On some semi-constructive theories related to Kripke-Platek set theory

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Abstract

We consider some very robust semi-constructive theories related to Kripke-Platek set theory, with and without the powerset operation. These theories include the law of excluded middle for bounded formulas, a form of Markov's principle, the unrestricted collection scheme and, also, the classical contrapositive of the bounded collection scheme. We analyse these theories using forms of a functional interpretation which work in tandem with the constructible hierarchy (or the cumulative hierarchy, if the powerset operation is present). The main feature of these functional interpretations is to treat bounded quantifications as "computationally empty." Our analysis is extended to a second-order setting enjoying some forms of class comprehension, including strict-Π^1_1 reflection. The key idea of the extended analysis is to treat second-order (class) quantifiers as bounded quantifiers and strict-Π^1_1 reflection as a form of collection. We will be able to extract some effective bounds from proofs in these systems in terms of the constructive tree ordinals up to the Bachmann-Howard ordinal.

Key words and phrases. Intuitionistic Kripke-Platek set theory, functional interpretations, Σ-ordinal, strict-Π^1_1 reflection, power Kripke-Platek set theory

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1 Introduction

In recent writings, Solomon Feferman was urging the study of semi-construtive theories. His papers "On the strength of some semi-constructive theories" [8] and "Logic,

[*]I would like to thank the editors of this volume, Gerhard Jäger and Wilfried Sieg, for inviting me to contribute to this volume dedicated to Solomon Feferman and his work. It is an honor and a joy to participate. Over several decades, Sol was a role model for a certain approach to mathematical logic and the foundations of mathematics, combining high technical proficiency with sustained philosophical reflection. On a more personal level, he was always supportive of my work, ever since he told me more than a quarter of century ago that he had studied with interest my (then recent) doctoral dissertation. Thank you Sol! You'll be remembered. [The financial support of FCT, by way of grant PEst-OE/MAT/UI0209/2013 to the research center CMAFIO, is hereby duly acknowledged.]

mathematics, and conceptual pluralism" [9] (especially section 6) are examples of these writings. These theories are a blend of intuitionistic and classical logic and their philosophical rationale can be described succinctly: according to some philosophical conceptions, there are good reasons to treat certain collections as constituting a definite totality (and, therefore, membership in them as abiding by the the law of excluded middle) and others as open ended. For instance, one may want to see sets defined by bounded formulas as definite, and unbounded set-theoretic quantifications as open ended (and, as a consequence, treated intuitionistically). The consideration of intuitionistic subsystems of set theory with the law of excluded middle for bounded formulas was apparently first given by Lawrence Pozsgay in [16] (see [8] for more information on this regard). Even though the present paper studies semi-constructive theories of this kind, its rationale is mainly technical. It is an exploitation of a form of functional interpretation that was introduced in the classical setting in [10] and whose roots can be found in a seminal paper of Jeremy Avigad and Henry Towsner [2]. This form of functional interpretation works in tandem with Gödel's constructible hierarchy (or with the cumulative hierarchy, in case the powerset operation is present) and treats bounded and second-order (class) quantifications as "computationally empty." As it turns out, the theory of these functional interpretations is very natural and satisfying.

The layout of this paper is the following. In Section 2 we introduce and draw some simple but fundamental consequences of the basic semi-constructive theory that we will analyse. Section 3 studies with some detail the term calculus of the primitive recursive functionals, introduced by William Howard in [14]. The Ω-type tree terms q of this calculus give the means to refer to the various (countable) stages L_q of the constructible hierarchy (or of the cumulative hierarchy V_q). The most important section is the fifth, where the main functional interpretation is defined and where a pertinent soundness theorem is proved. Departing from tradition, the verifications of the functional interpretations of this paper do not take place within formal theories, but are rather seen to hold semantically. A previous Section 4 introduces the two basic semantical structures with which we will be working with. We will be able to extract constructive information from proofs of sentences of the form $\forall x \exists y\, \phi(x, y)$, where ϕ can take various forms. The constructive information is given by a closed term t of type $\Omega \to \Omega$ such that $\forall c^{\Omega} \forall x \in L_c \exists y \in L_{tc}\, \phi(x, y)$ holds. In particular, if the semi-constructive theory proves a Σ_1-sentence then it follows that this sentence already holds in L_α, for α an ordinal less than the Bachmann-Howard ordinal. Our methods are able to provide a Σ-ordinal analysis in this sense.

We extend the analysis to a second-order setting with a form of bounded comprehension and with strict-Π^1_1 reflection. In a first study, the extension keeps the original separation scheme. This is done in Section 6 and the conclusion is that an ordinary Σ-ordinal analysis still goes through. In the last section, we allow second-order parameters in the separation scheme. This simple modification entails a major change because, as Vincenzo Salipante has observed in [19], with this form of separation one can prove the totality of the powerset operation. Nevertheless, a functional interpretation of the second-order theory with the extended separation scheme can still be made. There is a crucial difference, though. We now need the (countable) stages of the *cumulative* hierarchy. We are also able to extract some constructive information and to perform a *relativized* Σ-ordinal analysis (in the sense of Michael Rathjen in [17]). As a last item, we recuperate Rathjen's relativized Σ-ordinal analysis of the classical theory dubbed *power Kripke-Platek set theory*.

2 Intuitionistic Kripke-Platek set theory

Kripke-Platek set theory with infinity, with acronym $\mathsf{KP}\omega$, is a well-known theory framed in the language of set theory. It is a theory of classical logic whose axioms are extensionality, (unordered) pair, union, infinity, and the schemata of Δ_0-separation, Δ_0-collection and (unrestricted) foundation. The reader can consult [5] for a precise formulation. Since we are interested in (semi) intuitionistic versions of $\mathsf{KP}\omega$, the primitive logical symbols are absurdity, conjunction, disjunction, implication, and the universal and existential quantifiers. It is also convenient in our setting (as it was in [10]) to include bounded quantifiers as a primitive syntactic device. Both $\forall x \in z\, \phi$ and $\exists x \in z\, \phi$ are part of the primitive syntactic apparatus and are not considered as abbreviations of $\forall x\, (x \in z \to \phi)$ and $\exists x\, (x \in x \wedge \phi)$, respectively. Instead, our axioms include the corresponding equivalences between these formulas. The class of bounded or Δ_0-formulas is the smallest class of formulas that includes the atomic formulas (including the absurdity) and which is closed under propositional connectives and bounded quantifiers. For the record (and because of its importance), we state the scheme Δ_0-Coll of bounded collection:

$$\forall y \in w\, \exists x\, \phi(x, y) \to \exists z\, \forall y \in w\, \exists x \in z\, \phi(x, y),$$

where $\phi(x, y)$ is a bounded formula, possibly with parameters. The scheme of foundation is formulated in its inductive form (which is the form appropriate for intuitionistic studies):

$$\forall x\, (\forall y \in x\, \phi(y) \to \phi(x)) \to \forall x\, \phi(x),$$

for *every* formula $\phi(x)$, possibly with parameters. Since the scheme is unrestricted, it is easy to see that the scheme is (intuitionistically) equivalent to the rule

$$\frac{\forall x\, (\forall y \in x\, \phi(y) \to \phi(x))}{\forall x\, \phi(x)}$$

where $\phi(x)$ is any formula (possibly with other free variables besides x). The proof of the soundness theorem (Theorem 2) simplifies if we consider the rule instead of the axiom scheme.

Following [8], let $\mathsf{IKP}\omega$ be the system $\mathsf{KP}\omega$ with the logic restricted to be intuitionistic. Let Δ_0-LEM be the scheme $\phi \vee \neg\phi$ of excluded middle for bounded formulas ϕ. Our basic intuitionistic theory is $\mathsf{IKP}\omega + \Delta_0$-$\mathsf{LEM}$. In the remaining of this section we present a series of four definitions that introduce principles that the functional interpretation of Section 5 is able to interpret.

Definition 1. *Markov's principle* MP *is the scheme* $\neg\forall x\, \phi(x) \to \exists x\, \neg\phi(x)$*, for* ϕ *a bounded formula (possibly with parameters).*

Proposition 1. $\mathsf{IKP}\omega + \Delta_0$-$\mathsf{LEM} + \mathsf{MP}$ *proves the following scheme for bounded formulas* ϕ *and* ψ *(parameters are allowed):* $(\forall x\, \phi(x) \to \psi) \to \exists x\, (\phi(x) \to \psi)$.

Proof. Assume $\forall x\, \phi(x) \to \psi$. By Δ_0-LEM, there are two cases to consider. If ψ holds then any x will do. Otherwise, we have $\neg\forall x\, \phi(x)$. By MP take x_0 such that $\neg\phi(x_0)$. Of course, $\phi(x_0) \to \psi$. We are done. □

A Σ_1-formula ϕ is a formula of the form $\exists z\, \psi(z)$, where $\psi(z)$ is a bounded formula (possibly with parameters). Up to provability in $\mathsf{IKP}\omega$, this class of formulas is closed under conjunction, disjunction, bounded quantifications and existential quantifications. This is well-known in the classical setting, but the argument also goes through in $\mathsf{IKP}\omega$. We defer the discussion of the dual class Π_1 until the end of this section. A Π_2-formula is a formula of the form $\forall w\, \phi(w)$, where $\phi(w)$ is a Σ_1-formula. We have only defined these formulas with a single universal quantifier '$\forall w$', but it is clear (using the pair axiom and the closure properties of bounded formulas) that a tuple of universal quantifications yields a formula equivalent (in $\mathsf{IKP}\omega$) to a Π_2-formula. The following theorem shows that the theory $\mathsf{IKP}\omega + \Delta_0\text{-}\mathsf{LEM} + \mathsf{MP}$ has a certain robustness.

Theorem 1. *The theory* $\mathsf{KP}\omega$ *is* Π_2*-conservative over* $\mathsf{IKP}\omega + \Delta_0\text{-}\mathsf{LEM} + \mathsf{MP}$.

