{\omega}_{\mathrm{bd}}[\trianglelefteq]$ and are straightforward. Finally, the interpretation of the majorizability axiom is

$$\tilde{\exists} f \tilde{\forall} a \forall x \trianglelefteq a \exists y \trianglelefteq fa (x \trianglelefteq y),$$

which is also interpreted by the identity functional. □

4.2 The Characterization Theorem

In this section we show that the principles $\mathsf{P}[\trianglelefteq]$ are exactly the ones needed for proving the equivalence $A \leftrightarrow (A)^{\mathrm{B}}$, for arbitrary formulas A.

Proposition 4 (Monotone Axiom of Choice) $\mathsf{IL}^{\omega}_{\trianglelefteq} + \mathsf{P}[\trianglelefteq]$ *proves*

$$\Big(\tilde{\forall} \underline{a} \tilde{\forall} \underline{b} \tilde{\forall} \underline{b'} \trianglelefteq \underline{b} [A(\underline{a}, \underline{b'}) \to A(\underline{a}, \underline{b})] \wedge \tilde{\forall} \underline{a} \tilde{\exists} \underline{b} A(\underline{a}, \underline{b}) \Big) \to \tilde{\exists} \underline{f} \tilde{\forall} \underline{a} A(\underline{a}, \underline{f}(\underline{a})),$$

where A is an arbitrary formula of the language $\mathcal{L}^{\omega}_{\trianglelefteq}$.

Proof. We argue the above for single variables, instead of tuples (the general case reduces to this one by induction on the number of variables). Assume the antecedent. In particular, we have $\forall a(a \trianglelefteq a \to \exists b(b \trianglelefteq b \wedge A(a,b)))$. By $\mathsf{bIP}^{\omega}_{\forall \mathrm{bd}}[\trianglelefteq]$ we get

$$\forall a \tilde{\exists} b(a \trianglelefteq a \to \exists b' \trianglelefteq b(b' \trianglelefteq b' \wedge A(a,b')))$$

which implies, by the monotonicity of A, $\forall a \tilde{\exists} b(a \trianglelefteq a \to A(a,b))$. By $\mathsf{bAC}^{\omega}[\trianglelefteq]$ we have

$$\tilde{\exists} f \tilde{\forall} a \forall a' \trianglelefteq a \tilde{\exists} b \trianglelefteq f(a)(a' \trianglelefteq a' \to A(a',b)).$$

By the monotonicity of A, $\tilde{\exists} f \tilde{\forall} a \tilde{\forall} a' \trianglelefteq a A(a', f(a))$ follows. We now conclude that $\tilde{\exists} f \tilde{\forall} a A(a, f(a))$. $\square$

Theorem 3 (Characterization) *Let A be an arbitrary formula of $\mathcal{L}^{\omega}_{\trianglelefteq}$. Then*

$$\mathsf{IL}^{\omega}_{\trianglelefteq} + \mathsf{P}[\trianglelefteq] \vdash A \leftrightarrow (A)^{\mathrm{B}}.$$

Proof. The proof is by induction on the logical structure of A. Let us assume that $A \leftrightarrow (A)^{\mathrm{B}}$ and $B \leftrightarrow (B)^{\mathrm{B}}$. It is easy to see that $(A \to B)^{\mathrm{B}}$ implies $A \to B$. For the reverse implication, we have that

$$\tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b,c) \to \tilde{\exists} d \tilde{\forall} e B_{\mathrm{B}}(d,e),$$

implies (by intuitionistic logic)

$$\tilde{\forall} b(\tilde{\forall} c A_{\mathrm{B}}(b,c) \to \tilde{\exists} d \tilde{\forall} e B_{\mathrm{B}}(d,e)).$$

By $\mathsf{bIP}^{\omega}_{\forall \mathrm{bd}}[\trianglelefteq]$ and the monotonicity property we get

$$\tilde{\forall} b \tilde{\exists} d(\tilde{\forall} c A_{\mathrm{B}}(b,c) \to \tilde{\forall} e B_{\mathrm{B}}(d,e)).$$

Again by intuitionistic logic we obtain

$$\tilde{\forall} b \tilde{\exists} d \tilde{\forall} e(\tilde{\forall} c A_{\mathrm{B}}(b,c) \to B_{\mathrm{B}}(d,e)).$$

By $\mathsf{bMP}^{\omega}_{\mathrm{bd}}[\trianglelefteq]$ we have

$$\tilde{\forall} b \tilde{\exists} d \tilde{\forall} e \tilde{\exists} c(\tilde{\forall} c' \trianglelefteq c A_{\mathrm{B}}(b,c) \to B_{\mathrm{B}}(d,e)).$$

By two applications of Proposition 4 we have

$$\tilde{\exists} f, g \tilde{\forall} b, e(\tilde{\forall} c' \trianglelefteq g(b,e) A_{\mathrm{B}}(b,c) \to B_{\mathrm{B}}(f(b),e)).$$

The equivalence between $A \vee B$ and $(A \vee B)^{\mathrm{B}}$ can be shown in $\mathsf{IL}^{\omega}_{\trianglelefteq} + \mathsf{bUD}^{\omega}_{\forall \mathrm{bd}}[\trianglelefteq]$, while the equivalence between $A \wedge B$ and $(A \wedge B)^{\mathrm{B}}$ relies purely on intuitionistic logic. The equivalence $\forall x A(x) \leftrightarrow (\forall x A(x))^{\mathrm{B}}$ depends on $\mathsf{MAJ}^{\omega}[\trianglelefteq]$ and $\mathsf{bAC}^{\omega}[\trianglelefteq]$ while $\exists x A(x) \leftrightarrow (\exists x A(x))^{\mathrm{B}}$ relies on $\mathsf{MAJ}^{\omega}[\trianglelefteq]$ and $\mathsf{bBCC}^{\omega}_{\mathrm{bd}}[\trianglelefteq]$. For bounded quantifiers $\forall x \trianglelefteq t A(x)$ and $\exists x \trianglelefteq t A(x)$ we use $\mathsf{bBC}^\omega[\trianglelefteq]$ and $\mathsf{bBCC}^\omega_{\mathrm{bd}}[\trianglelefteq]$ respectively. $\square$

5 The Negative Translation

We extend a version of the 'negative' translation of classical logic into intuitionistic logic to the language $\mathcal{L}^\omega_\trianglelefteq$ of bounded quantifiers, and show that the translation of some important principles are implied by the principles themselves (with the help of a form of Markov's principle and a stability condition). We use a 'negative' translation due to S. Kuroda [29]. This translation is defined in two steps. Firstly, it translates a formula A into a formula $A^\dagger$ by maintaining unchanged atomic formulas, conjunctions, disjunctions, implications and existential quantifications and inserting a double negation *after* each universal quantification. The 'negative' translation A' of A is, by definition, $\neg\neg A^\dagger$. We extend this translation to the language $\mathcal{L}^\omega_\trianglelefteq$ of the bounded quantifiers in the obvious way:

(1) $(\exists x \trianglelefteq tA)^\dagger$ is $\exists x \trianglelefteq t\, A^\dagger$.
(2) $(\forall x \trianglelefteq tA)^\dagger$ is $\forall x \trianglelefteq t \neg\neg A^\dagger$.

The *Stability Axiom* S is the statement $\forall x^0, y^0(\neg\neg(x \trianglelefteq_0 y) \to x \trianglelefteq_0 y)$. Note that this axiom holds in theories of arithmetic because in these theories the relation $\leq_0$ is decidable. The Stability Axiom lifts to all types:

Lemma 7 *For all types σ, $\mathsf{IL}^\omega_\trianglelefteq + \mathsf{S} \vdash \neg\neg(x \trianglelefteq_\sigma y) \to x \trianglelefteq_\sigma y$.*

Proof. The proof is by induction on the type. The base case is the definition of S. We must show that $\mathsf{IL}^\omega_\trianglelefteq + \mathsf{S} \vdash \neg\neg(x \trianglelefteq_{\rho\to\sigma} y) \to x \trianglelefteq_{\rho\to\sigma} y$. According to the rule $\mathsf{RL}_\trianglelefteq$, it is sufficient to show that $\mathsf{IL}^\omega_\trianglelefteq + \mathsf{S}$ proves

$$(\ddagger) \quad \neg\neg(x \trianglelefteq_{\rho\to\sigma} y) \wedge u \trianglelefteq_\rho v \to xu \trianglelefteq_\sigma yv \wedge yu \trianglelefteq_\sigma yv.$$

Well, the implication $x \trianglelefteq_{\rho\to\sigma} y \wedge u \trianglelefteq_\rho v \to xu \trianglelefteq_\sigma yv \wedge yu \trianglelefteq_\sigma yv$ is an axiom of $\mathsf{IL}^\omega_\trianglelefteq$. By intuitionistic logic, $\mathsf{IL}^\omega_\trianglelefteq$ proves

$$\neg\neg(x \trianglelefteq_{\rho\to\sigma} y \wedge u \trianglelefteq_\rho v) \to \neg\neg(xu \trianglelefteq_\sigma yv \wedge yu \trianglelefteq_\sigma yv).$$

Now, using the fact that $\neg\neg(A \wedge B) \leftrightarrow \neg\neg A \wedge \neg\neg B$ and the induction hypothesis, $(\ddagger)$ follows. $\square$

Let $\mathsf{bAC}^\omega_{\mathrm{bd}}[\trianglelefteq]$ be the version of the Bounded Choice Principle in which the matrix A is bounded. The acronym $\mathsf{P}_{\mathrm{bd}}[\trianglelefteq]$ denotes the modification of $\mathsf{P}[\trianglelefteq]$ in which $\mathsf{bAC}^\omega[\trianglelefteq]$ is substituted by $\mathsf{bAC}^\omega_{\mathrm{bd}}[\trianglelefteq]$. By the proof of Proposition 3, the theory $\mathsf{IL}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq]$ guarantees the Bounded Collection Principle $\mathsf{bBC}^\omega[\trianglelefteq]$

for *bounded* formulas A (this restriction is denoted by $\mathsf{bBC}^{\omega}_{\mathrm{bd}}[\trianglelefteq]$).

Proposition 5 *If* $\mathsf{CL}^{\omega}_{\trianglelefteq} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash A$ *then* $\mathsf{IL}^{\omega}_{\trianglelefteq} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] + \mathsf{S} \vdash A'$.

