Categoricity and Mathematical Knowledge

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In memory of Solomon Feferman (1928-2016)[*]

Abstract

We argue that the basic notions of mathematics (number, set, etcetera) can only be properly formulated in an informal way. Mathematical notions transcend formalizations and their study involves the consideration of other mathematical notions. We explain the fundamental role of categoricity theorems in making these studies possible. We arrive at the conclusion that the notion of mathematical consequence is ultimately defeasible, and that we must rely on degrees of evidence.

Keywords: categoricity theorems, platonism, epistemology of mathematics, formal and informal reasoning, infallibility.

1. Introduction

The classical structure of the natural numbers with 0 and the successor function has a second-order categorical axiomatization. The axioms say that no two numbers have the same successor, that 0 is not a successor and that for every set, (i) if 0 belongs to the set and (ii) the successor of each element of the set belongs to the set, then all numbers belong to the set. These axioms are due to Richard Dedekind in his famous essay *Was sind und was sollen die Zahlen*.[1] Dedekind proved that any two structures that satisfy his axioms are isomorphic. One is tempted by this categoricity result to conclude that Dedekind's axiomatization is complete (and, as a result, that it proves all true number-theoretic statements). The reasoning is as follows. Suppose that P is a number-theoretic statement and that neither P nor its negation are consequences of Dedekind's axioms. Since P is not a consequence of the axioms, there is a model of the axioms in which P is false. Similarly, there is also a model of the axioms in which P is true. This is impossible because these two models couldn't be isomorphic. Is this argument correct? What is its significance? The first thing to notice is that this argument can be turned into an

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[*] In his last email message to me, Sol wrote about (a first version of) this paper of mine saying that "(he) had (finally) a chance to read it more seriously. It is beautifully written with a lot of stimulating ideas, but I will still need to read it once more to see if I have any useful comments." The kind words were characteristic of him. Unfortunately, his passing some seven months after this exchange cut the dialogue short. The present paper is dedicated to his memory.

[1] Richard Dedekind, *Was sind und was sollen die Zahlen?* (Braunschweig: Vieweg, 1888). Based on Dedekind's notion of simply infinite system, Giuseppe Peano gave in his *Arithmetices principia, nova methodo exposita* (Torino: Bocca, 1889) a formulation of these axioms using a symbolic system close to the one used today in formal languages. The well known formal theory of first-order arithmetic PA (Peano Arithmetic) is named after Peano.

impeccable argument within set theory. In set theory one can define with mathematical rigor the notion of (full) second-order structure for a formal second-order language, as well as the accompanying notions of truth (of a sentence with respect to a given structure) and of semantical consequence. The parenthetic 'full' means that the range of the second-order variables is constituted by the full power set of the first-order domain (and not, as in Henkin models, by subcollections of the power set). With these mathematical notions in place, the above proof can be given with mathematical precision.

The structure of the real numbers also has a second-order categorical axiomatization. Even set theory enjoys a result of this kind. In an important paper of 1930, Ernst Zermelo showed that the second-order version of the formal theory ZFC is quasi-categorical in the sense that given any two models of second-order ZFC, one of them is isomorphic to an initial segment of the cumulative hierarchy of the other.[2] It goes without saying that all these results can be proved with mathematical rigor. In fact, they can be formalized in first-order ZFC.[3]

The notion of consequence at play in these rigorous mathematical results is semantical. By definition, a sentence is a consequence of certain (second-order) axioms if it is true in all models of the axioms. It is this notion that justifies a crucial step in the above "completeness result" when it is said that if P is not a consequence of the axioms then there is a model of the axioms in which P is false. This step is just a rephrasing of the notion of semantical consequence. Now, it has been known since the nineteen thirties that this semantical notion cannot be identified with a notion of formal derivability (technically, it is not a Σ^0_1-notion). The "completeness result" above does not accomplish much from an epistemological point of view because it operates with an inappropriate notion of consequence, not with a derivability notion as it is used by ordinary mathematicians to prove theorems and gain mathematical knowledge. Georg Kreisel has remarked that Hilbert's program would never have been pursued if one was not convinced that second-order consequence was equivalent, at least in suitable contexts, to formal derivability.[4] This is yet another facet of the failure of Hilbert's program and one that has cast a dark shadow on second-order logic. As a logical tool it has been said that, at best, it is an equivocation (second-order logic is set theory in sheep's clothing, as Willard Quine famously quipped) and, at worst, a cheap trick.[5]