Proof. This is an easy consequence of the (Gödel and Gentzen) negative translation. The translation is extended to the bounded quantifiers in the natural way: $(\forall x \in z\, \phi(x))^g$ is $\forall x \in z\, \phi^g(x)$, and $(\exists x \in z\, \phi(x))^g$ is $\neg\neg\exists x \in z\, \phi^g(x)$. Note that the translation of a bounded formula is still a bounded formula. Therefore, a bounded formula is equivalent to its negative translation in $\mathsf{IKP}\omega + \Delta_0\text{-}\mathsf{LEM}$. From this it is clear that the translations of the axioms of extensionality, pair, union, infinity and Δ_0-separation are theorems of $\mathsf{IKP}\omega + \Delta_0\text{-}\mathsf{LEM}$. The negative translation of an instance of the scheme of foundation is still an instance of foundation. In order to argue that the negative translation of $\mathsf{KP}\omega$ is contained in $\mathsf{IKP}\omega + \Delta_0\text{-}\mathsf{LEM} + \mathsf{MP}$, it remains to study the scheme of Δ_0-collection. Well, the negative translation of an instance of Δ_0-collection has the form $\forall x \in z \neg\neg\exists y\, \phi^g(x, y) \to \neg\neg\exists w \forall x \in z \neg\neg\exists y \in w\, \phi^g(x, y)$, where ϕ is a bounded formula. In the presence of MP, the antecedent of the above implication is equivalent to $\forall x \in z \exists y \phi^g(x, y)$. Now, by an application of bounded collection, we get something stronger than the consequent of the implication.

We are ready to prove the theorem. Suppose that $\mathsf{KP}\omega$ proves $\forall x \exists y\, \phi(x, y)$, with ϕ a bounded formula. By the properties of the negative translation, the theory $\mathsf{IKP}\omega + \Delta_0\text{-}\mathsf{LEM} + \mathsf{MP}$ proves $\forall x \neg\neg\exists y\, \phi(x, y)$. Using MP, we get the desired conclusion. □

Definition 2. *The independence of premises principle* bIP_{Π_1} *is the scheme*

$$(\forall x\, \phi(x) \to \exists y\, \psi(y)) \to \exists y\, (\forall x\, \phi(x) \to \exists z \in y\, \psi(z)),$$

where ϕ *is a bounded formula and* ψ *is any formula (parameters are allowed).*

This principle is reminiscent of the independence of premises principle of Gödel's *dialectica* interpretation (cf. [1]). The analogy is not total because of the intrusion of a bounded quantification in the consequent of the existential consequent above. This is a crucial feature and it is in line with the bounded functional interpretation [11]. However, when the formula ψ is bounded then the bounded quantifier is not needed.

Lemma 1. $\mathsf{IKP}\omega + \Delta_0\text{-}\mathsf{LEM} + \mathsf{MP} + \mathsf{bIP}_{\Pi_1}$ *proves the following scheme for bounded formulas* ϕ *and* ψ *(parameters are allowed):* $(\forall x\, \phi(x) \to \exists y\, \psi(y)) \to \exists y \exists x\, (\phi(x) \to \psi(y))$.

Observation 1. *Note that* $(\forall x\, \phi(x) \to \exists y\, \psi(y)) \to \exists y\, (\forall x\, \phi(x) \to \psi(y))$ *follows intuitionistically.*

Proof of Lemma 1. Assume $\forall x\, \phi(x) \to \exists y\, \psi(y)$. By bIP_{Π_1}, take y_0 so that $\forall x\, \phi(x) \to \exists z \in y_0\, \psi(z)$. Since the consequent of the latter formula is bounded, by Proposition 1 there is x_0 such that $\phi(x_0) \to \exists z \in y_0\, \psi(z)$. By Δ_0-LEM, there are two cases to consider. If $\phi(x_0)$ holds, let z_0 be an element of y_0 such that $\psi(z_0)$. Otherwise, let z_0 be $\emptyset$. Clearly, $\phi(x_0) \to \psi(z_0)$. □

Proposition 2. *The theory* $\mathsf{IKP}\omega + \Delta_0$-$\mathsf{LEM} + \mathsf{MP} + \mathsf{bIP}_{\Pi_1}$ *proves* Δ_1*-separation, i.e., it proves* $\forall x\, (\forall u\, \phi(u, x) \leftrightarrow \exists v\, \psi(v, x)) \to \forall z \exists y \forall x\, (x \in y \leftrightarrow (x \in z \wedge \exists v\, \psi(v, x)))$*, for bounded formulas* ϕ *and* ψ *(possibly with parameters).*

Proof. Suppose that $\forall x\, (\forall u\, \phi(u, x) \leftrightarrow \exists v\, \psi(v, x))$ and fix z. From the left-to-right direction and the above lemma, $\forall x \in z \exists u, v\, (\phi(u, x) \to \psi(v, x))$. By bounded collection, there is w such that

$$\forall x \in z \exists u, v \in w\, (\phi(u, x) \to \psi(v, x)).$$

It is easy to see that we can take $y = \{x \in z : \exists v \in w\, \psi(v, x)\}$. □

Corollary 1. *The theory* $\mathsf{IKP}\omega + \Delta_0$-$\mathsf{LEM} + \mathsf{bIP}_{\Pi_1} + \mathsf{MP}$ *proves the* Δ_1 *law of excluded middle, i.e., it proves* $(\forall u\, \phi(u) \leftrightarrow \exists v\, \psi(v)) \to (\forall u\, \phi(u) \vee \neg\forall u\, \phi(u))$*, for bounded formulas* ϕ *and* ψ *(possibly with parameters).*

Proof. Let $y_0 = \{x \in \{0, 1\} : (x = 0 \wedge \exists u\, \neg\phi(u)) \vee (x = 1 \wedge \exists v\, \psi(v))\}$. This set exists by the previous proposition. It is clear that $1 \in y_0 \leftrightarrow \forall u\, \phi(u)$. We are done. □

In a personal communication, Makoto Fujiwara observed that the Δ_1 law of excluded middle is a consequence of $\mathsf{IKP}\omega + \Delta_0$-$\mathsf{LEM} + \mathsf{bIP}_{\Pi_1}$ (MP is not needed). We will see this in the similar situation of Proposition 6.

One of the characteristic principles of the bounded functional interpretation [11] is a generalization of weak Kőnig's lemma. In the second-order setting of Section 6 ahead, the classical contrapositive of (a restriction of) strict-Π^1_1 reflection (see chapter VIII of [5]) takes the place of weak Kőnig's lemma. However, at a more fundamental level, the above mentioned characteristic principle of the bounded functional interpretation is better seen as the classical contrapositive of a collection scheme (it was dubbed bounded *contra-collection* scheme in [11]). Functional interpretations which treat bounded quantifications as computationally empty, as it is the case of the bounded functional interpretation [11] and the functional interpretation of this paper (see Section 5), enjoy the novelty of interpretating a bounded contra-collection scheme. As a consequence, they are able to interpret a semi-intuitionism that is able to accomodate principles like the lesser limited principle of omniscience LLPO of Errett Bishop (see [6] and also [7]). It also gives us a good theory of Δ_1-predicates. Let us first look at these matters in our first-order Kripke-Platek framework.

Definition 3. *The principle of bounded contra-collection* Δ_0-CColl *is the scheme*

$$\forall z \exists y \in w \forall x \in z\, \phi(x, y) \to \exists y \in w \forall x\, \phi(x, y),$$

where ϕ *is a bounded formula (possibly with parameters).*

Note that, classically, this is just the bounded collection scheme. Bounded contra-collection easily generalizes for a tuple of z's. The proof of the following lemma presents the argument for a pair of z's:

Lemma 2. *For each bounded formula ϕ, the theory* $\mathsf{IKP}\omega + \Delta_0$-$\mathsf{CColl}$ *proves*

$$\forall x, z \exists y \in w \forall u \in x \forall v \in z\, \phi(u, v, y) \to \exists y \in w\, \forall u, v\, \phi(u, v, y).$$

Proof. Suppose that $\forall x, z \exists y \in w\, \forall u \in x \forall v \in z\, \phi(u, v, y)$. We claim that

$$\forall s \exists y \in w \forall r \in s \forall u \in r \forall v \in r\, \phi(u, v, y).$$

Let s be given. Using the assumption with x and z taking the common value $\cup s$, we get $\exists y \in w \forall u \in \cup s\, \forall v \in \cup s\, \phi(u, v, y)$ and, as a consequence, the claim. By Δ_0-CColl, $\exists y \in w \forall r \forall u \in r\, \forall v \in r\, \phi(u, v, y)$. The result follows using the pair axiom. □

Proposition 3. *For bounded formulas ϕ and ψ, the theory* $\mathsf{IKP}\omega + \Delta_0$-$\mathsf{CColl}$ *proves*

$$\forall x, z\, (\forall u \in x\, \phi(u) \vee \forall v \in z\, \psi(v)) \to \forall u\, \phi(u) \vee \forall v\, \psi(v).$$

Proof. Suppose that $\forall x, z\, (\forall u \in x\, \phi(x) \vee \forall v \in z\, \psi(z))$. It clearly follows that

$$\forall x, z \exists y \in \{0, 1\} \forall u \in x\, \forall v \in z\, ((y = 0 \wedge \phi(u)) \vee (y = 1 \wedge \psi(v))).$$

The result follows from the previous lemma. □

We are now able to derive the analogue of the lesser limited principle of omniscience in our setting (parameters are allowed):

Corollary 2. *For bounded formulas ϕ and ψ, the theory* $\mathsf{IKP}\omega +\ \Delta_0$-$\mathsf{LEM} + \Delta_0$-$\mathsf{CColl}$ *proves*

$$\forall u, v\, (\phi(u) \vee \psi(v)) \to \forall u\, \phi(u) \vee \forall v\, \psi(v).$$

Proof. In the presence of Δ_0-LEM, it can easily be argued that $\forall u, v\, (\phi(u) \vee \psi(v))$ entails $\forall x, z\, (\forall u \in x\, \phi(x) \vee \forall v \in z\, \psi(z))$. Now, apply the previous proposition. □

The theory $\mathsf{IKP}\omega +\ \Delta_0$-$\mathsf{LEM} + \mathsf{MP} + \mathsf{bIP}_{\Pi_1} + \Delta_0$-$\mathsf{CColl}$ is a rather robust theory. For instance, this theory is able to prove the results described by Jon Barwise between sections 3 and 6 of chapter I of [5]. These results include the existence of ordered pairs, cartesian products and transitive closures as well as various forms of reflection and replacement. More importantly, this theory is well behaved regarding the introduction of Δ_1-relation symbols and Σ_1-function symbols. The arguments of the referred sections of [5] rely crucially on the fact that Σ_1-formulas are closed under conjunctions, disjunctions, bounded quantifications and existential quantifications. As observed, this is also the case in our intuitionistic setting. It *also* relies crucially on corresponding dual properties of Π_1-formulas. This is immediate in the classical setting. However, in our semi-constructive setting, one must proceed with some care. A Π_1-formula is a formula of the form $\forall z\, \psi(z)$, where ψ is a bounded formula. Note that a negation of a Π_1-formula is (equivalent to) a Σ_1-formula, thanks to MP. We claim that Π_1-formulas are closed (up to equivalence in $\mathsf{IKP}\omega + \Delta_0\text{-}\mathsf{LEM} + \mathsf{MP} + \mathsf{bIP}_{\Pi_1} + \Delta_0\text{-}\mathsf{CColl}$) under conjunctions, disjunctions, bounded quantifications and universal quantifications. This is clear for conjunctions and universal quantifications (bounded and unbounded). Corollary 2 entails that Π_1-formulas are close under disjunctions. The closure under existential bounded quantifications follows from Δ_0-CColl. One last word, the theory $\mathsf{IKP}\omega + \Delta_0\text{-}\mathsf{LEM} + \mathsf{MP} + \mathsf{bIP}_{\Pi_1} + \Delta_0\text{-}\mathsf{CColl}$ also allows definitions by Σ-recursion.