Notation 2 *The theory* $\mathsf{CL}^{\omega}_{\trianglelefteq}$ *is the classical version of* $\mathsf{IL}^{\omega}_{\trianglelefteq}$*, i.e., it is obtained from it by the adjunction of the law of excluded middle. In general, if* T *is a intuitionistic theory,* CT *denotes its classical counterpart.*

Proof. The proof is by induction on the derivation of A. The axioms M_1 and M_2 are universal and, hence, their negative translations are consequences of themselves. In order to deal with the rule, the other axioms and the principles $\mathsf{P}_{\mathrm{bd}}[\trianglelefteq]$ observe, as a preliminary, that if A_{b} is a bounded formula, then so is $A^{\dagger}_{\mathrm{b}}$. Let us now check the axioms for the bounded quantifiers. The negative translation of $\mathsf{B}_{\forall}$ is the double negation of

$$\forall x \trianglelefteq t \neg\neg A^{\dagger}(x) \leftrightarrow \forall x \neg\neg(x \trianglelefteq t \rightarrow A^{\dagger}(x)).$$

It is clear that the above follows (intuitionistically) from $\mathsf{B}_{\forall}$ itself. The case $\mathsf{B}_{\exists}$ is even simpler. Let us now study the behaviour of the rule $\mathsf{RL}_{\trianglelefteq}$ under the negative translation. Suppose that the theory $\mathsf{CL}^{\omega}_{\trianglelefteq} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq]$ proves the premise (of an instance) of rule $\mathsf{RL}_{\trianglelefteq}$:

$$A_{\mathrm{b}} \wedge u \trianglelefteq v \rightarrow su \trianglelefteq tv \wedge tu \trianglelefteq tv,$$

where A_{b} is a bounded formula. By induction hypothesis, the theory $\mathsf{IL}^{\omega}_{\trianglelefteq} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] + \mathsf{S}$ derives

$$A^{\dagger}_{\mathrm{b}} \wedge u \trianglelefteq v \rightarrow \neg\neg(su \trianglelefteq tv \wedge tu \trianglelefteq tv).$$

By intuitionistic logic and the lemma above, we obtain a derivation of

$$A^{\dagger}_{\mathrm{b}} \wedge u \trianglelefteq v \rightarrow su \trianglelefteq tv \wedge tu \trianglelefteq tv.$$

By rule $\mathsf{RL}_{\trianglelefteq}$, $\mathsf{IL}^{\omega}_{\trianglelefteq} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] + \mathsf{S}$ derives $A^{\dagger}_{\mathrm{b}} \rightarrow s \trianglelefteq t$ and, hence, $A^{\dagger}_{\mathrm{b}} \rightarrow \neg\neg(s \trianglelefteq t)$, as wanted.

Clearly, the negative translation of a majorizability axiom $\forall x \exists y (x \trianglelefteq y)$ follows intuitionistically from itself. In the presence of the majorizability axioms, the Bounded Independence of Premises Principle and the Bounded Markov's Principle are classically true. We now show that $\mathsf{IL}^{\omega}_{\trianglelefteq} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] + \mathsf{S} \vdash (\mathsf{bAC}^{\omega}_{\mathrm{bd}}[\trianglelefteq])'$. An instance of $(\mathsf{bAC}^{\omega}_{\mathrm{bd}}[\trianglelefteq])'$ is intuitionistically equivalent to:

$$(*) \quad \forall x \neg\neg\exists y (A_{\mathrm{b}}(x,y))^{\dagger} \rightarrow \neg\neg\tilde{\exists} f \tilde{\forall} a \forall x \trianglelefteq a \neg\neg\exists y \trianglelefteq fa (A_{\mathrm{b}}(x,y))^{\dagger}$$

where A_{b} is a bounded formula. By $\mathsf{bMP}^{\omega}_{\mathrm{bd}}[\trianglelefteq]$, the antecedent of the above formula $(*)$ implies $\forall x \tilde{\exists} b \neg\neg\exists y \trianglelefteq b (A_{\mathrm{b}}(x,y))^{\dagger}$. By $\mathsf{bAC}^{\omega}_{\mathrm{bd}}[\trianglelefteq]$, we may conclude that $\tilde{\exists} f \tilde{\forall} a \forall x \trianglelefteq a \tilde{\exists} b \trianglelefteq fa \neg\neg\exists y \trianglelefteq b (A_{\mathrm{b}}(x,y))^{\dagger}$. Using the transitivity of $\trianglelefteq$, the following implication is intuitionistically valid:

$$\tilde{\exists} b \trianglelefteq fa \neg\neg\exists y \trianglelefteq b (A_{\mathrm{b}}(x,y))^{\dagger} \to \neg\neg\exists y \trianglelefteq fa (A_{\mathrm{b}}(x,y))^{\dagger}.$$

This implies the consequent of $(*)$, as wanted. In a similar vein, we can show that $\mathsf{IL}^{\omega}_{\trianglelefteq} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] + \mathsf{S} \vdash (\mathsf{bBC}^{\omega}_{\mathrm{bd}}[\trianglelefteq])'$. Finally the Bounded Contra Collection Principle follows classically from $\mathsf{bBC}^{\omega}_{\mathrm{bd}}[\trianglelefteq]$. □

6 Arithmetic

With a view to applications, in this section we associate to each theory of arithmetic HA^{ω}, $\mathsf{PRA}^{\omega}_{i}$ and $\mathsf{G}_n\mathsf{A}^{\omega}_{i}$ ($n \geq 2$) a majorizability theory $\mathsf{HA}^{\omega}_{\trianglelefteq}$, $\mathsf{PRA}^{\omega}_{i,\trianglelefteq}$ and $\mathsf{G}_n\mathsf{A}^{\omega}_{i,\trianglelefteq}$ (respectively). We prove some basic facts concerning these majorizability theories and see how they relate to the original ones.

6.1 Theories of Arithmetic

The theory HA^{ω} is a version of Gödel's quantifier-free calculus T with quantifiers ranging over each finite type, with the axioms and rules of intuitionistic predicate logic and induction for all formulas of the new language. We keep, however, Troelstra's minimal treatment of equality as described in Section 2. [4] In HA^{ω}, we can define the usual less than or equal numerical relation $\leq_0$, and the usual term $\max^{0\to(0\to0)}$, giving the maximum of two numbers. Under HA^{ω}, $\leq_0$ is a reflexive and transitive relation and max satisfies the axioms A_1 and A_2. We may suppose that $\leq_0$ and max are primitive symbols of the language, and may take 0 as the distinguished type 0 constant (just add new constants $\leq_0$ and m and adjoin universal numerical axioms characterizing them according to the usual definitions).

Definition 5 *The theory $\mathsf{HA}^{\omega}_{\trianglelefteq}$ in the language $\mathcal{L}^{\omega}_{\trianglelefteq}$ is the extension of HA^{ω} which has the additional axioms $\mathsf{B}_{\exists}$, $\mathsf{B}_{\forall}$, M_1, M_2 and the rule $\mathsf{RL}_{\trianglelefteq}$. Moreover, the induction axiom*

$$A(0) \wedge \forall x (A(x) \to A(\mathrm{S}x)) \to \forall x A(x)$$

is extended to all formulas of $\mathcal{L}^{\omega}_{\trianglelefteq}$ (i.e., the new relation symbols $\trianglelefteq$ may occur in A).

[4] Troelstra denotes the theory with the minimal treatment of equality by HA^{ω}_{0}. For simplicity, we use the simpler notation, since other treatments of equality will not be discussed in this paper.

Observation 3 *Note that, as part of the theory* HA^ω *(and, hence, as part of the theory* $\mathsf{HA}^\omega_\trianglelefteq$*), one has the following "equality" axioms for the recursors* R*,*

$$\mathsf{E}_{\mathrm{R}} \ : \ \phi[\mathrm{R}0yz/w] \leftrightarrow \phi[y/w], \quad \phi[\mathrm{R}(\mathrm{S}x)yz/w] \leftrightarrow \phi[z(\mathrm{R}xyz, x)/w],$$

where ϕ *is an atomic formula of* $\mathcal{L}^\omega_\leq$ *with a distinguished variable* w*, and* x*,* y *and* z *are variables of appropriate type. Proposition 2 can be extended accordingly.*

The following result is an adaptation of a result due to Howard in [11]:

Proposition 6 $\mathsf{HA}^\omega_\trianglelefteq$ *is a majorizability theory.*

Proof. As discussed above, $\mathsf{HA}^\omega_\trianglelefteq$ extends $\mathsf{IL}^\omega_\trianglelefteq$ (where m is max). Clearly, the arithmetical constants 0^0 and S^1 are self-majorizing (one uses rule $\mathsf{RL}_\trianglelefteq$ to check that $\mathrm{S} \trianglelefteq_1 \mathrm{S}$). It remains to see that the recursors R can be majorized (in the sense of $\trianglelefteq$). The basic observation is that Howard's proof that the recursors can be majorized (in the sense of $\leq^*$) only needs the rule $\mathsf{RL}_\trianglelefteq$, not the unwarranted implication. □

The intuitionistic theories of arithmetic $\mathsf{G}_n\mathsf{A}^\omega_i$ ($n \geq 2$) were introduced by Kohlenbach (see [19]) in suitable languages of all finite types. They form a sequence of increasing strength, closely related to the levels of Grzegorczyk's hierarchy of primitive recursive functions (first defined in [10]). These theories include: (a) a minimization functional μ_b of type $(0 \to 1) \to 1$ with (universal) axioms stating that $\mu_b f^{0\to 1} n^0 =_0 \min_0 k \leq_0 n(fnk =_0 0)$ if such a $k \leq_0 n$ exists, and $=_0 0$ otherwise; and (b) a maximization functional M of type $1 \to 1$ satisfying the equations $Mf0 =_0 f0$ and $Mf(n+1) =_0 \max_0(Mfn, f(n+1))$. They also have suitable recursors meant to define functions by bounded "predicative" recursion. There are only two places in which we do not follow Kohlenbach. Firstly (and importantly), we do not include Spector's weak extensionality rule in these theories,[5] and opt instead for the minimal treatment of equality already discussed above. Secondly (although not essentially), in order to keep in tune with the usual presentation of arithmetical theories, we do not include in $\mathsf{G}_n\mathsf{A}^\omega_i$ ($n \geq 2$) all the true purely universal sentences $\forall \underline{x} A_0(\underline{x})$, where $\underline{x}$ is a tuple of variables whose types have degree ≤ 2 (it is a known observation of G. Kreisel that the addition of true universal sentences does not have any effect on the bounds extracted – see Subsection 7.1 for a gener-

[5] The use of the weak extensionality rule was proposed by C. Spector in [35], since the standard extensionality axiom was shown by Howard [11] not to have a Dialectica interpretation. It seems, however, that even Spector's weak rule of extensionality does not have a bounded functional interpretation, which forces us to consider systems where equality is treated in a minimal fashion, as Section 2 illustrates. The reason for this apparent failure is the fact that the soundness theorem for b.f.i. makes use of a relativization to majorizable functionals.

alization of this observation). Due to this last modification, we must explicitly include in the theories $\mathsf{G}_n\mathsf{A}_i^\omega$ $(n \geq 2)$ the axiom of quantifier-free induction:

$$\forall f^1, x^0(f(0) = 0 \wedge \forall z \leq_0 x(f(z) = 0 \to f(\mathrm{S}z) = 0) \to f(x) = 0).$$

The theory PRA_i^ω is obtained from the union of the theories $\mathsf{G}_n\mathsf{A}_i^\omega$ $(n \geq 2)$ by adding the "predicative" recursors $\hat{\mathrm{R}}_\sigma$ due to Kleene (see [1] for a description of these recursors). [6]

Definition 6 *The theories* $\mathsf{PRA}_{i,\trianglelefteq}^\omega$ *and* $\mathsf{G}_n\mathsf{A}_{i,\trianglelefteq}^\omega$ *$(n \geq 2)$ in the language* $\mathcal{L}_\trianglelefteq^\omega$ *are the extensions of* PRA_i^ω *and* $\mathsf{G}_n\mathsf{A}_i^\omega$ *$(n \geq 2)$, respectively, obtained by adding the axioms* $\mathsf{B}_\exists$, $\mathsf{B}_\forall$, M_1, M_2 *and the rule* $\mathsf{RL}_\trianglelefteq$.