The fact that the set-theoretic notion of second-order consequence is not derivability-based disqualifies it as a good epistemological notion (in contrast, first-order consequence is a derivability notion by Gödel's completeness theorem). Our main thesis is that, appropriately

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[2] Ernst Zermelo, 'Über Grenzzahlen und Mengenbereiche. Neue Untersuchungen über die Grundlagen der Mengenlehre', *Fundamenta Mathematicae* 16 (1930): 26-47.

[3] A good reference for ZFC is the book *Introduction to Set Theory*, by Karel Hrbacek and Thomas Jech (New York: Marcel-Dekker, 1999). For the results of mathematical logic mentioned in this paper (e.g., Gödel's completeness theorem, Gödel's second incompleteness theorrem, the Löwenheim-Skolem theorems and Tarski's truth definition), I recommend *A Mathematical Introduction to Logic*, by Herbert Enderton (San Diego: Academic Press, 2001).

[4] Georg Kreisel, 'Informal rigour and completeness proofs', in *Problems in the Philosophy of Mathematics*, edited by I. Lakatos (Amsterdam: North-Holland, 1967), 146.

[5] Quine's quip appears in his *Philosophy of Logic* (Cambridge, Massachusetts: Harvard University Press, 1986), 66. The "cheap trick" remark is taken from Daniel Isaacson's 'The reality of mathematics and the case of set theory', in *Truth, Reference, and Realism*, edited by Z. Novak and A. Simonyi (Budapest: Central European University Press, 2011), 46. Isaacson does not endorse the remark.

understood, second-order categorical axiomatizations are nevertheless highly significant for the epistemology of mathematics.

It is important to separate the proof of Dedekind's categoricity theorem from the question of the axiomatization of the first-order structure of the natural numbers.[6] In the discussion above these two issue were treated in tandem relying, as it were, on the semantical notions of truth and (full) second-order consequence. However, Dedekind's categoricity theorem need not be treated in this full-blown semantical way. It can be stated as follows: Given $(N_1,0_1,S_1)$ and $(N_2,0_2,S_2)$ two first-order natural number structures, the map h: $N_1 \rightarrow N_2$ defined recursively by $h(0_1) = 0_2$ and $h(S_1(x)) = S_2(h(x))$ is an isomorphism. For the strict purpose of proving the categoricity theorem, what really matters is that we accept the existence of the function h and enough induction to be able to show that h, as abiding by the above recursive clauses, is an isomorphism. Essentially, the proof of Dedekind's categoricity theorem proceeds in a first-order fashion *given* the two natural number structures.[7]

2. The epistemological chasm

This paper is neutral with regard to object platonism, even in the robust version according to which there is a world of independent abstract mathematical objects and that, somehow, our mathematical talk is about the objects of this world. This view is sometimes dubbed naive realism and it is mostly seen with condescension (in spite of some illustrious defenders, like Kurt Gödel).[8] Nevertheless, it does solve a cluster of (related) problems. One is the objectivity of mathematics. Under the robust platonistic view, the world of abstract mathematical objects is independent of human interests or conceptions and it is this world that mathematicians try to describe. The other is that mathematical assertions have a determinate truth value. For instance, the continuum hypothesis (of set theory) has a determinate truth value. At the present moment we do not know which one and it cannot be ruled out that we may never know. It also solves a perceived problem regarding the set-theoretic notion of second-order consequence. Even though it is a poor notion from the epistemological point of view, the criticism runs, it also does not do what it claims that it does, viz. to characterize a mathematical structures up to isomorphism. In effect, when reasoning set-theoretically within a formal theory like ZFC, well known model-theoretic results show that there are models of these theories with different

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[6] As it is well-known, an infinite first-order structure can never be characterized up to isomorphism by a first-order axiomatization. This is a consequence of the upward Löwenheim-Skolem theorem. Therefore, a categoric axiomatization of the first-order structure of the natural numbers must use resources beyond first-order logic.