The last principle of this section is the *unrestricted* collection scheme. The functional interpretation of Section 5 is able to interpret it. This is possible because we are in an intuitionistic setting (this is analogous to the functional interpretation of [8]). In the classical setting of [10], only the bounded collection scheme is interpretable.

Definition 4. *The principle of (unrestricted) collection* Coll *is the scheme*

$$\forall y \in w \exists x\, \phi(x, y) \to \exists z \forall y \in w \exists x \in z\, \phi(x, y),$$

where ϕ *is* any *formula (possibly with parameters).*

3 On the term calculus of the primitive recursive tree functionals

It was said in the introduction that the functional interpretations of this paper uses the combinatory term calculus $\mathcal{L}_\Omega$ of the primitive recursive tree functionals. This term calculus is due to Howard (cf. [14]) and in [10] we have presented a streamlined version of it. Let us briefly go through $\mathcal{L}_\Omega$. We expand Gödel's language of "primitive recursive functions of finite-type" (see [1]) with a new ground type Ω for the countable constructive tree ordinals. The ground type of the natural numbers is denoted by N. The complex types are obtained from the ground types by closing under arrow. We use the Greek letters ρ, τ, σ, ... to denote the types. The language has a denumerable stock of variables a, b, c, ... for each type. We follow [1] for notations and conventions concerning omission of parentheses. The constants of $\mathcal{L}_\Omega$ are:

(a) *Logical constants* or *combinators*. For each pair of types ρ, τ there is a combinator of type $\rho \to \tau \to \rho$ denoted by $\Pi_{\rho,\tau}$. For each triple of types δ, ρ, τ there is a combinator of type

$$(\delta \to \rho \to \tau) \to (\delta \to \rho) \to (\delta \to \tau)$$

denoted by $\Sigma_{\delta,\rho,\tau}$.

(b) *Arithmetical constants*. The constant 0_N of type N. The *successor* constant S of type $N \to N$. For each type ρ, a *(number) recursor* constant of type

$$N \to \rho \to (N \to \rho \to \rho) \to \rho$$

denoted by R^N_ρ.

(c) *Tree constants*. The constant 0_Ω of type Ω. The *supremum* constant *Sup* of type $(N \to \Omega) \to \Omega$. For each type ρ, a *tree recursor* constant of type

$$\Omega \to \rho \to ((N \to \Omega) \to (N \to \rho) \to \rho) \to \rho$$

denoted by R^{Ω}_{ρ}.

The above treatment is not completely rigorous because the recursors must operate simultaneously on tuples of variables (simultaneous recursion), and not only on a single variable. A rigorous treatment for arithmetic with simultaneous recursions is given in [15]. Another option would be to permit product types $\rho \times \tau$. We will not worry in this paper about these fine details. In any circumstance, a combinatory calculus with a notion of weak equality $=_w$ can be associated with the above. The conversions for the the tree recursors can be found in [10], but can also be read from their set-theoretical interpretation (definition) in the next section. In this paper we follow the treatment of [13], including the way of defining lambda terms (abstration). Let us introduce some important terms (see [10] for more information).

i. $q^{\Omega} + 1 :\equiv Sup(\lambda x^N.q)$.

ii. It is possible to define by number recursion a closed term $d^{\Omega\to\Omega\to N\to\Omega}$ such that $d(q, s, 0) =_w q$ and $d(q, s, Sn) =_w s$. Define $\max(a, b) + 1 :\equiv Sup(\lambda x^N.d(a, b, x))$ (the notation '$\max(a, b) + 1$' should be viewed syncategorematically).

iii. We can define by number recursion a closed term $q^{N\to\Omega}$ such that $q0 =_w 0_{\Omega}$ and $q(Sn) =_w S(qn)$. We write n_{Ω} instead of qn.

iv. $\omega_{\Omega} :\equiv Sup(\lambda x^N.x_{\Omega})$.

v. It is possible to define by tree recursion a term Sup^{-1} of type $\Omega \to (N \to \Omega)$ such that $Sup^{-1}(Sup(t)) =_w t$ for each term t of type $N \to \Omega$. We abbreviate $Sup^{-1}(q)(n)$ by $q\langle n\rangle$, for terms q^{Ω} and n^N. Clearly, $Sup(t)\langle n\rangle =_w tn$. We have also $(q + 1)\langle n\rangle =_w q$, $(\max(q, s) + 1)\langle 0\rangle =_w q$ and $(\max(q, s) + 1)\langle Sn\rangle =_w s$.

vi. Fix p a pairing term of type $N \to (N \to N)$ with inverse functions l and r, both of type $N \to N$. Hence, $p(l(n), r(n)) =_w n$, $l(p(m, k)) =_w m$ and $r(p(m, k)) =_w k$, for terms n, m, k of type N. Given $t^{N\to\Omega}$, we define $\bigsqcup t :\equiv Sup\,(\lambda y^N.(t(ly))\langle ry\rangle)$. This is a term of type Ω. An important particular case of this "square union" is $q \sqcup s :\equiv \bigsqcup(\lambda x^N.d(q, s, x))$, where q and s are of type Ω and d is the term introduced in (ii) above.

The next definition is a refinement of a similar definition in [2]:

Definition 5. *Let t, q be terms of type Ω and r a term of type $N \to N$. We say that $t \sqsubseteq_r q$ if $t\langle x\rangle =_w q\langle rx\rangle$, where x is a fresh variable of type N.*

Sometimes we write only $t \sqsubseteq q$ when the witnessing term r is presupposed.

Lemma 3. *Let t be a term of type $N \to \Omega$ and k^N. Then, for each term n of type N, $(\bigsqcup t)\langle p(k, n)\rangle =_w (tk)\langle n\rangle$ and (therefore) $tk \sqsubseteq_{\lambda x^N.p(k,x)} \bigsqcup t$. (Here, p is the pairing function of* (vi) *above.) In particular, for q and s of type Ω, $q \sqsubseteq q \sqcup s$ and $s \sqsubseteq q \sqcup s$.*

Proof. $(\bigsqcup t)\langle p(k, n)\rangle =_w (\lambda y^N.(t(ly))\langle ry\rangle)(p(k, n)) =_w (tk)\langle n\rangle$. □

In order to deal with the functional interpretation of Section 5, we need to lift some of the above notions from the type Ω to so-called *pure Ω-types* (i.e., types obtained only from the ground type Ω by means of the arrow). The lifting is done pointwise. As a consequence, we need to have a stronger notion of equality: one that incorporates some extensionality. We use *combinatory extensional equality*, as explained in [13]. This notion allows for the so-called ζ-rule: from $tx = qx$ infer $t = q$, where x is a fresh variable (of appropriate type). This rule is equivalent to the ξ-rule: from $t = q$ infer $\lambda x.t = \lambda x.q$. This is proved in [13]. Note that the ξ-rule is automatic in a direct treatment (i.e., not via combinators) of the lambda calculus. In a direct treatment of the lambda calculus, the extensionality needed is given by the η-axiom scheme: $\lambda x.qx = q$, where q is a term in which the variable x does not occur free. On the other hand, in combinatory logic, with a proper definition of abstraction (as in [13], and which we follow), the η-axiom scheme is automatic. These are somewhat technical issues relating the combinatory calculus with the lambda calculus. To cut through the fog, the proper way to state extensional equality in the combinatory calculus is the above ζ-rule. For more information, consult [13]. Risking confusion (but essentially following the notation of [13]), we use the notation $=_{\beta\eta}$ for combinatory extensional equality. Of course, if $t =_w q$ then $t =_{\beta\eta} q$.

vii. If q is a term of pure Ω-type τ of the form $\tau_1 \to \cdots \to \tau_k \to \Omega$, one defines $q + 1 :\equiv \lambda \underline{z}.((q\underline{z}) + 1)$, where $\underline{z}$ abbreviates a tuple of variables $z^{\tau_1}, \ldots, z^{\tau_k}$. If t is a term of type $N \to \tau$, we let $Sup(t) :\equiv \lambda \underline{z}.Sup(\lambda n.tn\underline{z})$.

viii. If q is as above and n is of type N, let $q\langle n\rangle :\equiv \lambda \underline{z}.((q\underline{z})\langle n\rangle)$. We claim that $(q+1)\langle n\rangle =_{\beta\eta} q$. Well, $(q+1)\langle n\rangle =_{\beta\eta} (\lambda \underline{z}.((q\underline{z})+1))\langle n\rangle =_{\beta\eta} \lambda \underline{z}.(((q\underline{z})+1)\langle n\rangle) =_{\beta\eta} \lambda \underline{z}.q\underline{z} =_{\beta\eta} q$. It is in the penultimate equality that extensionality is used (a ξ-rule application). If t is a term of type $N \to \tau$, where τ is the pure Ω-type above, we have $Sup(t)\langle n\rangle =_{\beta\eta} \lambda \underline{z}.tn\underline{z}$. To see this, notice that $Sup(t)\langle n\rangle =_{\beta\eta} \lambda \underline{z}.((Sup(\lambda n.tn\underline{z}))\langle n\rangle) =_{\beta\eta} \lambda \underline{z}.tn\underline{z}$. We are using (v) and the ξ-rule application in the last equality.

ix. Again, if t is a term of type $N \to \tau$, where τ is the pure Ω-type above, we define $\bigsqcup t :\equiv \lambda \underline{z}.(\bigsqcup \lambda x^N.(tx\underline{z}))$. If q and s are of type τ, we define $q \sqcup s$ pointwise in analogy to (vi) above: $q \sqcup s :\equiv \lambda \underline{z}. \bigsqcup \lambda x^N.d(q\underline{z}, s\underline{z}, x)$.