Warning 2 *The induction available in the new extended theories* $\mathsf{PRA}_{i,\trianglelefteq}^\omega$ *and* $\mathsf{G}_n\mathsf{A}_{i,\trianglelefteq}^\omega$ *$(n \geq 2)$ is exactly the same as that of the original theories, i.e., it does* not *include induction for quantifier-free formulas in which the new predicate symbols* $\trianglelefteq$ *occur.*

By the work of Kohlenbach in [19], the following is clear:

Proposition 7 *The theories* $\mathsf{PRA}_{i,\trianglelefteq}^\omega$ *and* $\mathsf{G}_n\mathsf{A}_{i,\trianglelefteq}^\omega$ *$(n \geq 2)$ are majorizability theories.*

Theorem 4 (Soundness, Arithmetical Extension) *Let* $\mathsf{T}_\trianglelefteq^\omega$ *be one of the theories* $\mathsf{HA}_\trianglelefteq^\omega$, $\mathsf{PRA}_{i,\trianglelefteq}^\omega$ *or* $\mathsf{G}_n\mathsf{A}_{i,\trianglelefteq}^\omega$ *$(n \geq 2)$, and assume* $(A(\underline{z}))^{\mathrm{B}} :\equiv \tilde{\exists}\underline{b}\tilde{\forall}\underline{c}A_{\mathrm{B}}(\underline{b}, \underline{c}, \underline{z})$, *where $A(\underline{z})$ is an arbitrary formula of* $\mathcal{L}_\trianglelefteq^\omega$ *with its free variables as displayed. If*

$$\mathsf{T}_\trianglelefteq^\omega + \mathsf{P}[\trianglelefteq] \vdash A(\underline{z}),$$

then there are closed monotone terms $\underline{t}$ of appropriate types of the language of $\mathsf{T}_\trianglelefteq^\omega$ *such that*

$$\mathsf{T}_\trianglelefteq^\omega \vdash \tilde{\forall}\underline{a}\forall\underline{z} \trianglelefteq \underline{a}\,\tilde{\forall}\underline{c}A_{\mathrm{B}}(\underline{t}\underline{a}, \underline{c}, \underline{z}).$$

Proof. By the Soundness Theorems, it is enough to see that the arithmetical axioms are interpreted in $\mathsf{T}_\trianglelefteq^\omega$. This is clearly the case for the arithmetical axioms that are universal statements, and this includes the quantifier-free induction axioms of the theories $\mathsf{PRA}_{i,\trianglelefteq}^\omega$ and $\mathsf{G}_n\mathsf{A}_{i,\trianglelefteq}^\omega$ $(n \geq 2)$.

It remains to see that the scheme of unrestricted induction can be interpreted in $\mathsf{HA}_\trianglelefteq^\omega$. It is easier to verify the equivalent induction rule. Let $(A(x))^{\mathrm{B}}$ be

[6] Except for the treatment of equality, PRA_i^ω is the theory *restricted*-$\hat{\mathsf{Z}}_i^\omega$ of Solomon Feferman in [6]. This theory is also denoted by $\widehat{\mathsf{HA}}^\omega\!\restriction$ in the literature. Charles Parsons' quantifier-free theory T_0 (see [33]) is an earlier version of these theories.

$\tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b, c, x)$, where A is an arbitrary formula. Assume that we have already monotone terms r, s and t such that

(i) $\tilde{\forall} c A_{\mathrm{B}}(r, c, 0)$

and

(ii) $\tilde{\forall} a^0, b, c \forall x \trianglelefteq a(\tilde{\forall} c' \trianglelefteq sabc A_{\mathrm{B}}(b, c', x) \to A_{\mathrm{B}}(tab, c, \mathrm{S}x))$.

Notice that (ii) implies

(iii) $\tilde{\forall} a, b \forall x \leq_0 a(\tilde{\forall} c A_{\mathrm{B}}(b, c, x) \to \tilde{\forall} c A_{\mathrm{B}}(tab, c, \mathrm{S}x))$.

By (i) and (iii), the scheme of induction and the monotonicity of A_{B} on the first argument, we get

$$\tilde{\forall} a, c \forall x \leq_0 a A_{\mathrm{B}}(\Phi(a), c, x),$$

where $\Phi(a) = \Psi(a, a)$, and $\Psi(x, a)$ is the iteration functional defined according to the following recursive clauses:

$$\begin{cases} \Psi(0, a) = r \\ \Psi(\mathrm{S}x, a) = \max\{\Psi(x, a), t(a, \Psi(x, a))\}. \end{cases}$$

Note that Φ is monotone (by the scheme of induction) and, hence, so is Ψ. □

6.2 *Two distinctive facts*

The next result shows that the presence of the rule $\mathsf{RL}_{\trianglelefteq}$ (and a minimal amount of arithmetic) entails the failure of the deduction theorem:

Proposition 8 *Let* $\mathsf{T}^{\omega}_{\trianglelefteq}$ *be one of the theories* $\mathsf{HA}^{\omega}_{\trianglelefteq}$, $\mathsf{PRA}^{\omega}_{i,\trianglelefteq}$ *or* $\mathsf{G}_n\mathsf{A}^{\omega}_{i,\trianglelefteq}$ *(*$n \geq 2$*). The deduction theorem fails for* $\mathsf{T}^{\omega}_{\trianglelefteq}$*.*

Proof. If the deduction theorem were valid for $\mathsf{T}^{\omega}_{\trianglelefteq}$ one could prove

$$\forall f, g\Big(\forall n \forall m \leq n(fm \leq gn \wedge gm \leq gn) \to f \trianglelefteq g\Big).$$

By the soundness theorem we would have a term t satisfying

$$\tilde{\forall} f^*, g^* \forall f \trianglelefteq f^* \forall g \trianglelefteq g^* \Big(\forall n \leq t f^* g^* \forall m \leq n(fm \leq gn \wedge gm \leq gn) \to f \trianglelefteq g\Big).$$

Let $f^* = g^* = 1$ and $k = t11$. We get

$$\forall f \trianglelefteq 1 \forall g \trianglelefteq 1 \Big(\forall n \leq k \forall m \leq n (fm \leq gn \wedge gm \leq gn) \to f \trianglelefteq g \Big).$$

Let f be constant zero function up to k and one otherwise, whereas g is one up to k and zero otherwise. Using rule $\mathsf{RL}_{\trianglelefteq}$ it is easy to show in $\mathsf{T}^{\omega}_{\trianglelefteq}$ that $f \trianglelefteq 1$ and $g \trianglelefteq 1$. We can also prove the premise of the implication above in $\mathsf{T}^{\omega}_{\trianglelefteq}$, which would entail $\mathsf{T}^{\omega}_{\trianglelefteq} \vdash f \trianglelefteq g$. This implies, however,

$$\mathsf{T}^{\omega}_{\trianglelefteq} \vdash \forall n \forall m \leq n (fm \leq gn \wedge gm \leq gn).$$

Clearly a contradiction given the way f and g are defined. $\Box$

Intuitionistic mathematics accepts some results which the classical mathematician rejects, e.g. some versions of the FAN theorem (see [5] for a recent introduction to intuitionistic mathematics). The derivation of the non-classical FAN theorem within intuitionistic mathematics relies upon continuity principles peculiar to the intuitionistic philosophy of the continuum. From these continuity principles, one can obtain *refutations* of laws of classical logic. Although b.f.i. bypasses the intuitionistic principles of continuity via majorizability (as we have already observed in the introduction) it vindicates a very general form of the FAN theorem. It so happens that this general form already refutes laws of classical logic (this seems to be a folklore result). We show next that this folklore result can be formalized in the theory interpretable by b.f.i., via the minimal amount of Markov principle available there, which implies that b.f.i. is sound for classically false principles.

Proposition 9 *Let* $\mathsf{T}^{\omega}_{\trianglelefteq}$ *be one of the theories* $\mathsf{HA}^{\omega}_{\trianglelefteq}$, $\mathsf{PRA}^{\omega}_{i,\trianglelefteq}$ *or* $\mathsf{G}_n\mathsf{A}^{\omega}_{i,\trianglelefteq}$ *(*$n \geq 2$*). The theory* $\mathsf{T}^{\omega}_{\trianglelefteq} + \mathsf{P}[\trianglelefteq]$ *is inconsistent with* classical *logic.*

Proof. We show that $\mathsf{T}^{\omega}_{\trianglelefteq} + \mathsf{P}[\trianglelefteq]$ proves

$$\neg \forall x^1 (\forall n^0 (xn = 0) \vee \neg \forall n^0 (xn = 0)).$$

To see this, assume that the sentence prefixed by the negation sign holds. By $\mathsf{bMP}^0_{\mathrm{bd}}[\trianglelefteq]$ and a bit of intuitionistic arithmetic we may conclude

$$\forall x^1 \exists n^0 (\forall n^0 (xn = 0) \vee xn \neq 0).$$

In particular,

$$\forall x \trianglelefteq_1 1 \exists n^0 (\forall n^0 (xn = 0) \vee xn \neq 0).$$

Hence, by $\mathsf{bBC}^{1,0}[\trianglelefteq]$,

$$\exists k \forall x \trianglelefteq_1 1 \exists n \leq_0 k (\forall n^0 (xn = 0) \vee xn \neq 0).$$

This is clearly a contradiction: Just consider the function x^1 which has the value zero for the natural numbers up to k and is equal to one afterwards (note that $\mathsf{T}^{\omega}_{\trianglelefteq}$ proves that $x \trianglelefteq_1 1$). $\Box$

Using the terminology of Errett Bishop in [3], the theory interpretable by b.f.i. refutes the principle of limited omniscience LPO but, according to the results of Section 4.1, proves the lesser limited principle of omniscience (LLPO).