[7] In 'The uniqueness of the natural numbers', *Iyyun* 39 (1990): 13-44, Charles Parsons says that Dedekind's theorem and its proof are "essentially *first-order*" (italics as in the original). In Parsons's words, the function h is introduced "in keeping with Skolem's recursive arithmetic", and the proof that it is an isomorphism only needs first-order instances of induction (using, of course, the data N_1, 0_1, S_1, N_2, 0_2 and S_2). Alternatively, Dedekind's categoricity theorem can be proved within relatively weak *deductive* system for second-order logic (note the underlining - it makes the system really of first-order character). For further information and discussion on these matters, see the recent survey 'Structure and categoricity: determinacy of reference and truth value in the philosophy of mathematics', by Tim Button and Sean Walsh, *Philosophia Mathematica* 24, no. 3 (2006): 283-307.

[8] The *locus classicus* of Gödel's defense of platonism is 'What is Cantor's continuum problem?', *American Mathematical Monthly* 54 (1947): 515-525. Some years later (1963/1964), Gödel wrote a revised and expanded version for *Philosophy of Mathematics (Selected Readings)*, edited by P. Benacerraf and H. Putnam (Cambridge: Cambridge University Press, 1983), 470-485.

power set operations. For instance, the reals in one first-order model of ZFC need not be isomorphic with the reals in another model, even though the categoricity theorem for the real numbers holds within each model. Some set-theoretic notions are relative to models of first-order set theory[9] and, for this reason, it is claimed that the categoricity theorems do not say as much as they seem to be saying. For the robust platonist, this argument is of no relevance because when reasoning set-theoretically (in the sense of the ordinary mathematician) one is not reasoning about this or that model of first-order set theory but about an abstract world of independent mathematical objects (sets).

The mathematical objects of this abstract world are not located in spacetime. They do not causally interact with us. This can be a source of perplexity. How can we know anything about the mathematical world? How is reference to the objects of the mathematical world attained? Gödel himself defended that we possess a special faculty akin to sense perception that allows us to "perceive" the objects of set theory.[10] In a much quoted essay on these issues, Paul Benacerraf has a speculative historical note in which he sees Plato's concept of *anamnesis* as bridging the chasm between an abstract world of mathematics and the human knower.[11] In past lifes we have been in touch with such an abstract reality and, with proper training, tutoring and effort, we can remember this world. Even though I consider these solutions unreasonable, they are - on their own terms - answers to the epistemological chasm. I may even suggest a Leibnizian solution of my own. There are no interactions between the abstract world of mathematical objects and the sublunar world of human affairs. However, we do possess some mathematical conceptions like those of number, set, etc. and, due to a pre-established harmony, these conceptions happen to describe the abstract world of mathematical objects. It is not that we interact with an independent reality (our notions run in strict parallelism with the world of mathematical objects without any mutual interference). It is that, nevertheless, an independent reality happens to be described by our notions. In Leibnizian terms, the abstract world of mathematical objects and our sublunar world are "windowless", but our mathematical conceptions "mirror" the abstract world by a kind of pre-established harmony.[12]

Our Leibnizian solution is also unreasonable. We introduced it as a philosophical *Gedankenexperiment* with the purpose of illustrating that an absolute conception of number (i.e., one that transcends any formalism) makes sense without presupposing - not even in the thought experiment - interactions with an independent reality. The disengagement of mathematical knowledge from interactions with an independent reality of abstract mathematical objects coheres with the *modus operandi* of the pure mathematician. When doing mathematics, pure mathematicians lay down proofs which, in the end, report to some first principles, be they implicit or explicit. Mathematical knowledge is not obtained by interacting with the objects of some abstract world but rather by relating concepts and notions through

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[9] By the downward Löwenheim-Skolem theorem, every transitive model of ZFC has a countable transitive submodel M. Since M is a model of ZFC, by Cantor's theorem the set R of its real numbers is not countable-in-M. But, of course, every set of M is countable and, in particular, so is R. Therefore, countability is not the same as countability-in-M (countability is a relative set-theoretic notion).

[10] Gödel, 'What is Cantor's continuum problem', revised and expanded version, 483-484.

[11] Paul Benacerraf, 'Mathematical truth', in *Philosophy of Mathematics. Selected Readings*, 416. The article was first published in 1973.