Definition 6. *Let t, q be terms of pure Ω-type τ and let r a term of type $N \to N$. We say that $t \sqsubseteq_r q$ if $t\langle x\rangle =_{\beta\eta} q\langle rx\rangle$, where x is a fresh variable of type N.*

One should see this definition as superseding Definition 5 and the next lemma as superseding Lemma 3:

Lemma 4. *Let t be a term of type $N \to \tau$, where τ is a pure Ω-type. Let k and n be terms of type N. Then $(\bigsqcup t)\langle p(k,n)\rangle =_{\beta\eta} (tk)\langle n\rangle$ and (therefore) $tk \sqsubseteq_{\lambda x^N.p(k,x)} \bigsqcup t$. In particular, for q and s of type τ, $q \sqsubseteq q \sqcup s$ and $s \sqsubseteq q \sqcup s$.*

Proof. Suppose τ is $\tau_1 \to \cdots \to \tau_r \to \Omega$. Then, if $\underline{z}$ stands for a tuple of variables $z_1^{\tau_1}, \ldots, z_r^{\tau_r}$, we have

$$(\bigsqcup t)\langle p(k,n)\rangle =_{\beta\eta} (\lambda \underline{z}. \bigsqcup \lambda x^N.(tx\underline{z}))\langle p(k,n)\rangle =_{\beta\eta} \lambda \underline{z}.((\bigsqcup x^N.(tx\underline{z}))\langle p(k,n)\rangle) =_{\beta\eta}$$

$$\lambda \underline{z}.((tk\underline{z})\langle n\rangle) =_{\beta\eta} (tk)\langle n\rangle$$

Extensionality is used in the penultimate equality. □

We were somewhat careful (perhaps even pedantic) in the discussion of extensionality because there is a *faux pas* in section 6 of [10]. In that paper, we discussed a so-called internalization of an interpretation introduced in a previous section (in the present paper we do not discuss internalizations). The referred internalization is given by the intensional model of section 9.3 of [1] (a version of the hereditarily recursive operations for our setting). However, the conclusion of the soundness theorem is not verified in this structure because of a lack of extensionality: the above discussed $\beta\eta$ equalities do not hold in the intensional model. The problem is easily fixed, though. The internalization should have been done with the analogue of the hereditarily effective operations for our setting (also discussed in 9.3 of [1]).

4 Brief semantical considerations

In the next section we define a functional interpretation of the theory $\mathsf{IKP}\omega + \Delta_0\text{-}\mathsf{LEM} + \mathsf{MP} + \mathsf{bIP}_{\Pi_1} + \Delta_0\text{-}\mathsf{CColl} + \mathsf{Coll}$ and prove an appropriate soundness theorem. The conclusion of the soundness theorem is not verified in a formal theory (as it is the tradition in functional interpretations) but it is rather semantically verified, i.e., seen to be true in a certain structure. This is a simplificatory option of this paper. In principle, one can refine the soundness theorem in order to have a verification in a suitable formal theory, as we did in section 6 of [10] with an "internalization" of the semantical interpretation presented there in sections 2 and 3. With verifications in formal theories, one can obtain stronger results (viz., conservation results) but, as noted in the previous section, one has to proceed carefully and this is would make the present paper too long and perhaps even distracting from its main objective. So, in this paper, we will proceed semantically in the verification of the soundness theorems.

The interpretation of the term calculus of this paper is common to all the interpretations of this paper. It is the (full and extensional) set-theoretical structure $\langle S_\rho\rangle$ of section 9.4 of [1] (see also [10]). The variables of each type ρ of $\mathcal{L}_\Omega$ range over S_ρ. These sets are defined thus:

1. $S_N = \omega$
2. S_Ω is the smallest set W which contains 0 and is such that, whenever f is a function that maps ω into W, then the ordered pair $(1, f)$ is in W.
3. $S_{\rho\to\tau} = \{f : f \text{ is a set-theoretic function that maps } S_\rho \text{ into } S_\tau\}$

It is clear how the terms of $\mathcal{L}_\Omega$ are interpreted in the set-theoretical model (see [10] for some details). We only note that the constant *Sup* is interpreted by the function which, on input $f \in S_{N\to\Omega}$, outputs the element $(1, f)$ of W. We also remark that, to each element c of W, we can associate a countable set-theoretical ordinal $|c|$ so that $|0| = 0$ and, for a function $f : \omega \to W$, $|Sup(f)| = \sup\{|f(n)| + 1 : n \in \omega\}$. Observe that $|f(n)| < |Sup(f)|$, for each natural number n. It is well-known that the first uncountable ordinal ω_1 is the supremum of all $|c|$, with $c \in W$. The previous discussion also permits to define, by classical ordinal recursion, the interpretations of the tree recursors R^{Ω}_{ρ} so that:

$$R^{\Omega}_{\rho}(0_{\Omega}, a, F) = a \text{ and } R^{\Omega}_{\rho}(Sup(f), a, F) = F(f, \lambda x^N.R^{\Omega}_{\rho}(f(x), a, F)),$$

for all $a \in S_{\rho}$ and F a function that maps $S_{N\to\rho}$ into $(S_{\rho})^{S_{N\to\rho}}$.

Clearly, the equalities (both $=_w$ and $=_{\beta\eta}$) established in the previous section between terms of $\mathcal{L}_{\Omega}$ give rise to set-theoretical equalities in $\langle S_{\rho}\rangle$.

Lemma 5. (i) *If $c \in W$ and $c \neq 0$, then* $|c| = \mathrm{Sup}_{n\in\omega}(|c\langle n\rangle| + 1)$.

(ii) *If $c, d \in W$ and $c \sqsubseteq d$ then* $|c| \leq |d|$.

Proof. (ii) is an immediate consequence of (i). Given $c \neq 0$, let $c = \sup f$, for a certain $f : \omega \to W$. Then $|c| = \sup\{|f(n)| + 1 : n \in \omega\} = \sup\{|c\langle n\rangle| + 1 : n \in \omega\}$. □

Lemma 6. *Let $b, c : \rho \to \tau$ and $a : \rho$ be term variables of pure Ω-types and f be a term variable of type $N \to N$. Then the implication $b \sqsubseteq_f c \to ba \sqsubseteq_f ca$ is true in the set-theoretical structure $\langle S_{\rho}\rangle$ for any values of the variables.*

Proof. Let τ be $\tau_1 \to \cdots \to \tau_k \to \Omega$. Take n a natural number. Since $b \sqsubseteq_f c$, then $b\langle n\rangle = c\langle fn\rangle$ holds set-theoretically. Note that $\rho \to \tau$ is $\rho \to \tau_1 \to \cdots \to \tau_k \to \Omega$. Hence, $\lambda w^{\rho}, \underline{z}.((bw\underline{z})\langle n\rangle) = \lambda w^{\rho}, \underline{z}.((cw\underline{z})\langle fn\rangle)$ holds set-theoretically (where $\underline{z}$ is a tuple of variables $z_1^{\tau_1}, \ldots, z_k^{\tau_k}$). Therefore, $(\lambda w^{\rho}, \underline{z}.((bw\underline{z})\langle n\rangle))a = (\lambda w^{\rho}, \underline{z}.((cw\underline{z})\langle fn\rangle))a$ holds set-theoretically and, hence, $\lambda\underline{z}.((ba\underline{z})\langle n\rangle) = \lambda\underline{z}.((ca\underline{z})\langle fn\rangle)$ holds set-theoretically. In sum, $ba \sqsubseteq_f ca$. □

Finally, before discussing our so-called mixed structures, observe that the interpretation of a *closed* term t of ground type Ω is an element of W and, therefore, has an associated set-theoretical ordinal which, with abuse of notation, we denote by $|t|$. The Bachmann-Howard ordinal is the supremum of all these ordinals.

The functional interpretation of the next section translates a formula ϕ of the language of set theory into formulas ϕ^{B} and ϕ_{B} of a mixed language $\mathcal{L}^{\mathrm{mix}}_{\Omega}$. A version of this language was introduced in [10]. Let us briefly describe it. The mixed language $\mathcal{L}^{\mathrm{mix}}_{\Omega}$ has three kinds of terms: the terms of $\mathcal{L}_{\Omega}$ (including, of course, the variables a, b, c, etc. of $\mathcal{L}_{\Omega}$), the set-theoretical variables x, y, z, etc. and (set) terms M_t, where t is a term of $\mathcal{L}_{\Omega}$. The *atomic formulas* of $\mathcal{L}^{\mathrm{mix}}_{\Omega}$ are the formulas of the form $x = y$, $x \in y$ or $x \in M_t$, for x and y set-theoretical variables and t a term of $\mathcal{L}_{\Omega}$ of type Ω. The *bounded mixed formulas* are generated from the atomic formulas by means of the propositional logical connectives $\neg$ and $\wedge$ and quantifications of the form $\forall x \in y$, $\forall x \in M_t$ and $\forall n^N$ (note that this last quantifier is classified as bounded). Since our semantics is classical, there is no need to introduce other connectives and quantifiers, as they can be defined. The *formulas* of $\mathcal{L}^{\mathrm{mix}}_{\Omega}$ are generated from the bounded mixed formulas by means of propositional connectives and quantifications of the form $\forall a^{\rho}$, where a is a term variable (of a certain type ρ) of the term language $\mathcal{L}_{\Omega}$. Observe that we do not need unbounded set-theoretic quantifications.

The two basic interpretations for $\mathcal{L}^{\mathrm{mix}}_{\Omega}$ are the structures $L^{\mathrm{mix}}_{\omega_1}$ and $V^{\mathrm{mix}}_{\omega_1}$. In both of these structures, the terms of $\mathcal{L}_{\Omega}$ (and, therefore, the range of term variables) are interpreted set-theoretically in $\langle S_\rho \rangle$ (as described by points 1, 2 and 3 above). The set-theoretic variables range over L_{ω_1}, respectively V_{ω_1}. The terms M_t are interpreted as $L_{|t|}$ in $L^{\text{mix}}_{\omega_1}$ and as $V_{|t|}$ in $V^{\text{mix}}_{\omega_1}$. Abusing language, we often replace the term M_t by the notations L_t or V_t according to the interpretation that we have in mind.