6.3 Towards Applications

In this subsection, we go through some basic assorted facts that are needed for the applications.

Lemma 8 *The following are derivable in* $\mathsf{G_2A}^{\omega}_{i,\trianglelefteq}$*:*

(i) $M \trianglelefteq_{1\to 1} M$
(ii) $f \trianglelefteq_1 Mf$
(iii) $x \trianglelefteq_\tau z \to \min_\tau(x,y) \trianglelefteq_\tau z$

Observation 4 *In the above,* $\min_0$ *is the usual minimum function of the natural number system, whereas* $\min_{\rho\to\sigma}(x,y)$ *is* $\lambda u^\rho. \min_\sigma(xu, yu)$*.*

Observation 5 *Instead of* Mf *we usually write* f^M*. With this notation, property 2 above becomes* $f \trianglelefteq f^M$*.*

Proof. We just have to be careful and make sure that we use the rule $\mathsf{RL}_{\trianglelefteq}$, not the unwarranted implication $\forall u, v(u \trianglelefteq v \to xu \trianglelefteq yv \wedge yu \trianglelefteq yv) \to x \trianglelefteq y$. By $\mathsf{RL}_{\trianglelefteq}$, in order to obtain $M \trianglelefteq M$ it is enough to prove that $f \trianglelefteq_1 g \to Mf \trianglelefteq_1 Mg$. This, on the other hand, follows (again by $\mathsf{RL}_{\trianglelefteq}$) from the provability of the implication $f \trianglelefteq_1 g \wedge n \trianglelefteq_0 m \to Mfn \trianglelefteq_0 Mgm \wedge Mgn \trianglelefteq_0 Mgm$. This is clear. The other claims of the lemma are also easy. For instance, the last claim is proved by induction on the types. The base case is trivial. The conditional

$$x \trianglelefteq_{\rho\to\sigma} z \to \min_{\rho\to\sigma}(x,y) \trianglelefteq_{\rho\to\sigma} z$$

follows from the provability of

$$x \trianglelefteq_{\rho\to\sigma} z \wedge u \trianglelefteq_\rho v \to \min_\sigma(xu, yu) \trianglelefteq_\sigma zv \wedge zu \trianglelefteq_\sigma zv.$$

This holds because, under the antecedent, the induction hypothesis yields $\min_\sigma(xu, yu) \trianglelefteq_\sigma zv$. $\Box$

Definition 7 *Let* $i, j \in \{0,1\}$*. By the acronym* $\mathsf{bAC}_0^{i,j}$ *we mean the following* bounded *choice principle:*

$$\forall x^i \exists y^j A_0(x,y) \to \exists \Phi^{i\to j} \forall x^i \exists y \leq_j \Phi x\, A_0(x,y),$$

where A_0 is a quantifier-free formula.

Proposition 10 *For $i, j \in \{0,1\}$, $\mathsf{G_2A}^\omega_{i,\trianglelefteq} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash \mathsf{bAC}^{i,j}_0$.*

Proof. We argue for $\mathsf{bAC}^{1,1}_0$ (the other choice principles follow from this one). Assume $\forall x^1 \exists y^1 A_0(x,y)$. By $\mathsf{bAC}^\omega_{\mathrm{bd}}[\trianglelefteq]$, $\tilde{\exists}\Phi^{1\to1}\tilde{\forall}a^1 \forall x \trianglelefteq_1 a \exists y \trianglelefteq_1 \Phi a\, A_0(x,y)$. Pick such a $\Phi^{1\to1}$. Given x^1, put a^1 as x^M and use part (ii) of Lemma 8 and part (iii) of Lemma 2, to obtain the desired conclusion. □

For $i, j \in \{0,1\}$, the usual quantifier-free choice principle $\mathsf{AC}^{i,j}_0$ is

$$\forall x^i \exists y^j A_0(x,y) \to \exists \Phi^{i\to j} \forall x^i \Phi\, A_0(x, \Phi x),$$

where A_0 is a quantifier-free formula. In virtue of the existence of the minimization functional μ_b, it is clear that $\mathsf{bAC}^{i,0}_0 \Rightarrow \mathsf{AC}^{i,0}_0$, for $i \in \{0,1\}$.

Theories of arithmetic satisfy the Stability Axiom (they even decide $x \leq_0 y$). The corollary below is a consequence of Proposition 5 together with the fact that the theories considered are closed under the negative translation:

Corollary 1 *Let $\mathsf{T}^\omega_\trianglelefteq$ be one of the theories $\mathsf{HA}^\omega_\trianglelefteq$, $\mathsf{PRA}^\omega_{i,\trianglelefteq}$ or $\mathsf{G}_n\mathsf{A}^\omega_{i,\trianglelefteq}$ $(n \geq 2)$. If $\mathsf{CT}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash A$ then $\mathsf{T}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash A'$.*

Each of the theories of arithmetic T^ω that we are considering was extended by a corresponding majorizability theory $\mathsf{T}^\omega_\trianglelefteq$. Theorems of these extended theories give rise, in a natural way, to theorems of the original theories. Let us see how. Given A any formula of the language $\mathcal{L}^\omega_\trianglelefteq$, A^* denotes the formula obtained from A by replacing each intensional symbol $\trianglelefteq_\sigma$ by the corresponding extensional relation $\leq^*_\sigma$ and, afterwards, unravelling (in the obvious way) the bounded quantifiers obtained thereof. Formally,

Definition 8 *For any given formula A in the language $\mathcal{L}^\omega_\trianglelefteq$, we define the formula A^* of the language $\mathcal{L}^\omega_\leq$ by recursion on A as follows:*

- (a) *If A is an atomic formula in which $\trianglelefteq$ does not occur, A^* is A.*
- (b) *For any given type σ, $(t \trianglelefteq_\sigma q)^*$ is $t \leq^*_\sigma q$.*
- (c) *$(A \Box B)^*$ is $A^* \Box B^*$, for $\Box \in \{\wedge, \vee, \to\}$.*
- (d) *$(QxA)^*$ is QxA^*, for $Q \in \{\forall, \exists\}$*
- (e) *For any given type σ, $(\forall x \trianglelefteq_\sigma tA)^*$ is $\forall x(x \leq^*_\sigma t \to A^*)$ and $(\exists x \trianglelefteq_\sigma tA)^*$ is $\exists x(x \leq^*_\sigma t \wedge A^*)$.*

The following result is clear:

Proposition 11 *Let T^ω be one of the theories HA^ω, PRA^ω_i or $\mathsf{G}_n\mathsf{A}^\omega_i$ $(n \geq 2)$, and let $A(\underline{z})$ be an arbitrary formula of $\mathcal{L}^\omega_\trianglelefteq$, with its free variables as displayed. We have:*

$\mathsf{T}^\omega_\trianglelefteq \vdash A(\underline{z}) \Rightarrow \mathsf{T}^\omega \vdash A^*(\underline{z})$.

Notice that (as remarked in Warning 2) the amount of induction available in theories with restricted induction must remain the same when the extension to the language $\mathcal{L}^\omega_\trianglelefteq$ is made. This is essential for Proposition 11 to hold, since via the transformation $(\cdot)^*$, the prime formulas $\trianglelefteq$ become formulas of high complexity. This is no problem, however, for theories already containing unrestricted induction as HA^ω.

7 Application: Interpreting UWKL

Weak König's Lemma, WKL for short, is the well-known principle saying that every infinite tree of finite sequences of 0's and 1's has an infinite path. We formalize this axiom as follows:

$$\forall f^1(\forall k \exists s \in \{0,1\}^k f^{\mathrm{tree}}(s) =_0 0 \to \exists x \leq_1 1 \forall k\, f^{\mathrm{tree}}(\overline{x}(k)) =_0 0),$$

where we are using the notation of [1]. This notation is explained swiftly. Given f^1, f^{tree} is a functional of type 1 so that, for any s^0,

$$f^{\mathrm{tree}}(s) =_0 0 \leftrightarrow (s \in \{0,1\}^{<\omega} \wedge \forall q \subseteq s(f^{\mathrm{tree}}(q) =_0 0)),$$

where $s \in \{0,1\}^{<\omega}$ means that s is (the code of) a binary sequence, and $q \subseteq s$ means that q is (the code of) an initial sequence of s. The functional f^{tree} itself is obtained from f by pruning away extraneous sequences (formally, $\lambda f^1.f^{\mathrm{tree}}$ is a functional of type $1 \to 1$). The expression $s \in \{0,1\}^k$ means that s is (the code of) a binary sequence of length k (note that the quantification $\exists s \in \{0,1\}^k(\ldots)$ is first-order bounded). Finally, given k^0 and x^1, $\overline{x}(k)$ is the (code of the) sequence $\langle x(0), x(1), \ldots, x(k-1)\rangle$. *Uniform weak König's lemma* (first introduced in [26], def. 2.3), UWKL for short, is the following strengthening of weak König's lemma[7]:

$$\exists \Phi^{1\to 1} \forall f^1(\forall k \exists s \in \{0,1\}^k f^{\mathrm{tree}}(s) =_0 0 \to \forall k\, f^{\mathrm{tree}}(\overline{\Phi f}(k)) =_0 0).$$

Lemma 9 *The following are derivable in* $\mathsf{G}_3\mathsf{A}^\omega_{i,\trianglelefteq}$*:*

(i) $\forall f^1(f^{\mathrm{tree}} \trianglelefteq_1 1)$.
(ii) $\forall s^0(\hat{s} \trianglelefteq_1 1)$*, where* $\hat{s}$ *is the functional of type 1 with the same values as the binary sequence* s *up to its length, and zero otherwise (in case* $s \notin \{0,1\}^{<\omega}$*,* $\hat{s}$ *is constantly zero).*

[7] Note that uniform weak König's lemma is no longer "weak" in the presence of full extensionality, as shown in [26].

Notation 3 *The '1' on the right-hand side of '$\trianglelefteq_1$' denotes the constant functional $\lambda n^0.1^0$.*

Observation 6 *Formally, $\lambda s.\hat{s}$ is a functional of type $0 \to 1$.*

Proof. We argue (ii) (part (i) is similar). According to rule $\mathsf{RL}_{\trianglelefteq}$, in order to prove that $\hat{s} \trianglelefteq_1 1$, it is enough to prove the implication $u \leq_0 v \to \hat{s}(u) \leq_0 1$. This is clear by the definition of $\hat{s}$. $\square$

Lemma 10 $\mathsf{G}_3\mathsf{A}^{\omega}_{i,\trianglelefteq} + \mathsf{bBCC}^{\omega}_{\mathrm{bd}}[\trianglelefteq] \vdash \mathsf{UWKL}$.