[12] See specially the paragraphs 78 et seq. of Leibniz's *Principes de la Philosophie ou Monadologie* (1714).

mathematical proof. There is nevertheless a kind of methodological platonism underlying the work of ordinary mathematicians, according to which mathematical notions transcend formalizations (e.g., first-order axiomatizations). When speaking of the natural numbers, ordinary mathematicians are not speaking of numbers in this or that nonstandard model of first-order Peano Arithmetic (PA). Confronted with two non-isomorphic models of Peano Arithmetic, a mathematician will immediately investigate the failure of the induction principle in one (or both) of them. Rather than putting into question his notion of natural number, he will say *a priori* that there must be something wrong with one (or both) of the models. Similarly, when ordinary mathematicians consider the power set of the natural numbers, they are not considering it in this or that model of first-order ZFC. Their consideration of the real numbers transcends first-order models. They have an absolute conception of the power set operation.

3. Informal reasoning

The work of ordinary mathematicians is developed within a completely extensional framework, where the objects of mathematics are cut off from all links with the reflecting subject. Mathematicians do not construct sets, they prove the existence of sets.[13] As a result, a certain indefiniteness normally associated with notions like property or concept (we use these words more or less interchangeably) is absent from ordinary mathematics, and this is certainly part of the success of the mathematical enterprise. However, there are a few apparent exceptions to this view. Even ordinary mathematicians find quite natural to state the induction principle by saying that if a property holds of 0, and if it holds of S(n) whenever it holds of n, then it holds of every natural number. Induction can be stated with sets instead and this was, in fact, how we presented Dedekind's axiomatization of the natural numbers. That notwithstanding, mathematicians do not find it unnatural to use the intensional terminology in formulating the induction principle. The following example is more incisive. Mathematicians also say that if a "definite property" holds of some elements of a given set X, then there is a set whose elements are exactly those elements of X for which the given property holds. This is Zermelo's *Aussonderungsaxiom*. In this case it is not possible to replace the mention of properties by the mention of sets.

Properties or concepts are not the subject matter of mathematics. Sets, numbers, groups, orders, topological spaces, differential manifolds, etcetera, are. Of course, mathematicians apply concepts like everybody else. In their work towards gaining mathematical knowledge, mathematicians apply concepts to mathematical objects and lay down proofs of mathematical facts. This kind of work must be presented *in concreto* if it has to have any epistemological significance. The induction principle '$P(0) \wedge \forall n(P(n) \rightarrow P(S(n))) \rightarrow \forall nP(n)$' is a schema where the letter 'P' stands in place of an unspecified numerical predicate. The important thing to say concerning this schema is that it is not restricted to a given formal language (as it is the case with studies in mathematical logic). It is an informal schema. The induction schema contributes to the articulation of the notion of natural number in the guise of an *inference ticket*: whenever a predicate is well-defined for the natural numbers, then it can be used in an inductive

[13] I am adapting some words of Paul Bernays taken from his well known essay on platonism, 'Sur le platonisme dans les mathématiques', *L'Enseignment Mathématique* 34 (1935): 52-69.

argument. The proper use of a natural number predicate in an inductive argument relies, as we will illustrate, on the endorsement of certain mathematical notions. Such endorsements constitute the grounds which, ultimately, warrant the use of numerical predicates.

The mathematization of logic is but an approximation of the view of logic as a philosophical discipline concerned with concepts, propositions and demonstrations. In their stead, mathematical logic operates with formulas, sentences and proofs within the setting of formal languages (and their interpretations). It is a central claim of this paper that the basic notions of mathematics, like those of number and set, can only be properly formulated in an informal manner. As Kreisel observes in *Informal rigour and completeness proofs*, the evidence of a first-order schema derives from a corresponding second-order (informal) principle. In the case of the first-order theory PA, with zero, successor and also the primitive operations of addition and multiplication, the induction schema is restricted to arithmetical predicates (in the technical sense of being predicates defined by formulas of the language of first-order Peano Arithmetic) and our grounds for accepting this schema is that predicates given by arithmetical formulas are well-defined for the natural numbers. The evidence of the schema of induction of PA stems from the informal notion of natural number together with the conviction that first-order formulas of the language of PA express definite properties of the natural numbers.