5 The main functional interpretation

We are going to associate to each formula $\phi(x_1, \ldots, x_n)$ of the language of set theory (free variables as shown) a bounded mixed formula ϕ_B of the form

$$\phi_B(a_1, \ldots, a_k, b_1, \ldots, b_m, x_1, \ldots, x_n),$$

with the free variables as shown (the a's and the b's are variables of $\mathcal{L}_\Omega$ of pure Ω-type) and also the following formula $\phi^B(x_1, \ldots, x_n)$ of the language $\mathcal{L}^{\text{mix}}_\Omega$:

$$\exists a_1 \cdots \exists a_k \forall b_1 \cdots \forall b_m \, \phi_B(a_1, \ldots, a_k, b_1, \ldots, b_m, x_1, \ldots, x_n).$$

Note that k or m (or both) can be zero. For notational simplicity, we simply write $\phi(x)$, $\phi^B(x)$ and $\phi_B(a, b, x)$, instead of carrying the tuple notation. Many times we also omit the parameters x.

Definition 7. *To each formula ϕ of the language of set theory (possibly with parameters), we assign formulas ϕ^B and ϕ_B so that ϕ^B is of the form $\exists a \forall b \, \phi_B(a, b)$, with $\phi_B(a, b)$ a bounded mixed formula, according to the following clauses:*

1. *ϕ^B and ϕ_B are simply ϕ, for bounded formulas ϕ of the language of set theory.*

For the remaining *cases, if we have already interpretations for ϕ and ψ given by $\exists a \forall b \, \phi_B(a, b)$ and $\exists d \forall e \, \psi_B(d, e)$ (respectively), then we define:*

2. *$(\phi \wedge \psi)^B$ is $\exists a, d \forall b, e \, [\phi_B(a, b) \wedge \psi_B(d, e)]$,*

3. *$(\phi \vee \psi)^B$ is $\exists a, d \forall b, e \, [\forall n \, \phi_B(a, b\langle n \rangle) \vee \forall m \, \psi_B(d, e\langle m \rangle)]$,*

4. *$(\phi \to \psi)^B$ is $\exists B, D \forall a, e \, [\forall n \, \phi_B(a, (Bae)\langle n \rangle) \to \psi_B(Da, e)]$,*

5. *$(\forall x \in z \, \phi(x, z))^B$ is $\exists a \forall b \, [\forall x \in z \, \phi_B(a, b, x, z)]$,*

6. *$(\exists x \in z \, \phi(x, z))^B$ is $\exists a \forall b \, [\exists x \in z \forall n \, \phi_B(a, b\langle n \rangle, x, z)]$,*

7. *$(\forall x \phi(x))^B$ is $\exists A \forall c^\Omega, b \, [\forall x \in L_c \, \phi_B(Ac, b, x)]$,*

8. *$(\exists x \phi(x))^B$ is $\exists c^\Omega, a \forall b \, [\exists x \in L_c \forall n \, \phi_B(a, b\langle n \rangle, x)]$.*

The lower B-translations are displayed inside the square parentheses. Note that they are bounded mixed formulas. The following lemma is an immediate consequence of the definitions.

Lemma 7. *Let ϕ be a Δ_0-formula. Then:*

(i) $(\exists x \phi(x))^B$ *is* $\exists c^\Omega [\exists x \in L_c\, \phi(x)]$.

(ii) $(\forall x \exists y \phi(x,y))^B$ *is* $\exists A^{\Omega\to\Omega} \forall c^\Omega [\forall x \in L_c \exists y \in L_{Ac}\, \phi(x,y)]$.

As it is usual with this kind of functional interpretations, we have the following crucial property:

Lemma 8 (Monotonicity property)**.** *Let $\phi(x)$ be a formula of the language of set theory. In $L^{\rm mix}_{\omega_1}$ one has the implication $a \sqsubseteq_f c \wedge \phi_B(a,b,x) \to \phi_B(c,b,x)$.*

Proof. It is clear that the clauses 2, 3, 5 and 6 preserve this property. By Lemma 6, clause 7 preserves the property, and by Lemma 5(ii), so does clause 8. Let us look now at clause 4. Suppose that $\forall n\, \phi_{\rm B}(a, (Bae)\langle n\rangle) \to \psi_{\rm B}(Da, e)$ holds in $L^{\rm mix}_{\omega_1}$ and that $B \sqsubseteq_f B'$ and $D \sqsubseteq_g D'$. Assume that $\forall n\, \phi_{\rm B}(a, (B'ae)\langle n\rangle)$. Let k be an arbitrary natural number. Since $B \sqsubseteq_f B'$, by two applications of Lemma 6, we get $Bae \sqsubseteq_f B'ae$. Hence, $(Bae)\langle k\rangle = (B'ae)\langle fk\rangle$. By the assumption, we have $\phi_{\rm B}(a, (B'ae)\langle fk\rangle)$. Now, by the arbitrariness of k, we may conclude that $\forall k\, \phi_{\rm B}(a, (Bae)\langle k\rangle)$. By hypothesis we may infer $\psi_{\rm B}(Da, e)$ and, therefore by Lemma 6 and the induction hypothesis, $\psi_{\rm B}(D'a, e)$. We are done. □

We are now ready to state and prove the soundness theorem of the functional interpretation.

Theorem 2 (Soundness Theorem)**.** *Let ϕ be a sentence of the language of set theory. Suppose that* $\mathsf{IKP}\omega + \Delta_0\text{-}\mathsf{LEM} + \mathsf{MP} + \mathsf{bIP}_{\Pi_1} + \Delta_0\text{-}\mathsf{CColl} + \mathsf{Coll} \vdash \phi$*. Then there are closed terms t of $\mathcal{L}_\Omega$ such that, for appropriate types ρ,*

$$L^{\rm mix}_{\omega_1} \models \forall b^\rho\, \phi_B(t,b).$$

Proof. The proof is by induction on the length of the derivation. We show that if a formula $\phi(w)$ is provable in the theory of the theorem, then there are closed terms t of $\mathcal{L}_\Omega$ such that, for appropriate types ρ, we have

$$L^{\rm mix}_{\omega_1} \models \forall c^\Omega \forall b^\rho\, [\forall w \in L_c\, \phi_{\rm B}(tc, b, w)],$$

where $\phi(w)^{\rm B}$ is $\exists a \forall b\, \phi_{\rm B}(a,b,w)$.

For ease of reading, in the following we ignore parameters that do not play an important role in the proof of the theorem. For the logical part of the theory, we rely on the formalization of intuitionistic logic given in [1]. The verification of these axioms and rules has some rough similarities with the verifications of [11]. In the following, we take ϕ and ψ as in Definition 7, and γ with $\gamma^{\rm B}$ given by $\exists u \forall v\, \gamma_{\rm B}(u,v)$. Let us now discuss each rule and axiom:

1. $\phi, \phi \to \psi \Rightarrow \psi$. By induction hypothesis, there are terms t, r and s such that $\forall b\, \phi_{\rm B}(t,b)$ and $\forall a, e\, [\forall n\, \phi_{\rm B}(a, (rae)\langle n\rangle) \to \psi_{\rm B}(sa, e)]$ hold in $L^{\rm mix}_{\omega_1}$. Let $q :\equiv st$. It is clear that we can conclude $\forall e\, \psi_{\rm B}(q, e)$.

2. $\phi \to \psi, \psi \to \gamma \Rightarrow \phi \to \gamma$. By hypothesis we have terms t, s, r and q such that the following holds in $L^{\rm mix}_{\omega_1}$: (i) $\forall a, e\, [\forall n\, \phi_{\rm B}(a, (sae)\langle n\rangle) \to \psi_{\rm B}(ta, e)]$ and (ii) $\forall d, v\, [\forall m\, \psi_{\rm B}(d, (rdv)\langle m\rangle) \to \gamma_{\rm B}(qd, v)]$. Take $l :\equiv \lambda a, v. \bigsqcup_m s(a, (r(ta, v))\langle m\rangle)$ and

$o :\equiv \lambda a.q(ta)$. We show that $L^{\text{mix}}_{\omega_1} \models \forall a, v\, [\forall k\, \phi_{\mathrm{B}}(a, (lav)\langle k\rangle) \to \gamma(oa, v)]$. Take a and v and suppose that $\forall k\, \phi_{\mathrm{B}}(a, (lav)\langle k\rangle)$. Then, for every n, m, we have $\phi_{\mathrm{B}}(a, (lav)\langle p(m, n)\rangle)$ (here p is the pairing term). By Lemma 4, $(lav)\langle p(m, n)\rangle = (s(a, (r(ta, v))\langle m\rangle))\langle n\rangle$. Hence, fixing m, we have $\forall n\, \phi_{\mathrm{B}}(a, (s(a, (r(ta, v))\langle m\rangle))\langle n\rangle)$. Particularizing (i) with e as the element $(r(ta, v))\langle m\rangle$, we get $\psi_{\mathrm{B}}(ta, (r(ta, v))\langle m\rangle)$. By the arbitrariness of m and (ii), we conclude $\gamma(q(ta), v)$. That is what we want.

3a. $\phi \vee \phi \to \phi$. A simple computation of $(\phi \vee \phi \to \phi)^{\mathrm{B}}$ shows that we must find terms q_1, q_2 and t such that, for all a, a', b'' the following holds in $L^{\text{mix}}_{\omega_1}$:

$$\forall k, k' (\forall n\, \phi_{\mathrm{B}}(a, (q_1 aa'b'')\langle k\rangle\langle n\rangle) \vee \forall m\, \phi_{\mathrm{B}}(a', (q_2 aa'b''')\langle k'\rangle\langle m\rangle) \to \phi_{\mathrm{B}}(taa', b'').$$

We claim that the above is true with $t :\equiv \lambda a, a'.(a \sqcup a')$, and with both q_1 and q_2 as $\lambda a, a', b''.(b'' + 2)$. In effect, the antecedent above entails $\phi_{\mathrm{B}}(a, b'') \vee \phi_{\mathrm{B}}(a', b'')$. By the monotonicity property, we get $\phi(a \sqcup a', b'')$, as wanted.