Proof. Let Bounded(f^1, k^0) abbreviate $\forall s \in \{0,1\}^k\, f(s) \neq_0 0$. $\mathsf{UWKL}^{\star}$ is the following principle:

$$\exists \Phi^{1\to 1} \forall f^1 \forall k^0 (\neg \mathrm{Bounded}(f^{\mathrm{tree}}, k) \to f^{\mathrm{tree}}(\overline{\Phi f}(k)) =_0 0).$$

It is clear that $\mathsf{UWKL}^{\star}$ entails UWKL. We show that $\mathsf{G}_3\mathsf{A}^{\omega}_{i,\trianglelefteq} + \mathsf{bBCC}^{\omega}_{\mathrm{bd}}[\trianglelefteq] \vdash \mathsf{UWKL}^{\star}$. Firstly, we claim that $\mathsf{G}_3\mathsf{A}^{\omega}_{i,\trianglelefteq}$ proves

$$\forall k^0 \exists \Phi \trianglelefteq_{1\to 1} \mathbb{1} \forall g \trianglelefteq_1 1 \forall n \leq_0 k (\neg \mathrm{Bounded}(g^{\mathrm{tree}}, n) \to g^{\mathrm{tree}}(\overline{\Phi g}(n)) =_0 0),$$

where $\mathbb{1}^{1\to 1} := \lambda f^1, k^0.1^0$. The proof of the claim follows closely an argument of Avigad and Feferman in [1]. Take an arbitrary k^0. Define $\phi^{1\to 0}$ as follows: Given g^1, if the empty sequence ϵ is not in g^1 (i.e., g applied to the code of ϵ is not zero), let ϕg be (the code of) ϵ; if not, consider the greatest length $\ell < k+1$ for which there is a sequence $s \in \{0,1\}^{\ell}$ such that $g^{\mathrm{tree}}(s) = 0$, and take ϕg to be (say) the leftmost such sequence. Define $\Phi^{1\to 1} g := \widehat{\phi g}$. By (ii) of the previous lemma, it is easy to argue that $\Phi \trianglelefteq_{1\to 1} \mathbb{1}$. Let g^1 be given. It does not take much reflection to conclude that

$$\forall n \leq_0 k (\neg \mathrm{Bounded}(g^{\mathrm{tree}}, n) \to g^{\mathrm{tree}}(\overline{\Phi g}(n)) = 0).$$

The claim follows. Using (i) of the previous lemma and the facts that $(g^{\mathrm{tree}})^{\mathrm{tree}}$ and g^{tree}, and Φg and Φg^{tree} are equal pointwise, the claim can be restated thus:

$$\tilde{\forall} f^1, k^0 \exists \Phi \trianglelefteq_{1\to 1} \mathbb{1} \forall g \trianglelefteq f \forall n \trianglelefteq k (\neg \mathrm{Bounded}(g^{\mathrm{tree}}, n) \to g^{\mathrm{tree}}(\overline{\Phi g}(n)) = 0).$$

By $\mathsf{bBCC}^{\omega}_{\mathrm{bd}}[\trianglelefteq]$, we get $\mathsf{UWKL}^{\star}$. $\square$

Theorem 5 and Corollary 2 below are similar to Theorem 4.8 of [14] and Theorem 3.2 of [26].

Theorem 5 *Let T^{ω} be one of the theories HA^{ω}, PRA^{ω}_i or $\mathsf{G}_n\mathsf{A}^{\omega}_i$ $(n \geq 3)$. If*

$$\mathsf{CT}^{\omega} + \mathsf{bAC}^{1,1}_0 + \mathsf{UWKL} \vdash \forall x^{\tau} \exists y^{\rho} A_0(x, y),$$

where τ and ρ are arbitrary types and A_0 is a quantifier-free formula (its free variables as displayed), then there is a closed monotone term $q^{\tau\to\rho}$ such that

$$\mathsf{CT}^\omega \vdash \forall a^\tau \forall x \leq^*_\tau a \exists y \leq^*_\rho qa\, A_0(x,y).$$

Proof. Suppose that $\mathsf{CT}^\omega + \mathsf{bAC}_0^{1,1} + \mathsf{UWKL} \vdash \forall x^\tau \exists y^\rho A_0(x,y)$. By Proposition 10 and the previous lemma, $\mathsf{CT}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash \forall x^\tau \exists y^\rho A_0(x,y)$. By Corollary 1, $\mathsf{T}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash \forall x^\tau \neg\neg\exists y^\rho A_0(x,y)$. Using $\mathsf{bMP}^\omega_{\mathrm{bd}}[\trianglelefteq]$, we get

$$\mathsf{T}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash \forall x^\tau \exists w^\rho \neg\neg\exists y \trianglelefteq_\rho w\, A_0(x,y).$$

By the Soundness Theorem (Arithmetical Extension) and Proposition 11, there is a closed monotone term $q^{\tau\to\rho}$ such that,

$$(++)\quad \mathsf{T}^\omega \vdash \forall a^\tau \forall x \leq^*_\tau a \neg\neg\exists y \leq^*_\rho qa\, A_0(x,y).$$

The result follows. □

Corollary 2 *Let T^ω be one of the theories HA^ω, PRA^ω_i or $\mathsf{G}_n\mathsf{A}^\omega_i (n \geq 3)$. If $\mathsf{CT}^\omega + \mathsf{bAC}_0^{1,1} + \mathsf{UWKL} \vdash \forall x^0 \exists y^0 A_0(x,y)$, where A_0 is a quantifier-free formula (its free variables as displayed), then there is a closed term t^1 such that $\mathsf{T}^\omega \vdash \forall x^0 A(x,tx)$.*

Proof. We follow the proof of the previous theorem until (++). In our case, by letting a^0 be x^0, we get $\mathsf{T}^\omega \vdash \forall x^0 \neg\neg\exists y \leq_0 qx\, A_0(x,y)$. The double negation sign may disappear because first-order bounded formulas are decidable. The term t can be obtained by bounded search. □

The theory $\mathsf{PRA}^\omega + \mathsf{bAC}_0^{1,1}$ contains the well-known second-order theory RCA_0 from the studies in *Reverse Mathematics* [34]. This is clear because the presence of $\mathsf{AC}_0^{0,0}$ allows the deduction of Σ^0_1-induction and Δ^0_1-comprehension. Therefore, the second-order theory WKL_0 – obtained from RCA_0 by the adjunction of weak König's lemma – is a sub-theory of $\mathsf{PRA}^\omega + \mathsf{bAC}_0^{1,1} + \mathsf{UWKL}$. [8] The following result is essentially due to Kleene [12] (see [1] for an exposition): There is a natural translation of type 0 terms t of the language of PRA^ω_i whose only free variables are of type 0 to terms t^{PRA} of the language of Primitive Recursive Arithmetic PRA, such that if $\mathsf{PRA}^\omega_i \vdash t = q$, then $\mathsf{PRA} \vdash t^{PRA} = q^{PRA}$. The discussion of this paragraph, together with the corollary above, yields the following result of Harvey Friedman (1976, unpublished):

Corollary 3 *If $\mathsf{WKL}_0 \vdash \forall n \exists m A_0(n,m)$, where A_0 is a quantifier-free formula (its variables as displayed) and n and m are numerical variables, then there is a functional symbol f of the language of PRA such that $\mathsf{PRA} \vdash \forall n A_0(n, f(n))$.*

---

[8] The systems RCA_0 and WKL_0 were introduced by Harvey Friedman in [8] using a somewhat different language and axioms.

Ulrich Kohlenbach has been calling attention to the importance (for applications) of the extraction of bounds for consequences having the general form $\forall x^1 \forall z \leq_\sigma sx \exists y^\rho A_0(x, y, z)$, where A_0 is a quantifier-free formula, $\rho \leq 2$ and σ arbitrary (the special case $\sigma = 1$, $\rho = 0$ is the most important case for applications). Here follows a result based on section 3 of [18]:

Corollary 4 *Let* T^ω *be one of the theories* HA^ω, PRA_i^ω *or* $\mathsf{G}_n\mathsf{A}_i^\omega$ *(*$n \geq 3$*). If* $\mathsf{CT}^\omega + \mathsf{bAC}_0^{1,1} + \mathsf{UWKL} \vdash \forall x^1 \forall z \leq_\sigma sx \exists y^\rho A_0(x, y, z)$*, where* $\rho \leq 2$*,* $s^{1\to\sigma}$ *is a closed term, and* A_0 *is a quantifier-free formula (its free variables as displayed), then there is a closed monotone term* t *such that*

$$\mathsf{CT}^\omega \vdash \forall x^1 \forall z \leq_\sigma sx \exists y \leq_\rho tx\, A_0(x, y, z).$$

Proof. Write the universal formula $z \leq_\rho sx$ as $\forall \underline{u} B_0(\underline{u}, x, z)$, with B_0 quantifier-free. Then

$$\mathsf{CT}^\omega + \mathsf{bAC}_0^{1,1} + \mathsf{UWKL} \vdash \forall x^1, z^\sigma \exists y^\rho, \underline{u}(B_0(\underline{u}, x, z) \to A_0(x, y, z)).$$

Let $\tilde{s}$ be a closed term such that $\mathsf{T}^\omega \vdash s \leq^*_{1\to\sigma} \tilde{s}$. We apply Theorem 5 with a the pair $x^M, \tilde{s}x^M$, and infer

$$\mathsf{CT}^\omega \vdash \forall x^1 \forall z \leq^*_\sigma \tilde{s}x^M \exists y \leq^*_\rho qx^M(\tilde{s}x^M) \exists \underline{u}(B_0(\underline{u}, x, z) \to A_0(x, y, z)),$$

for a certain closed monotone term q (we disregard the bound on $\underline{u}$). Using the fact that $sx \leq^*_\sigma \tilde{s}x^M$ (and hence $z \leq_\sigma sx \to z \leq^*_\sigma \tilde{s}x^M$), we get

$$\mathsf{CT}^\omega \vdash \forall x^1 \forall z \leq_\sigma sx \exists y \leq^*_\rho qx^M(\tilde{s}x^M)\, A_0(x, y, z).$$

Suppose that ρ is 2 (the other cases are simpler). It is clear that the term $t^{1\to 2}$ defined by $tx^1 := \lambda f^1.(qx^M(\tilde{s}x^M))f^M$ does the job. $\square$

7.1 A Note on a Class of Postulates

Georg Kreisel has remarked in several papers that the use of lemmata constituted by true universal sentences in proofs of $\forall\exists$ theorems has no impact in the extraction of bounds. In [14], Kohlenbach generalized this observation by considering sets of sentences Δ of the form $\forall b \exists u \leq rb \forall v B_0(v, u, b)$, where v, u and b are of arbitrary type, r is a closed term and B_0 is quantifier-free (many principles of analysis, such as weak König's lemma or Brouwer's fixed point theorem, can be put in this form: see [17,20]). He introduced a weakening Δ_w of Δ (its ϵ-weakening) constituted essentially by sentences of the form $\forall b, v \exists u \leq rb \forall v' \leq^* v B_0(v', u, b)$, each one corresponding to a sentence of Δ. In Kohlenbach's work, the verification of the bounds obtained from the analysis of $\forall\exists$ theorems takes place (in general) in a theory that replaces Δ by a strengthening of Δ_w (obtained by a partial Skolemization of the sentences thereof). This strengthening seems to be necessary due to the fact that m.f.i. interprets (even in a classical context) the axiom of quantifier-free choice for arbitrary types. In our context, however, where only restricted forms of choice can be handled, Δ_w is all that is needed for the verification:

Theorem 6 *Let T^ω be one of the theories HA^ω, PRA^ω_i or $\mathsf{G}_n\mathsf{A}^\omega_i$ ($n \geq 2$). Suppose that*

$$\mathsf{CT}^\omega + \mathsf{bAC}^{1,1}_0 + \Delta \vdash \forall x^\tau \exists y^\rho A_0(x, y),$$

where τ and ρ are arbitrary types, A_0 is a quantifier-free formula (its free variables as displayed), and Δ is a class of sentences as described above. Then there is a closed monotone term $q^{\tau \to \rho}$ such that

$$\mathsf{CT}^\omega + \Delta_w \vdash \forall a^\tau \forall x \leq^*_\tau a \exists y \leq^*_\rho qa\, A_0(x, y),$$

where Δ_w is the weakening of Δ described above.