Evidence comes in degrees. Investigations in the foundations of mathematics have classified various foundational positions according to the evidence supporting them.[14] An early overview by Hao Wang in *Eighty years of foundational studies* discerns five positions of decreasing degree of evidence, from anthropologism to set-theoretic platonism.[15] When applying a schema which contributes towards the articulation of a basic notion of mathematics (be it the induction schema in number theory, the *Aussonderungsaxiom* in set theory, or other) we must first convince ourselves that the predicate used is well-defined and this conviction is based on the endorsement of certain mathematical notions. For instance, in number theory a further step (down) in the ladder of evidence from the first-order theory PA leads to the acceptance of the truth predicate for its sentences as construing a definite property of natural numbers (via a Gödel numbering of formulas). A simple inductive argument using the truth predicate proves the soundness of PA and, therefore, its consistency. By Gödel's second incompleteness theorem, this is a truth (stateable, but) unprovable in PA. Another way of formulating well-defined predicates and, therefore, of extending the use of the induction principle, is by considering new ontologies. Beyond the natural numbers, we may also consider arbitrary subsets of the natural numbers (yet another step down in the ladder of evidence). This extension of the domain and a corresponding extension of the language with the membership relation and a new sort for sets of natural numbers enables the formulation of new predicates

[14] The concept of evidence is relational. The aim of reductive proof theory is to give a precise mapping of reductions of certain formal theories to others. These reductions intend to capture mathematically the relational notion of evidence. For an overview see Feferman's 'What rests on what? The proof-theoretic analysis of mathematics', in *In the Light of Logic* (Oxford: Oxford University Press, 1998), 187-208. The article was first published in 1993.

[15] The referred article of Wang was published in *dialectica* 12 (1958): 466-497. Section 7.1 of Wang's book *A Logical Journey. From Gödel to Philosophy* (Cambridge, Massachusetts: MIT Press, 1996) has a very interesting discussion on the origin of mathematical notions. According to Wang, Gödel classified the acceptance of infinity (in the form of natural numbers) as the "big jump."

and, hence, permits one to work with new properties of the natural numbers. As a case in point, one can define the truth predicate for sentences of the language of PA if arbitrary sets of natural numbers are available (Tarski's truth definition), thereby recovering the previous step in the ladder of evidence.

The first extension of PA discussed above is ideological, in the Quinean sense of adding a new predicate to the language (no new objects are added).[16] The new predicate - the truth predicate - holds exactly of the Gödel numbers of true sentences of the (formal) language of PA. From a mathematical (and epistemological) point of view new axioms are accepted, expressing Tarski's truth conditions. The proof of the consistency of PA uses PA itself, the new axioms and the induction principle applied to formulas of the extended first-order language (i.e., in which the new truth predicate occurs). In what sense, if any, can we view this argument as an outcome of the informal Dedekind axioms? Note that in order to cash the inference ticket of informal induction with the truth predicate, we had to accept extra axioms. The second example is an ontological extension in which one considers, alongside natural numbers, also arbitrary sets of natural numbers. As mentioned, it is possible to exhibit a formula in the language of two-sorted arithmetic which is true of a number just in case that number is the Gödel number of a true sentence of the language of PA. The same inductive argument as before permits one to prove the consistency of PA. The second example seems to fare better. However, the new proof relies on the fact that the exhibited formula defines the truth predicate (with its usual properties). As a proof, it is a proof with a gap. The gap is easily filled if we accept axioms that articulate the notion of arbitrary subset of the natural numbers. Full comprehension for formulas of the two-sorted language is enough.[17] Once again, one does not avoid the predicament brought by the need to accept extra axioms in order to be able to cash the inference ticket made available by the informal schema of induction.

In spite of the problems of the above two examples, the arguments for the new arithmetical truth (the consistency of PA) are quite satisfying from the informal Dedekind-axiomatic point of view because their main thrust is based on an induction on the length of formal proofs applied to a formula involving the truth predicate. One sees exactly how the inference ticket of informal induction is cashed and, moreover, the induction itself constitutes the main line of the argument. However, the ties to the informal induction schema need not be explicit. Consider, for example, a number-theoretic statement provable in ZFC. Can we reasonably say, within this level of generality, that the statement is an outcome of the notion of natural number as given by the informal Dedekind axioms?