3b. $\phi \to \phi \wedge \phi$. We must find terms t_1, t_2 and q such that

$$\forall a, b', b''\, [\forall n\, \phi_{\mathrm{B}}(a, (qab'b'')\langle n\rangle) \to \phi_{\mathrm{B}}(t_1 a, b') \wedge \phi_{\mathrm{B}}(t_2 a, b'')]$$

holds in $L^{\text{mix}}_{\omega_1}$. Let t_1 and t_2 be the term $\lambda a.a$ and $q :\equiv \lambda a, b', b''.((b' + 1) \sqcup (b'' + 1))$. If $\forall n\, \phi_{\mathrm{B}}(a, ((b' + 1) \sqcup (b'' + 1))\langle n\rangle)$ we have, in particular $\phi_{\mathrm{B}}(a, b')$ and $\phi_{\mathrm{B}}(a, b'')$ because $((b' + 1) \sqcup (b'' + 1))\langle p(0, n)\rangle = b'$ and $((b' + 1) \sqcup (b'' + 1))\langle p(1, n)\rangle = b''$, where p is the pairing function.

4a. $\phi \to \phi \vee \psi$. We must find terms q, t and r such that

$$\forall a, b', e\, [\forall k\, \phi_{\mathrm{B}}(a, (rab'e)\langle k\rangle) \to \forall n\, \phi_{\mathrm{B}}(qa, b'\langle n\rangle) \vee \forall m\, \psi_{\mathrm{B}}(ta, e\langle m\rangle)]$$

holds in $L^{\text{mix}}_{\omega_1}$. It is clear that $q :\equiv \lambda a.a$, $t :\equiv \lambda a.0$, and $r :\equiv \lambda a, b', e.b'$ works (here, 0 is the usual zero constant of appropriate pure Ω-type).

4b. $\phi \wedge \psi \to \phi$. We must find terms q, t and r such that

$$\forall a, d, b'\, [\forall n \forall m\, (\phi_{\mathrm{B}}(a, (tadb')\langle n\rangle) \wedge \psi_{\mathrm{B}}(d, (radb')\langle m\rangle)) \to \phi_{\mathrm{B}}(qad, b')]$$

holds in $L^{\text{mix}}_{\omega_1}$. Clearly, $q :\equiv \lambda a, d.a$, $t :\equiv \lambda a, d, b'.(b' + 1)$ and $r :\equiv \lambda a, d, b'.0$ works.

5a. $\phi \vee \psi \to \psi \vee \phi$. We must find terms q, t, r and s such that

$$\forall a, b, b', e'\, [\forall k \forall k'\, (\forall n\, \phi_{\mathrm{B}}(a, (qabb'e')\langle k\rangle\langle n\rangle) \vee \forall m\, \psi_{\mathrm{B}}(d, (tabb'e')\langle k'\rangle\langle m\rangle)) \to$$

$$\forall n\, \phi_{\mathrm{B}}(rad, b'\langle n\rangle) \vee \forall m\, \psi_{\mathrm{B}}(sad, e'\langle m\rangle)]$$

holds in $L^{\text{mix}}_{\omega_1}$. It is clear that $r :\equiv \lambda a, d.a$, $s :\equiv \lambda a, d.d$, $q :\equiv \lambda a, b, b', e'.(b' + 1)$ and $t :\equiv \lambda a, b, b', e'.(e' + 1)$ work.

5b. $\phi \wedge \psi \to \psi \wedge \phi$. Clear.

6. $\phi \to \psi \Rightarrow (\gamma \vee \phi \to \gamma \vee \psi)$. By hypothesis we have terms t and q such that $\forall a, e\, [\forall k\, \phi_{\mathrm{B}}(a, (tae))\langle k\rangle \to \psi_{\mathrm{B}}(qa, e)]$. We must show that there are terms o, l, r and s such that

$$\forall u, a, v, e\, [\forall k \forall k' (\forall n\, \gamma_{\mathrm{B}}(u, (ruave)\langle k\rangle\langle n\rangle) \vee \forall m\, \phi_{\mathrm{B}}(a, (suave)\langle k'\rangle\langle m\rangle)) \to$$

$$\forall n\, \gamma(oua, v\langle n\rangle) \vee \forall m\, \psi(lua, e\langle m\rangle)]$$

holds in $L^{\mathrm{mix}}_{\omega_1}$. Let us define $o :\equiv \lambda u, a.u$, $l :\equiv \lambda u, a.qa$, $r :\equiv \lambda u, a, v, e.(v+1)$ and $s :\equiv \lambda u, a, v, e.((\bigsqcup_i t(a, e\langle i\rangle)) + 1)$. Suppose the antecedent. Particularizing for $k = k' = 0$, we get $\forall n\, \gamma_{\mathrm{B}}(u, v\langle n\rangle) \vee \forall m\, \phi_{\mathrm{B}}(a, (\bigsqcup_i t(a, e\langle i\rangle))\langle m\rangle)$. If we have the first disjunct, we are done. If we have the second then $\forall k, m\, \phi_{\mathrm{B}}(a, (\bigsqcup_i t(a, e\langle i\rangle))\langle p(m, k)\rangle)$, where p is the pairing term. Therefore, by Lemma 4, $\forall k, m\, \phi_{\mathrm{B}}(a, t(a, e\langle m\rangle)\langle k\rangle)$. The hypothesis now entails $\forall m\, \psi_{\mathrm{B}}(qa, e\langle m\rangle)$.

7a. $\phi \wedge \psi \to \gamma \Rightarrow \phi \to (\psi \to \gamma)$. By hypothesis, there are terms t, q and r such that $\forall a, d, v\, [\forall n \forall m\, (\phi_{\mathrm{B}}(a, (tadv)\langle n\rangle) \wedge \psi_{\mathrm{B}}(d, (qadv)\langle m\rangle)) \to \gamma_{\mathrm{B}}(rad, v)]$ holds in $L^{\mathrm{mix}}_{\omega_1}$, and we must obtain terms t', q' and r' such that the following also holds:

$$\forall a, d, v\, [\forall n\, \phi_{\mathrm{B}}(a, (t'adv)\langle n\rangle) \to (\forall m\, \psi_{\mathrm{B}}(d, (q'adv)\langle m\rangle) \to \gamma_{\mathrm{B}}(rad, v))].$$

Of course, $t' :\equiv t$, $q' :\equiv q$ and $r' :\equiv r$ work.

7b. $\phi \to (\psi \to \gamma) \Rightarrow \phi \wedge \psi \to \gamma$. Similar to 7a.

8. $\perp \to \phi$. Clear.

9. $\phi \to \psi(w) \Rightarrow \phi \to \forall w\, \psi(w)$, where w does not occur free in ϕ. By hypothesis, there are terms t and q such that

$$\forall c^{\Omega}, a, e\, [\forall w \in L_c\, (\forall n\, \phi_{\mathrm{B}}(a, (tcae)\langle n\rangle) \to \psi_{\mathrm{B}}(qca, e, w))]$$

holds in $L^{\mathrm{mix}}_{\omega_1}$. The interpretation asks for terms r and s such that

$$\forall c^{\Omega}, a, e\, [\forall n\, \phi_{\mathrm{B}}(a, (race)\langle n\rangle) \to \forall w \in L_c\, \psi_{\mathrm{B}}(sac, e, w)].$$

This is clear.

10. $\forall x\, \phi(x) \to \phi(w)$. A computation of the upper-B translation of this formula shows that we must find terms t, q and r such that for all $c \in W$ and A and b' of appropriate types in the set-theoretical structure $\langle S_\rho \rangle$,

$$\forall w \in L_c\, (\forall n, m \forall x \in L_{(qcAb')\langle n\rangle}\, \phi_{\mathrm{B}}(A((qcAb')\langle n\rangle), (rcAb')\langle m\rangle, x) \to \phi_{\mathrm{B}}(tcA, b', w))$$

holds in $L^{\mathrm{mix}}_{\omega_1}$. It is clear that the terms $t :\equiv \lambda c, A.Ac$, $q :\equiv \lambda c, A, b'.(c+1)$ and $r :\equiv \lambda c, A, b'.(b'+1)$ do the job.

11. $\phi(w) \to \exists x\, \phi(x)$. We must find terms t, q and r such that

$$\forall c^{\Omega}, a, b\, [\forall w \in L_c\, (\forall n\, \phi_{\mathrm{B}}(a, (qcab)\langle n\rangle, w) \to \exists x \in L_{tca} \forall n\, \phi_{\mathrm{B}}(rca, b\langle n\rangle, x))]$$

holds in $L^{\mathrm{mix}}_{\omega_1}$. It is clear that $t :\equiv \lambda c, a.c$, $q :\equiv \lambda c, a, b.b$ and $r :\equiv \lambda c, a.a$ do the job.

12. $\phi(w) \to \psi \Rightarrow \exists w\, \phi(w) \to \psi$, where w does not occur free in ψ. By hypothesis there are terms t and q such that

$$\forall c^{\Omega}, a, e\, [\forall w \in L_c\, (\forall n\, \phi_{\mathrm{B}}(a, (tcae)\langle n\rangle, w) \to \psi_{\mathrm{B}}(qca, e))]$$

holds in $L^{\mathrm{mix}}_{\omega_1}$. The interpretation asks for terms r and s such that

$$\forall c^{\Omega}, a, e\, [\forall m \exists w \in L_c\, \forall n\, \phi_{\mathrm{B}}(a, (rcae)\langle m\rangle\langle n\rangle, w) \to \psi_{\mathrm{B}}(sca, e)].$$

It is clear that $s :\equiv q$ and $r :\equiv \lambda c, a, e.((tcae) + 1)$ work.

If we do not count the axioms of equality of $\mathsf{IKP}\omega$, we are done with the logical part. The axioms of equality pose no problem because they can be taken as universal formulas and, hence, are interpreted by themselves. Before proceeding to the mathematical axioms of $\mathsf{IKP}\omega$, we must still pay attention to the four axioms that regulate the primitive bounded quantifiers $\forall x \in z\,(\ldots)$ and $\exists x \in z\,(...)$.

In order to study universal bounded quantification, we compute the upper-B translations of $\forall x \in z\,\phi(x,z)$ and $\forall x\,(x \in z \to \phi(x,z))$. They are $\exists a \forall b\,[\forall x \in z\,\phi_{\mathrm{B}}(a,b,x,z)]$ and $\exists A \forall c^{\Omega}, b\,[\forall x \in L_c\,(x \in z \to \phi_{\mathrm{B}}(Ac,b,x,z))]$, respectively.