Proof. Take $\tilde{r}$ such that $\mathsf{T}^\omega_\trianglelefteq \vdash r \trianglelefteq \tilde{r}$, and write the universal formula $u \leq rb$ as $\forall \underline{w} C_0(\underline{w}, u, b)$, with C_0 quantifier-free. Consider the class $\Delta_\trianglelefteq$ of sentences of $\mathcal{L}^\omega_\trianglelefteq$ formed by

$$\tilde{\forall} b \forall b' \trianglelefteq b \tilde{\forall} v \tilde{\forall} \underline{w} \exists u \trianglelefteq \tilde{r} b \forall v' \trianglelefteq v \forall \underline{w}' \trianglelefteq \underline{w} (C_0(\underline{w}', u, b') \wedge B_0(v', u, b')),$$

each one corresponding to a sentence of Δ. Using $\mathsf{bBCC}^\omega_{\mathrm{bd}}[\trianglelefteq]$, each of the above sentences implies $\tilde{\forall} b \forall b' \trianglelefteq b \exists u \trianglelefteq \tilde{r} b \forall v, \underline{w} (C_0(\underline{w}, u, b') \wedge B_0(v, u, b'))$. By $\mathsf{MAJ}^\omega[\trianglelefteq]$, each of these sentences implies, in turn, the corresponding one in Δ. Therefore,

$$\mathsf{CT}^\omega + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] + \Delta_\trianglelefteq \vdash \forall x \exists y A_0(x, y).$$

Since each sentence of $\Delta_\trianglelefteq$ implies its negative translation, we get

$$\mathsf{T}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] + \Delta_\trianglelefteq \vdash \forall x \exists z \neg\neg \exists y \trianglelefteq z\, A_0(x, y).$$

It is clear that the bounded functional interpretations of the sentences in $\Delta_\trianglelefteq$ are of the form $\tilde{\forall} F$, with F a bounded formula, and that they are implied by the original sentences themselves. Hence, by (an obvious extension of) the Soundness Theorem (Arithmetical Version), we infer that

$$\mathsf{T}^\omega_\trianglelefteq + \Delta_\trianglelefteq \vdash \forall a \forall x \trianglelefteq a \neg\neg \exists y \trianglelefteq qa\, A_0(x, y),$$

for a suitable closed monotone term q. By (an obvious extension of) Proposition 11, we conclude that

$$\mathsf{CT}^\omega + \Delta^*_\trianglelefteq \vdash \forall a \forall x \leq^* a \exists y \leq^* qa\, A_0(x, y).$$

It is clear that $\Delta_w \Rightarrow \Delta^*_{\trianglelefteq}$ (using the fact that $\alpha \leq \beta \wedge \beta \leq^* \gamma \rightarrow \alpha \leq^* \gamma$). We are done. $\square$

Weak König's lemma provides an instructive case. Kohlenbach showed in [14] that WKL can be replaced by a Δ-sentence, namely (the following is a slight simplification due to Avigad and Feferman in [1]):

$$\forall f^1 \exists x \leq_1 1 \forall k^0 (\neg \text{Bounded}(f^{\text{tree}}, k) \rightarrow f^{\text{tree}}(\overline{x}(k)) =_0 0).$$

Therefore, one can extract a bound from a proof of a $\forall\exists$ sentence in CT^ω + $\mathsf{bAC}_0^{1,1}$ + WKL, and verify it in CT^ω together with the following weakening of WKL:

$$\forall f^1 \forall n^0 \exists x \leq_1 1 \forall k \leq_0 n (\neg \text{Bounded}(f^{\text{tree}}, k) \rightarrow f^{\text{tree}}(\overline{x}(k)) =_0 0).$$

Observe that this weakening is already provable in CT^ω. This is in tune with what was proved in the previous section.

8 Application: Uniform Boundedness

The *Boundedness Principle* for type 2 functionals, abbreviated by BF^2, is the statement $\forall \Phi^2 \forall h^1 \exists n^0 \forall f \leq_1 h\, [\Phi(f) \leq_0 n]$. This principle is intuitionistically valid (it is a consequence of the FAN theorem) although it is easily seen to be classically false. In [19], Kohlenbach considered versions of this principle, and showed that in suitable *classical* settings with full extensionality certain of its consequences are true (Kohlenbach's result is actually a conservation result of a false theory over a true one).

In the following definition, we consider a very general boundedness principle, and prove a corresponding conservation result.

Definition 9 *Let τ be an arbitrary type. The* Uniform Boundedness Principle *for type $\tau \rightarrow 1$ functionals, which we abbreviate by $\mathsf{UBF}^{\tau\rightarrow 1}$, is*

$$\forall G^{0\rightarrow(\tau\rightarrow 1)} \forall B^{0\rightarrow\tau} \exists g^{0\rightarrow 1} \forall k^0 \forall \Phi \leq_\tau Bk\, [G(k, \Phi) \leq_1 gk].$$

Theorem 7 *Suppose that CT^ω is PA^ω, PRA^ω or $\mathsf{G}_n\mathsf{A}^\omega$ ($n \geq 2$). If*

$$\mathsf{E\text{-}CT}^\omega + \mathsf{bAC}_0^{0,1} + \mathsf{AC}_0^{1,0} + \mathsf{UBF}^{\tau\rightarrow 1} \vdash \forall x^{0/1} \exists y^\rho A_0(x, y),$$

where ρ is an arbitrary type and A_0 is a quantifier-free formula (its free variables as displayed), then there is a closed monotone term $q^{0/1\rightarrow\rho}$ such that

$$\mathsf{CT}^\omega \vdash \forall x^{0/1} \exists y \leq^*_\rho qxA_0(x, y).$$

Observation 7 *The theory* $\mathsf{E\text{-}CT}^\omega$ *is the theory* CT^ω *together with full extensionality. Given* s *and* t *terms of type* $\rho := \rho_1 \to (\ldots \to (\rho_k \to 0)\ldots)$ *we say that* $s =_\rho t$ *if* $\forall y_1^{\rho_1} \ldots \forall y_k^{\rho_k}(sy_1 \ldots y_k =_0 ty_1 \ldots y_k)$*.* Full extensionality *is the collection of axioms of the form* $\forall z^{\rho \to \tau} \forall x^\rho, y^\rho (x =_\rho y \to zx =_\tau zy)$*.*

Notation 4 *The expression* $x^{0/1}$ *means that* x *can be of type 0 or 1.*

Proof. Suppose $A := \forall x^{0/1} \exists y^\rho A_0(x, y)$ is a theorem of the theory $\mathsf{E\text{-}CT}^\omega + \mathsf{bAC}_0^{0,1} + \mathsf{AC}_0^{1,0} + \mathsf{UBF}^{\tau \to 1}$. In the presence of full extensionality, $\mathsf{UBF}^{\tau\to 1}$ is a consequence of (actually, it is equivalent to)

$$(**) \quad \forall G^{0\to(\tau\to 1)} \forall B^{0\to\tau} \exists g^{0\to 1} \forall k^0 \forall \Phi^\tau \, [G(k, \min_\tau(Bk, \Phi)) \leq_1 gk]$$

To see this, let $G^{0\to(\tau\to 1)}$ and $B^{0\to\tau}$ be given functionals. By $(**)$, there is $g^{0\to 1}$ so that $\forall k^0 \forall \Phi^\tau \, [G(k, \min_\tau(Bk, \Phi)) \leq_1 gk]$. Take Φ^τ with $\Phi \leq_\tau Bk$. It is clear that $\min_\tau(Bk, \Phi) =_\tau \Phi$. By extensionality, $G(k, \min_\tau(Bk, \Phi)) =_1 G(k, \Phi)$. Hence $G(k, \Phi) \leq_1 gk$.

By the above, $\mathsf{E\text{-}CT}^\omega + \mathsf{bAC}_0^{0,1} + \mathsf{AC}_0^{1,0} + (**) \vdash A$. Noticing that the quantifiers of type (essentially) greater than 1 in $(**)$ are universal and appear positively, it follows by elimination of extensionality (using a technique of H. Luckhardt in [30]) that $\mathsf{CT}^\omega + \mathsf{bAC}_0^{0,1} + \mathsf{AC}_0^{1,0} + (**) \vdash A$. Thus, by Proposition 10,

$$\mathsf{CT}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] + (**) \vdash A.$$

We now claim that $\mathsf{T}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash (**)$. Let us reason within $\mathsf{T}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq]$. Suppose that $G^{0\to(\tau\to 1)}$ and $B^{0\to\tau}$ are given. By the Majorization Axioms, we can take $\tilde{G}^{0\to(\tau\to 1)}$ and $\tilde{B}^{0\to\tau}$ with $G \trianglelefteq_{0\to(\tau\to 1)} \tilde{G}$ and $B \trianglelefteq_{0\to\tau} \tilde{B}$. Define $g^{0\to 1}$ as $\lambda k^0 . \tilde{G}(k, \tilde{B}k)$. Fix k^0 and take an arbitrary functional Φ^τ. By Lemma 8, $\min_\tau(Bk, \Phi) \trianglelefteq_\tau \tilde{B}k$. Thus, $G(k, \min_\tau(Bk, \Phi)) \trianglelefteq_1 \tilde{G}(k, \tilde{B}k)$. By part *(iii)* of Lemma 2, we conclude that $G(k, \min_\tau(Bk, \Phi)) \leq_1 gk$. The claim is proved. Therefore,

$$\mathsf{CT}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash \forall x^{0/1} \exists y^\rho A_0(x, y).$$