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[16] Properties of natural numbers are also obtained by means of definitions done with the help of particular elements of the ontologies available. In technical terms, parameters are allowed in the formulas that define properties and the induction principle also applies to formulas with parameters. The presence of parameters dramatizes the entanglement between ideology and ontology. My use here of the word 'ideology' comes from Quine's paper 'Ontology and ideology', *Philosophical Studies* 2 (1951): 11-15, but differs from Quine's usage in that I allow parameters in the construction of predicates. Moreover, for Quine, ontology and ideology are kept apart neatly. Not much attention has been paid to the entanglement between ontology and ideology (which, I believe, is very important in the philosophy of mathematics), as opposed to the entanglement between the analytic and the synthetic.

[17] It is enough, I mean, to prove that the presented formula satisfies Tarski's truth conditions. Full comprehension is, actually, an overkill. Technically, it is sufficient to have Δ^1_1-comprehension and Σ^1_1-induction.

4. Arithmetical knowledge

We tried to illustrate the consequences of regarding second-order principles as informal schemata (inference tickets). We argued that in order to cash these inference tickets we may need to help ourselves to mathematical notions other than the notions that are our immediate object of concern. New forms of number-theoretic induction are articulated by considering the truth predicate for arithmetical sentences or the quasi-combinatorial notion of arbitrary subset of the natural numbers.[18] By bringing along these new notions (and associated axioms) aren't we adding something to the notion of natural number as given by the informal Dedekind axioms? Couldn't then the notion be developed in different incompatible ways? Let me use the simile due to Gödel of "seeing a concept more clearly."[19] In order to see more clearly the concept of natural number it is necessary to use some auxiliary tools (like an astronomer who uses a telescope), namely other mathematical notions. Couldn't it be the case that, through the application of this process, a mathematician realizes that he is mixing two or more notions, that the notion of natural number bifurcates (like an astronomer who realizes that the celestial body that he is studying is actually a double star)? The significance of Dedekind's categoricity theorem is that this cannot happen. The work of the mathematician is, so to speak, confluent with respect to the notion of natural number. Dedekind's categoricity theorem ensures that the notion of natural number has a determinate direction (to use the phenomenological terminology of "directedness"). Let us say that the notion is *univocal*.[20]

The univocality of the notion of natural number is important for the epistemology of arithmetic because it justifies the use of other mathematical notions as a means of gaining arithmetical knowledge. Consider the notion of natural number N, 0, S as given by the informal

[18] According to Wang in *A Logical Journey. From Gödel to Philosophy*, Gödel commented on the situation just described in the main text (in which a given notion needs the help of other notions in order to be fully operational). He classifies the situation as "very strange" unless one is a platonist. In effect, on p. 222 of Wang's book, Gödel says that "so (...) in order to find out what properties *we* have given to certain objects of our imagination, [we] must first create other objects - a very strange situation indeed!"

[19] The simile is described in pp. 85-86 of Wang's book *From Mathematics to Philosophy* (London: Routledge & Kegan Paul, 1974).

[20] The same reasoning applies to other univocal notions. In the aftermath of Cohen's proof of the independence of the continuum hypothesis, Kreisel drew attention to the differences between first and second-order set theory and wrote that "denying the (alleged) *bifurcation* or *multifurcation of our notion of set of the cumulative hierarchy* is nothing else but asserting the properties of our intuitive conception of the cumulative type-structure" (italics as in the original). Op. cit. For instance, the ($\omega+\omega$)-level of the cumulative set-theoretical structure has a second-order categorical axiomatization (this follows from Gabriel Uzquiano's results in §5 of 'Models of second-order Zermelo set theory', *Bulletin of Symbolic Logic* 5, no. 3 (1999): 289-302). Since the continuum hypothesis can be formulated within this level by a sentence of the first-order language of set theory, this categoricity result entails that there cannot be a scenario of *actual* bifurcation in which the continuum hypothesis has different answers. Note that Gödel's thesis of robust platonism entails the stronger proposition that the continuum hypothesis has a determinate truth value. Combined with his rationalistic optimism, Gödel also thinks that *we can know* which one it is. On the other hand, an *optimistic* methodological platonist would say that *we can answer* (in the positive or in the negative) the question of the continuum hypothesis. According to our current knowledge, it is however compatible with methodological platonism that there is just no (positive or negative) answer to that question. Feferman's view that the continuum hypothesis is *inherently vague* (cf. his 'Why the programs for new axioms need to be questioned', *Bulletin Symbolic Logic* 6, no. 4 (2000): 401-413) can be seen as a defense that there is simply no (positive or negative) answer to the continuum hypothesis.