The axiom $\forall x \in z\,\phi(x,z) \to \forall x\,(x \in z \to \phi(x,z))$. It is easy to see that we must obtain terms t and q such that for all $e, c \in W$ and a and b of appropriate types in the set-theoretical structure $\langle S_\rho \rangle$, and for all $z \in L_{|e|}$, the statement

$$\forall n \forall x \in z\,\phi_{\mathrm{B}}(a,(tecab)\langle n\rangle,x,z) \to \forall x \in L_c\,(x \in z \to \phi_{\mathrm{B}}(qeac,b,x,z))$$

holds in $L^{\mathrm{mix}}_{\omega_1}$. This is clearly the case with $t :\equiv \lambda e,c,a,b.(b+1)$ and $q :\equiv \lambda e,a,c.a$.

The axiom $\forall x\,(x \in z \to \phi(x,z)) \to \forall x \in z\,\phi(x,z)$. We must obtain terms t, q and s such that for all for all $e \in W$ and A, b of appropriate types in the set-theoretical structure $\langle S_\rho \rangle$, and for all $z \in L_{|e|}$, the statement

$$\forall n,m \forall x \in L_{(teAb)\langle n\rangle}(x \in z \to \phi_{\mathrm{B}}(A((teAb)\langle n\rangle),(qeAb)\langle m\rangle,x,z)) \to \forall x \in z\,\phi_{\mathrm{B}}(seA,b,x,z)$$

holds in $L^{\mathrm{mix}}_{\omega_1}$. It is clear that $s :\equiv \lambda e,A.Ae$, $t :\equiv \lambda e,A,b.(e+1)$ and $q :\equiv \lambda e,A,b.(b+1)$ work.

Now we discuss the existential bounded quantifier. The upper B-translations of $\exists x \in z\,\phi(x,z)$ and $\exists x\,(x \in z \wedge \phi(x,z))$ are, respectively, $\exists a \forall b\,[\exists x \in z \forall n\,\phi_{\mathrm{B}}(a,b\langle n\rangle,x,z)]$ and $\exists c^{\Omega}, a \forall b\,[\exists x \in L_c \forall n\,(x \in z \wedge \phi_{\mathrm{B}}(a,b\langle n\rangle,x,z))]$.

The axiom $\exists x \in z\,\phi(x,z) \to \exists x(x \in z \wedge \phi(x,z))$. We must find terms t, q and s such that, for all $e \in W$ and a and b of appropriate types in the set-theoretical structure $\langle S_\rho \rangle$, and for all $z \in L_{|e|}$,

$$\forall m \exists x \in z \forall n\,\phi_{\mathrm{B}}(a,(teab)\langle m\rangle\langle n\rangle,x,z) \to \exists x \in L_{qea} \forall n\,(x \in z \wedge \phi_{\mathrm{B}}(sea,b\langle n\rangle,x,z))$$

holds in $L^{\mathrm{mix}}_{\omega_1}$. Clearly, $t :\equiv \lambda e,a,b.(b+1)$, $q :\equiv \lambda e,a.e$ and $s :\equiv \lambda e,a.a$ work.

The axiom $\exists x\,(x \in z \wedge \phi(x,z)) \to \exists x \in z\,\phi(x,z)$. We must obtain terms t and q such that, for all $e, c \in W$ and for all a, b of appropriate types in the set-theoretical structure $\langle S_\rho \rangle$, and for all $z \in L_{|e|}$, the statement

$$\forall m \exists x \in L_c \forall n\,(x \in z \wedge \phi_{\mathrm{B}}(a,(tecab)\langle m\rangle\langle n\rangle,x,z)) \to \exists x \in z \forall n\,\phi_{\mathrm{B}}(qeca,b\langle n\rangle,x,z)$$

holds in $L^{\mathrm{mix}}_{\omega_1}$. This is clearly the case with $t :\equiv \lambda e,a,c,b.(b+1)$ and $q :\equiv \lambda e,c,a.a$.

Let us now turn to the mathematical axioms of $\mathsf{IKP}\omega$. Extensionality poses no problem because it is the universal closure of a bounded formula. The verification of the pairing, union and infinity axioms is like the verification done in [10]. For completeness, for these three axioms we need (respectively) closed terms t, q and r such that $\forall c,e \forall x \in L_c \forall y \in L_e \exists z \in L_{tce}\,(x \in z \wedge y \in z)$, $\forall c \forall x \in L_c \exists z \in L_{qc} \forall y \in x \forall w \in y\,(w \in z)$ and $\exists x \in L_r\,Lim(x)$, where $Lim(x)$ is a bounded formula which expresses that

x is a limit ordinal. The terms $t :\equiv \lambda c, e.(\max(c, e) + 1)$, $q :\equiv \lambda c.c$ and $r :\equiv \omega_\Omega + 1$ do the job.

The separation scheme is $\forall w \forall y \exists z \forall x\, (x \in z \leftrightarrow x \in y \wedge \phi(x, w))$, where ϕ is a bounded formula in which the variable z does not occur. Note that the inner universal statement can be considered bounded. A straightforward computation of the upper B-translation of this formula shows that we need a closed term t of type $\Omega \to (\Omega \to \Omega)$ such that

$$\forall c^\Omega, e^\Omega\, [\forall y \in L_c \forall w \in L_e \exists z \in L_{tce} \forall x\, (x \in z \leftrightarrow x \in y \wedge \phi(x, w))]$$

holds in $L^{\text{mix}}_{\omega_1}$. It is clear that the term $t :\equiv \lambda c, e.(\max(c, e) + 1)$ does the job. To see this, just take z to be $\{x \in L_\alpha : x \in y \wedge \phi(x, w)\}$, where $\alpha = \max(|c|, |e|)$.

The bounded collection scheme is a sub-scheme of the scheme of unrestricted collection Coll that will be discussed later. Let us now study the foundation rule. A computation of the upper-B translation of the premise of the induction rule shows that, by induction hypothesis, there are terms t and q such that

$$(*) \qquad \forall c^\Omega, a, b\, [\forall x \in L_c\, (\forall n \forall y \in x\, \phi_{\text{B}}(a, (tcab)\langle n \rangle, y) \to \phi_{\text{B}}(qca, b, x))]$$

holds in $L^{\text{mix}}_{\omega_1}$. We want to find a term r such that

$$(\star) \qquad L^{\text{mix}}_{\omega_1} \models \forall c^\Omega, b\, [\forall x \in L_c\, \phi_{\text{B}}(rc, b, x)]$$

Define by tree recursion the term r as follows: $r0_\Omega = 0$ and

$$r(Sup(f)) = \bigsqcup_k q(fk, r(fk)).$$

We now check $(\star)$ by transfinite induction on $|c|$. If $|c| = 0$ there is nothing to prove. Suppose that $c = Sup(f)$. Take b appropriate of type in the set-theoretical structure $\langle S_\rho \rangle$ and $x \in L_{|c|}$. Take $y \in x$. Since $L_{|c|} = \bigcup_k L_{|f(k)|+1}$, there is a natural number k_0 (which we identify with the corresponding type N term) such that $x \in L_{|f(k_0)|+1}$. By transfinite induction hypothesis,

$$L^{\text{mix}}_{\omega_1} \models \forall b \forall y \in L_{f(k_0)}\, \phi_{\text{B}}(r(f(k_0)), b, y)$$

because $|f(k_0)| < |c|$. In particular,

$$L^{\text{mix}}_{\omega_1} \models \forall b \forall y \in L_{f(k_0)} \forall n\, \phi_{\text{B}}(r(f(k_0)), (t(f(k_0), r(f(k_0)), b))\langle n \rangle, y).$$

Using the hypothesis $(*)$, we may conclude that

$$L^{\text{mix}}_{\omega_1} \models \forall b\, \phi_{\text{B}}(q(f(k_0), r(f(k_0))), b, x).$$

But, by Lemma 4, $q(f(k_0), r(f(k_0))) \sqsubseteq \bigsqcup_k q(f(k), r(f(k)))$ holds in $L^{\text{mix}}_{\omega_1}$. By the definition of r and c, we have $L^{\text{mix}}_{\omega_1} \models q(f(k_0), r(f(k_0))) \sqsubseteq rc$. Now, $(\star)$ follows using the monotonicity property of the existencial entry of ϕ_{B}.

It remains to check the principles Δ_0-LEM, MP, bIP_{Π_1}, Δ_0-CColl and Coll. Of course, Δ_0-LEM is trivially interpreted by itself. We now check these four principles:

MP. We must find a term t such that $\forall c^{\Omega}\,[\neg\forall n\forall x \in L_{c\langle n\rangle}\,\phi(x) \to \exists x \in L_{tc}\,\neg\phi(x)]$ holds in $L^{\mathrm{mix}}_{\omega_1}$. The identity term in type Ω works.

bIP$_{\Pi_1}$. The upper B-translation of the antecedent of this principle is

$$\exists c^{\Omega}, d, B\forall e\,[\forall k\forall x \in L_{(Be)\langle k\rangle}\,\phi(x) \to \exists y \in L_c\forall n\,\psi_{\mathrm{B}}(d, e\langle n\rangle, y)],$$

whereas the upper B-translation of the consequent is

$$\exists c^{\Omega}, d, B\forall e\,[\exists y \in L_c\forall m\,(\forall k\forall x \in L_{(B(e\langle m\rangle))\langle k\rangle}\,\phi(x) \to \exists z \in y\forall n\,\psi_{\mathrm{B}}(d, e\langle m\rangle\langle n\rangle, z))].$$

Therefore, we must find terms t, q, r and s such that for all $c \in W$ and d, B and e of appropriate types in the set-theoretical structure $\langle S_\rho\rangle$,

$$\forall m\,(\forall k\forall x \in L_{(B((tcdBe)\langle m\rangle))\langle k\rangle}\,\phi(x) \to \exists z \in L_c\forall n\,\psi_{\mathrm{B}}(d, (tcdBe)\langle m\rangle\langle n\rangle, z)) \to$$

$$\exists y \in L_{qcdB}\forall m\,(\forall k\forall x \in L_{((scdB)(e\langle m\rangle))\langle k\rangle}\,\phi(x) \to \exists z \in y\forall n\,\psi_{\mathrm{B}}(rcdB, e\langle m\rangle\langle n\rangle, z))$$

holds in $L^{\mathrm{mix}}_{\omega_1}$. It does hold with $t :\equiv \lambda c, d, B, e.e$, $s :\equiv \lambda c, d, B.B$, $q :\equiv \lambda c, d, B.(c + 1)$ and $r :\equiv \lambda c, d, B.d$. To see this we have to check that

$$\forall c^{\Omega}, d, B, e\,[\forall m\,(\forall k\forall x \in L_{(B(e\langle m\rangle))\langle k\rangle}\,\phi(x) \to \exists z \in L_c\forall n\,\psi_{\mathrm{B}}(d, e\langle m\rangle\langle n\rangle, z)) \to$$

$$\exists y \in L_{c+1}\forall m\,(\forall k\forall x \in L_{(B(e\langle m\rangle))\langle k\rangle}\,\phi(x) \to \exists z \in y\forall n\,\psi_{\mathrm{B}}(d, e\langle m\rangle\langle n\rangle, z))]$$

holds in $L^{\mathrm{mix}}_{\omega_1}$. In effect, if we assume the antecedent, then the consequent is seen to immediately hold with $y = L_c$.