By Corollary 1, $\mathsf{T}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash (\forall x^{0/1} \exists y^\rho A_0(x, y))'$. By $\mathsf{bMP}^\omega_{\mathrm{bd}}[\trianglelefteq]$, the theory $\mathsf{T}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq]$ proves $\forall x^{0/1} \exists z^\rho \neg\neg \exists y \trianglelefteq_\rho z A_0(x, y)$. Now, the Soundness Theorem (Arithmetical Extension) and Proposition 11 yield a closed monotone term $q^{0/1\to\rho}$ such that

$$\mathsf{T}^\omega \vdash \forall a^{0/1} \forall x \leq^*_{0/1} a \exists z \leq^*_\rho qa \neg\neg \exists y \leq^*_\rho z \, A_0(x, y).$$

By the transitivity of $\leq^*_\rho$, $\mathsf{T}^\omega \vdash \forall a^{0/1} \forall x \leq^*_{0/1} a \neg\neg \exists y \leq^*_\rho qa \, A_0(x, y)$. If the type of x is 1 (the case 0 is even simpler), we infer that

$$\mathsf{T}^\omega \vdash \forall x^1 \neg\neg \exists y \leq^*_\rho tx \, A_0(x, y),$$

where $t := \lambda x.qx^M$. The conclusion of the theorem follows. $\square$

The above theorem also holds for functionals G of type $0 \to (\tau \to 0)$, yielding the so-called $\mathsf{UBF}^{\tau\to 0}$ principles. Plainly, in order to put to use the above theorem one must have interesting functionals G of appropriate type around. This is potentially the case when $\tau = 1$ because the presence of $\mathsf{AC}_0^{1,0}$ allows one to obtain functionals of type of type $1 \to 0$ from quantifier complexes of the form $\forall f^1 \exists n^0$. The principle, $\mathsf{UBF}^{1\to 0}$ (or UBF^2), is:

$$\forall \Phi^{0\to 2} \forall h^{0\to 1} \exists g^1 \forall k^0 \forall f \leq_1 hk[\Phi(k,f) \leq_0 gk].$$

It is a version of Kohlenbach's principle F:

$$\forall \Phi^{0\to 2} \forall h^{0\to 1} \exists \psi^{0\to 1} \forall k^0 \forall f \leq_1 hk[\Phi(k,f) \leq_0 \Phi(k,\psi k)].$$ [9]

Kohlenbach's principle is seemingly stronger than UBF^2, insofar as it guarantees the (uniform) attainment of the maximum by the functional Φ. Yet, we will show in Section 9 that they are equivalent over $\mathsf{E\text{-}G_3A}^\omega + \mathsf{AC}_0^{1,0}$.

The *Uniform* Σ^0_1*-Boundedness Principle*, Σ^0_1-UB for short, also introduced by Kohlenbach in [19], is

$$\forall h^{0\to 1}\Big[\forall k^0 \forall f \leq_1 hk \exists n^0 A_0(f,h,k,n) \to \exists g^1 \forall k^0 \forall f \leq_1 hk \exists n \leq_0 gk A_0(f,h,k,n)\Big]$$

where A_0 is a quantifier-free formula (which may contain parameters of arbitrary type). In [22], Kohlenbach showed that this *false* principle implies (even relative to systems as weak as $\mathsf{G_2A}^\omega$) that all functions f from $[0,1]$ into the real numbers are uniformly continuous (with a modulus of uniform continuity). Therefore, all continuity requirements may be dropped, i.e. functions f from $[0,1]$ into the real numbers can just be treated as arbitrary functionals of type $1 \to 1$ that respect the notion of equality between real numbers. It also allows for particularly simple proofs of

- the attainment of the maximum of a function in $C[0,1]$,
- Dini's theorem,

among others.

We finish this section with a (version of a) result of Kohlenbach.

Corollary 5 *Suppose that* CT^ω *is* PA^ω, PRA^ω *and* $\mathsf{G}_n\mathsf{A}^\omega$ *(*$n \geq 2$*). Let* $s^{1\to 1}$ *be a closed term, and* $A_0(x^1, y^\rho, z^1)$ *be a quantifier-free formula with its free*

[9] This principle was introduced in [19], along with a weaker version F^-, different from UBF^2 (see also the forerunner F_0 in [18]). Some years later, Kohlenbach showed (see [25]) that in the presence of full extensionality and a form of quantifier-free choice, F is actually implied by the weaker F^-.

variables as shown. If

$$\mathsf{E\text{-}CT}^\omega + \mathsf{bAC}_0^{0,1} + \mathsf{AC}_0^{1,0} + \Sigma_1^0\text{-}\mathsf{UB} \vdash \forall x^1 \forall z \leq_1 sx \exists y^\rho A_0(x,y,z),$$

where $\rho \leq 2$, then there is a closed monotone term $t^{1\to\rho}$ such that

$$\mathsf{CT}^\omega \vdash \forall x^1 \forall z \leq_1 sx \exists y \leq_\rho tx\, A_0(x,y,z).$$

Proof. We claim that Σ_1^0-UB follows from UBF^2 in the presence of $\mathsf{AC}_0^{1,0}$. Let $h^{0\to1}$ be given and assume that $\forall k^0 \forall f \leq_1 hk \exists n^0 A_0(f,h,k,n)$. Using $\mathsf{AC}_0^{1,0}$, it is easy to see that we may infer that

$$\exists \Phi^{0\to(1\to0)} \forall k^0 \forall f \leq_1 hk A_0(f,h,k,\Phi kf).$$

By UBF^2 there is g^1 such that $\forall k^0 \forall f \leq_1 hk[\Phi kf \leq_0 gk]$. The claim follows. By Theorem 7 there is a closed monotone term $q^{1\to(1\to\rho)}$ such that CT^ω proves that

$$\forall x^1 \forall z \leq_1 sx \exists y \leq_\rho^* qxz\, A_0(x,y,z).$$

Let $\tilde{s}^{1\to1}$ be a closed term such that $s \leq^* \tilde{s}$. Suppose that ρ is of type 2 (types 0 and 1 are easier). The term $t^{1\to2}$ defined by $tx^1f^1 := (qx^M(\tilde{s}x^M))f^M$ will do. $\square$

For the *connoisseur*, it should be noted that the theory T^ω in which the verification of the bounding term is made is the same as the underlying theory in the premise, even when the theory is $\mathsf{G_2A}^\omega$ (compare this with theorem 4.21 of [19]). Note, however, that our theories have a weaker version $\mathsf{bAC}_0^{0,1}$ of the quantifier-free axiom of choice $\mathsf{AC}_0^{0,1}$ (this is no real weakening, except perhaps for $\mathsf{G_2A}^\omega$ – see the next section).

9 Refinements

We refine the main theorem of the previous section in two ways. Given f a functional of type 1 and n a number, let $\overline{f,n}$ be the functional of type 1 that coincides with f for inputs $k <_0 n$, and is zero for $k \geq_0 n$. Formally, $\lambda f, n.\overline{f,n}$ is a functional of type $1 \to (0 \to 1)$.

The next proof relies on an argument of Kohlenbach in [18]:

Proposition 12 *The theory* $\mathsf{E\text{-}G_2A}^\omega + \mathsf{AC}_0^{1,0} + \mathsf{BF}^2$ *proves*

$$\forall \Phi^2 \forall f^1 \exists n^0 \forall r \geq_0 n[\Phi(f) =_0 \Phi(\overline{f,r})].$$

Proof. By extensionality, it follows that

$$\forall \Phi^2 \forall f^1, g^1 \exists n^0 (\forall k < n(fk =_0 gk) \to \Phi(f) =_0 \Phi(g)).$$

Fix Φ^2. It is an easy consequence of $\mathsf{AC}_0^{1,0}$ that there is μ of type $1 \to (1 \to 0)$ such that:

$$\forall f^1, g^1 (\forall k < \mu fg(fk =_0 gk) \to \Phi(f) =_0 \Phi(g)).$$

Fix now f^1. By BF^2, there is n^0 such that,

$$\forall g \leq_1 f(\mu fg \leq_0 n).$$

Therefore,

$$\forall g \leq_1 f(\forall k < n(fk =_0 gk) \to \Phi(f) =_0 \Phi(g)).$$

The proposition clearly follows. $\square$

We need to be able to do sequence coding in the proofs of the next two propositions. Hence, we work with the level 3 of the Grzegorczyk's hierarchy:

Proposition 13 $\mathsf{E\text{-}G_3A}^\omega + \mathsf{AC}_0^{1,0} + \mathsf{BF}^2 \vdash \mathsf{AC}_0^{1,1}$.

Proof. Assume that $\forall f^1 \exists g^1 A_0(f, g)$. By the previous proposition, this is equivalent to $\forall f^1 \exists n^0 A_0(f, \overline{g, n})$ which, in turn, implies

$$\forall f^1 \exists s^0 (s \in \omega^{<\omega} \wedge A_0(f, \hat{s})),$$

where $s \in \omega^{<\omega}$ says that s is a (finitely long) sequence of natural numbers. By $\mathsf{AC}_0^{1,0}$, take $\Phi^{1\to 0}$ so that $\forall f^1 A_0(f, \widehat{\Phi f})$. The functional $\Psi^{1\to 1} := \lambda f^1 . \widehat{\Phi f}$ does the job. $\square$

By the above proposition, in Theorem 7 we can substitute the hypothesis $\mathsf{bAC}_0^{0,1} + \mathsf{AC}_0^{1,0}$ by the more encompassing $\mathsf{AC}_0^{1,1}$ provided that we rule out the theory $\mathsf{G_2A}^\omega$. More interestingly, if we now *weaken* the hypothesis by removing the false (for τ non zero) principle $\mathsf{UBF}^{\tau\to 1}$ we get a *true* theory with the choice principle $\mathsf{AC}_0^{1,1}$ for which the theorem (again, with the exception of $\mathsf{G_2A}^\omega$) holds good. However, this result is obtained in a very roundabout way, via a false extension. Is there a more direct route?

Before we consider the second refinement of Theorem 7, we digress in order to show the following (as promised in the previous section):

Proposition 14 *Over the theory* $\mathsf{E\text{-}G_3A}^\omega + \mathsf{AC}_0^{1,0}$, UBF^2 *entails Kohlenbach's principle* F.