Dedekind axioms. Let P be a number-theoretical statement.[21] Take P^{set} its translation into the formal language of set theory (the quantifiers that occur in P^{set} are relativized to ω, the set of von Neumann finite ordinals, 0 is interpreted by the empty set $\varnothing$, and the successor of an ordinal α is given by $\alpha \cup \{\alpha\}$) and suppose that P^{set} is provable in (first-order) ZFC. *A fortiori*, P^{set} is provable in second-order ZFC, i.e., the version of ZFC in which the schemata of separation and replacement are informal. Now, the mathematical concept of von Neumann finite ordinal satisfies the informal Dedekind axioms (this follows from the informal schema of separation), and so we can invoke the isomorphism h: $N \rightarrow \omega$ (determined by the recursive clauses $h(0) = \varnothing$ and $h(S(n)) = h(n) \cup \{h(n)\}$) in order to argue (by induction on the complexity of the first-order formula P) that the biconditional $P \leftrightarrow P^{set}$ holds. Since we are endorsing the notion of set as given by second-order ZFC, we may conclude that P holds. In the sense just illustrated, Dedekind's categoricity theorem provides a passageway from number-theoretic statements proved in set theory to the notion of number as conceived according to Dedekind's informal axiomatization.

The previous discussions suggest an informal idea of mathematical consequence with respect to univocal notions. According to this idea, we may cash the inference tickets made available by the informal schemata which characterize a given univocal notion by helping ourselves to certain other mathematical notions.[22] The passageway provided by Dedekind's categoricity theorem justifies this form of reasoning. It is a form of mathematical reasoning that can, however, go astray. Mathematical consequence - as described here - is defeasible because the mathematical notions which are brought to the fore may harbor difficulties or, even, contradiction. Mathematics is not infallible and we have to get by with degrees of evidence.[23] When facing difficulties, or even trouble (contradiction), we have to analyze and clarify our notions and, in extreme cases, abandon them altogether. This happened historically with the hesitations concerning the axiom of choice and the bankruptcy of the logicist notion of set. For all of their visibility, these episodes are nevertheless very rare. Mathematical logic has taught us that ordinary mathematical work can (in principle) be developed in formal theories, where absolute rigor is attained. However, absolute rigor is not the same thing as absolute certainty. It is nevertheless worth emphasizing - as Solomon Feferman does - that mathematical logic has shown that *a little bit goes a long way* and that the bulk of ordinary mathematics can be formalized in theories enjoying a very high degree of evidence.[24]

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[21] To make the discussion interesting, we suppose that P is a sentence of the first-order language of PA (note that the first-order theory of the number-theoretic truths formulated in the language with only zero and successor is decidable). We nevertheless ignore the technical details arising from going beyond the primitive language with only zero and the successor operation.

[22] This notion of mathematical consequence is a form of absolute demonstrability as discussed by Gödel in pp. 268-269 of Wang's *A Logical Journey. From Gödel to Philosophy* op. cit. In quotation 8.4.21 of that book, Gödel suggests a kind of normal form for absolute demonstrability: "The idea of [absolute] proof may be nonconstructively equivalent to the concept of *set*: axioms of infinity and absolute proofs are more or less the same thing." I do not believe that there is compelling evidence for this bold thesis.

[23] According to entry 9.2.35 of Wang's *A Logical Journey. From Gödel to Philosophy*, Gödel defended that "We have no absolute knowledge of anything. There are degrees of evidence." Page 302 of the referred book has more material on Gödel's view on the fallibility of mathematical knowledge.

[24] S. Feferman, 'Why a little bit goes a long way: logical foundations of scientifically applicable mathematics', in *In the Light of Logic*, 284-298. The article was first published in 1992.

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