Proof. Let $\Phi^{0\to 2}$ and $h^{0\to 1}$ be given functionals. By UBF^2, there is g^1 such that

$$\forall k^0 \forall f \leq_1 hk\, [\Phi(k,f) \leq_0 gk].$$

According to Proposition 12, $\forall k^0, f^1 \exists n^0\, [\Phi(k,f) =_0 \Phi(k, \overline{f,n})]$. By $\mathsf{AC}_0^{1,0}$, it is easy to see that there is $M^{0\to 2}$ such that $\forall k^0, f^1\, [\Phi(k,f) =_0 \Phi(k, \overline{f, M(k,f)})]$. We now apply UBF^2 again (this time with respect to M and h) in order to get e^1 such that

$$\forall k^0 \forall f \leq_1 hk\, [M(k,f) \leq_0 ek].$$

Consider the formula A defined by,

$$A(k^0, m^0) \;:=\; \exists s \in \{0,1,\ldots,hk\}^{<ek}\, [\Phi(k,\hat{s}) = m],$$

where the above notation means that s is a sequence of length less than ek constituted by elements less than or equal to hk. Clearly, A is a first-order bounded formula and, thence, equivalent to a quantifier-free formula.

Fix k^0 momentarily. The theory $\mathsf{E\text{-}G_3A}^\omega$ has enough induction to show that there is the largest element $m_k \leq_0 gk$ such that $A(k, m_k)$. Therefore:

$$\begin{cases} \forall k^0 \exists m \leq_0 gk\, (\exists s \in \{0,1,\ldots,hk\}^{<ek}\, (\Phi(k,\hat{s}) = m) \wedge \\ \qquad \forall m' \leq_0 gk (A(k,m') \to m' \leq m)). \end{cases}$$

Using the minimization functional μ_b (to choose the numerically least s above), it is clear that there is ψ_* of type 1 such that

$$\forall k^0 (\psi_* k \in \{0,1,\ldots,hk\}^{<ek} \;\wedge\; \forall m' \leq_0 gk (A(k,m') \to m' \leq_0 \Phi(k, \widehat{\psi_* k})))$$

It is clear that $\psi^{0\to 1} := \lambda k.\widehat{\psi_* k}$ satisfies $\forall k^0 \forall f \leq_1 hk\, (\Phi(k,f) \leq_0 \Phi(k, \psi k))$. □

Let us refine Theorem 7 again, this time in a direction that is applicable to the theory $\mathsf{G_2A}^\omega$ as well. In this refinement, the matrix A of the theorem is allowed to be of a more general form:

Definition 10 *Let $\mathcal{L}$ be the language of arithmetic. We say that a formula is* 0-bounded *if all its quantifiers are of the form $\forall n^0 (n \leq_0 t \to \ldots)$ or $\exists n^0 (n \leq_0 t \wedge \ldots)$, where t is a term of type 0, with parameters of any type, in which the variable n does not occur. A formula is* 1-bounded *if all its quantifiers are like the quantifiers occurring in the 0-bounded formulas or, else, are of the form $\forall f^1 (f \leq_1 t \to \ldots)$ or $\exists f^1 (f \leq_1 t \wedge \ldots)$, where t is a term of type one, with parameters of any type, in which the variable f does not occur.*

In order to extend Theorem 7 to 1-bounded matrices, we perform a 'sandwich argument'.

Definition 11 *We associate to each 1-bounded formula A of the language $\mathcal{L}^\omega_\leq$*

formulas A_l, A_c and A_r. We define recursively A_l, A_c and A_r according to the following clauses:

(1) If A is atomic, A_l, A_c and A_r are all the same and equal to A.
(2) $(A \Box B)_\natural$ is $A_\natural \Box B_\natural$, where $\Box \in \{\wedge, \vee\}$ and $\natural \in \{l, c, r\}$.
(3)a) $(A \to B)_l$ is $A_r \to B_l$;
b) $(A \to B)_c$ is $A_c \to B_c$;
c) $(A \to B)_r$ is $A_l \to B_r$.
(4) $(Qz \leq_0 tA(z))_\natural$ is $Qz \trianglelefteq_0 t[A(z)]_\natural$, where $Q \in \{\forall, \exists\}$ and $\natural \in \{l, c, r\}$.
(5)a) $(\forall f \leq_1 tA(f))_l$ is $\forall f \trianglelefteq_1 t^M[A(\min_1(t, f))]_l$;
b) $(\forall f \leq_1 tA(f))_c$ is $\forall f[A(\min_1(t, \min_1(t^M, f)))]_c$;
c) $(\forall f \leq_1 tA(f))_r$ is $\forall f \trianglelefteq_1 t^M[A(\min_1(t, \min_1(t^M, f)))]_r$.
(6)a) $(\exists f \leq_1 tA(f))_l$ is $\exists f \trianglelefteq_1 t^M[A(\min_1(t, \min_1(t^M, f)))]_l$;
b) $(\exists f \leq_1 tA(f))_c$ is $\exists f[A(\min_1(t, \min_1(t^M, f)))]_c$;
c) $(\exists f \leq_1 tA(f))_r$ is $\exists f \trianglelefteq_1 t^M[A(\min_1(t, f))]_r$.

Observe that A_c is still a formula of $\mathcal{L}^\omega_\leq$, whereas A_l and A_r are formulas of the extended language $\mathcal{L}^\omega_\trianglelefteq$. We now remind the reader of Definition 8, where to each formula A of $\mathcal{L}^\omega_\trianglelefteq$ we associate the formula A^* of $\mathcal{L}^\omega_\leq$ by replacing each $\trianglelefteq_\sigma$ by the corresponding $\leq^*_\sigma$.

Lemma 11 *Let A be a 1-bounded formula of $\mathcal{L}$.*

(i) $\mathsf{E\text{-}G_2A}^\omega_i$ proves that A, A_c, $(A_l)^*$ and $(A_r)^*$ are all equivalent.
(ii) $\mathsf{G_2A}^\omega_{i,\trianglelefteq}$ proves $A_l \to A_c$ and $A_c \to A_r$.

Proof. It is clear that, in the presence of full extensionality, A and A_c are equivalent. The proof of the equivalences $A_c \leftrightarrow (A_l)^*$ and $A_c \leftrightarrow (A_r)^*$ require simultaneous induction. Let us look in some detail at equivalence $A_c \leftrightarrow (A_l)^*$ for the clause $\forall f \leq_1 tA(f)$. In this case,

(1) $(\forall f \leq_1 tA(f))_c$ is $\forall f[A(\min_1(t, \min_1(t^M, f)))]_c$;
(2) $[(\forall f \leq_1 tA(f))_l]^*$ is $\forall f \leq^*_1 t^M[A(\min_1(t, f))_l]^*$.

Since $\min_1(t^M, f) \leq^*_1 t^M$ holds, it is clear (by induction hypothesis) that $(A_l)^* \to A_c$. The reverse implication uses extensionality (and the induction hypothesis).

Claim (ii) is also proved by simultaneous induction. Let us look at the implication $A_l \to A_c$ for the clauses $\forall f \leq_1 tA(f)$ and $\exists f \leq_1 tA(f)$. We first consider the universal case. Suppose A_l, i.e., $\forall f \trianglelefteq_1 t^M[A(\min_1(t, f))]_l$. Take any f^1. By (ii) and (iii) of Lemma 8, $\min_1(t^M, f) \trianglelefteq_1 t^M$. Therefore, $[A(\min_1(t, \min_1(t^M, f)))]_l$. Using the induction hypothesis and the arbitrariness of f, we get A_c. The existential case is straightforward. $\Box$

We may now show that Theorem 7 holds good even when the formula proven is

$\forall x^{0/1}\exists y^\rho A_1(x,y)$ with A_1 a 1-bounded formula, *provided* that the verification of the conclusion takes place in $\mathsf{E\text{-}CT}^\omega$, where full extensionality is available (as opposed to plain CT^ω):

Theorem 8 *Suppose that* CT^ω *is* PA^ω, PRA^ω *or* $\mathsf{G}_n\mathsf{A}^\omega$ *(*$n \geq 2$*). If*

$$\mathsf{E\text{-}CT}^\omega + \mathsf{bAC}_0^{0,1} + \mathsf{AC}_0^{1,0} + \mathsf{UBF}^{\tau\to 1} \vdash \forall x^{0/1}\exists y^\rho A_1(x,y),$$

where ρ *is an arbitrary type and* A_1 *is a 1-bounded formula (its free variables as displayed), then there is a closed monotone term* $q^{0/1\to\rho}$ *such that*

$$\mathsf{E\text{-}CT}^\omega \vdash \forall x^{0/1}\exists y \leq^*_\rho qx\, A_1(x,y).$$

Proof. Suppose that

$$\mathsf{E\text{-}CT}^\omega + \mathsf{bAC}_0^{0,1} + \mathsf{AC}_0^{1,0} + \mathsf{UBF}^{\tau\to 1} \vdash \forall x^{0/1}\exists y^\rho A_1(x,y).$$

By part (i) of the previous proposition, $A_1(x,y)$ above can be replaced by $[A_1(x,y)]_c$. Following the proof of Theorem 7, we easily get to the point where

$$\mathsf{CT}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash \forall x^{0/1}\exists y^\rho [A_1(x,y)]_c$$

(notice that the formula $[A_1(x,y)]_c$ has only quantifiers of type 0 or 1 and, hence, remains unchanged in the process of elimination of extensionality). At this point we invoke part (ii) of the previous proposition to get

$$\mathsf{CT}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash \forall x^{0/1}\exists y^\rho [A_1(x,y)]_r.$$

By Corollary 1, $\mathsf{T}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash (\forall x^{0/1}\exists y^\rho [A_1(x,y)]_r)'$. Using the fact that $[[A_1(x,y)]_r]^\dagger$ is a bounded formula, an application of $\mathsf{bMP}^\omega_{\mathrm{bd}}[\trianglelefteq]$ shows that the theory $\mathsf{T}^\omega_\trianglelefteq + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq]$ proves $\forall x \exists z \neg\neg\exists y \trianglelefteq z [[A_1(x,y)]_r]^\dagger$. By the Soundness Theorem (Arithmetical Version), there is a closed monotone term $q^{0/1\to\rho}$ such that

$$\mathsf{T}^\omega \vdash \forall a^{0/1} \forall x \leq^*_\tau a \neg\neg\exists y \leq^*_\rho qa ([[A_1(x,y)]_r]^\dagger)^*.$$

If the type of x is 1 (the case 0 is easier), we infer

$$\mathsf{T}^\omega \vdash \forall x^1 \neg\neg\exists y \leq_\rho tx ([[A_1(x,y)]_r]^\dagger)^*,$$

where t is as in the proof of Theorem 7. It is clear that this entails

$$\mathsf{CT}^\omega \vdash \forall x^1 \exists y \leq^*_\rho tx\, ([A_1(x,y)]_r)^*.$$

Now, by (i) of the previous proposition we conclude that

$$\mathsf{E\text{-}CT}^\omega \vdash \forall x^1 \exists y \leq^*_\rho tx\, A_1(x,y). \qquad \Box$$

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