Δ_0-CColl. According to the upper B-translation of this principle, we need a term t such that

$$\forall d^{\Omega}, c^{\Omega}\,[\forall w \in L_d\,(\forall n\forall z \in L_{(tdc)\langle n\rangle}\exists y \in w\forall x \in z\,\phi(x, y) \to \exists y \in w\forall n\forall x \in L_{c\langle n\rangle}\,\phi(x, y))]$$

holds in $L^{\mathrm{mix}}_{\omega_1}$. It is easy to argue that $t :\equiv \lambda d, c.(c+2)$ works. One just has to instantiate the antecedent with $z = L_c$ (and $n = 0$, say) to see that the consequent holds.

Coll. The upper B-translation of the antecedent of this principle is

$$\exists c^{\Omega}, a\forall b\,[\forall y \in w\exists x \in L_c\forall n\,\phi_{\mathrm{B}}(a, b\langle n\rangle, x, y)],$$

and the upper B-translation of the consequent is

$$\exists d^{\Omega}, a\forall b\,[\exists z \in L_d\forall k\forall y \in w\exists x \in z\forall n\,\phi_{\mathrm{B}}(a, b\langle k\rangle\langle n\rangle, x, y)].$$

Therefore, we must obtain terms t, q and r such that

$$\forall e^{\Omega}, c^{\Omega}, a, b\,[\forall w \in L_e\,(\forall k\forall y \in w\exists x \in L_c\forall n\,\phi_{\mathrm{B}}(a, (tecab)\langle k\rangle\langle n\rangle, x, y) \to$$

$$\exists z \in L_{qeca}\forall k\forall y \in w\exists x \in z\forall n\,\phi_{\mathrm{B}}(reca, b\langle k\rangle\langle n\rangle, x, y))]$$

holds in $L^{\mathrm{mix}}_{\omega_1}$. Just take $t :\equiv \lambda e, c, a, b.b$, $q :\equiv \lambda e, c, a.(c + 1)$ and $r :\equiv \lambda e, c, a.a$. Obviously, the consequent becames true (given the antecedent) with $z = L_c$.

$\square$

The following proposition is an immediate consequence of the Soundness Theorem and of (ii) of Lemma 7:

Proposition 4. *If* $\mathsf{IKP}\omega + \Delta_0\text{-}\mathsf{LEM} + \mathsf{MP} + \mathsf{bIP}_{\Pi_1} + \Delta_0\text{-}\mathsf{CColl} + \mathsf{Coll} \vdash \forall x \exists y\, \phi(x, y)$*, where* ϕ *is a bounded formula (x and y are the only free variables), then there is a closed term t of type* $\Omega \to \Omega$ *such that*

$$L^{\text{mix}}_{\omega_1} \models \forall c^{\Omega} \forall x \in L_c \exists y \in L_{tc}\, \phi(x, y).$$

Moreover, $L_{\text{BH}} \models \forall x \exists y\, \phi(x, y)$*, where* BH *is the Bachmann-Howard ordinal.*

In the last conclusion, one uses the absoluteness of bounded formulas. The same for the following:

Corollary 3. *If* $\mathsf{IKP}\omega + \Delta_0\text{-}\mathsf{LEM} + \mathsf{MP} + \mathsf{bIP}_{\Pi_1} + \mathsf{Coll} + \Delta_0\text{-}\mathsf{CColl} \vdash \exists x\, \phi(x)$*, where* ϕ *is a bounded formula (x is the only free variable), then there is an ordinal* α *smaller than the Bachmann-Howard ordinal such that* $L_\alpha \models \exists x\, \phi(x)$*.*

A part of Proposition 4 can be much improved because the translations that define the functional interpretation are correct in $L^{\text{mix}}_{\omega_1}$ (however, the improvement does not seem susceptible to an internalization like the one in section 6 of [10]). In order to discuss this improvement it is convenient to permit also unbounded set-theoretic quantifications in the mixed language. With these quantifications, we refer to the language as the *extended* mixed language.

Lemma 9. *For every formula* ϕ *of the language of set theory,* $L^{\text{mix}}_{\omega_1} \models \phi \leftrightarrow \phi^{\text{B}}$*.*

Proof. The proof is by induction on the complexity of ϕ. The result is clear for ϕ bounded, for the conjunction and also for the disjunction. For the latter one, just use the fact that for every $b \in S_\rho$, the equality $(b+1)\langle n \rangle = b$ holds set-theoretically. The remaining cases follow from the following fact:

<u>Fact</u>. Let $z \in L_{\omega_1}$, ρ a pure Ω-type and ϕ a formula of the extended mixed language. Then

$$L^{\text{mix}}_{\omega_1} \models \forall x \in z\, \exists a^\rho\, \phi(a, x, z) \to \exists a\, \forall x \in z\, \exists n\, \phi(a\langle n \rangle, x, z).$$

The proof of the fact is easy. Let $z \in L_{\omega_1}$. Suppose that $L^{\text{mix}}_{\omega_1} \models \forall x \in z\, \exists a^\rho\, \phi(a, x, z)$. Since z is countable, we can take an enumeration $(x_n)_{n \in \omega}$ of the elements of z. For each $n \in \omega$, choose $a_n \in S_\rho$ be such that $L^{\text{mix}}_{\omega_1} \models \phi(a_n, x_n, z)$. Call f this function $n \rightsquigarrow a_n$. By (viii) of Section 3, $L^{\text{mix}}_{\omega_1} \models \forall n^N ((Sup f)\langle n \rangle = a_n)$. It is now clear that we can take for a the element $Sup f$. □ (end of proof of fact)

Let us study the universal bounded quantifier. We need to show that

$$L^{\text{mix}}_{\omega_1} \models \forall x \in z\, \exists a \forall b\, \phi_{\text{B}}(a, b, x, z) \leftrightarrow \exists a \forall b \forall x \in z\, \phi_{\text{B}}(a, b, x, z).$$

Only the left-to-right direction needs to be argued. Assume the antecedent. By the Fact, $\exists a \forall x \in z\, \exists n \forall b\, \phi_{\text{B}}(a\langle n \rangle, b, x, z)$. By Lemma 4 and the monotonicity lemma, we easily get $\forall b \forall x \in z\, \phi_{\text{B}}(\tilde{a}, b, x, z)$, where $\tilde{a}$ is $\bigsqcup \lambda x^N.(a\langle x \rangle)$.

Regarding the existential bounded quantifier, we must show that

$$L^{\mathrm{mix}}_{\omega_1} \models \exists x \in z \exists a \forall b\, \phi_{\mathrm{B}}(a,b,x,z) \leftrightarrow \exists a \forall b\, \exists x \in z \forall n\, \phi_{\mathrm{B}}(a, b\langle n\rangle, x, z).$$

This time, the left-to-right direction is trivial. So, assume the right-hand side and take an element a such that $\forall b\, \exists x \in z \forall n\, \phi_{\mathrm{B}}(a, b\langle n\rangle, x, z)$. We use the contrapositive of the Fact in order to we get what we want.

Let us consider the universal quantifier. We must show that

$$L^{\mathrm{mix}}_{\omega_1} \models \forall x \exists a^{\rho} \forall b\, \phi_{\mathrm{B}}(a,b,x) \leftrightarrow \exists A \forall c^{\Omega}, b \forall x \in L_c\, \phi_{\mathrm{B}}(Ac, b, x).$$

Suppose the antecendent. Hence, for all elements $c \in W$, $\forall x \in L_c \exists a \forall b\, \phi_{\mathrm{B}}(a,b,x)$. By the discussion of the universal bounded quantifier case, we get that for all $c \in W$ there is $a \in S_{\rho}$ such that $\forall x \in L_c \forall b\, \phi_{\mathrm{B}}(a,b,x)$. Take a function $A : W \to S_{\rho}$ (that is, an element of $S_{\Omega\to\rho}$) such that, for all elements c in W, $\forall b \forall x \in L_c\, \phi_{\mathrm{B}}(Ac, b, x)$. This A works. Now, assume the right-hand side. Let $A \in S_{\Omega\to\rho}$ such that, for all $c \in W$ one has $\forall b \forall x \in L_c\, \phi_{\mathrm{B}}(A(c), b, x)$. Take x an arbitrary element of L_{ω_1}. Then there is $c \in W$ such that $x \in L_{|c|}$ (we are using the fact that $\omega_1 = \sup_{c\in W} |c|$). It is clear that if we take a to be $A(c)$, we get $\forall b\, \phi_{\mathrm{B}}(a,b,x)$.

For the existential quantifier we must show that

$$L^{\mathrm{mix}}_{\omega_1} \models \exists x \exists a^{\rho} \forall b\, \phi_{\mathrm{B}}(a,b,x) \leftrightarrow \exists c^{\Omega}, a \forall b \exists x \in L_c \forall n\, \phi_{\mathrm{B}}(a, b\langle n\rangle, x).$$

The left-to-right direction follows from the fact $L_{\omega_1} = \bigcup_{c\in W} L_{|c|}$. To see the other direction, take $c \in W$ and $a \in S_{\rho}$ such that $\forall b \exists x \in L_c \forall n\, \phi_{\mathrm{B}}(a, b\langle n\rangle, x)$. By the contrapositive of the Fact, we obtain $\exists x \in L_c \forall b\, \phi_{\mathrm{B}}(a,b,x)$ and, therefore, the left-hand side.

It remains to check the implication. We must argue that the following holds in $L^{\mathrm{mix}}_{\omega_1}$:

$$(\exists a \forall b\, \phi_{\mathrm{B}}(a,b) \to \exists d \forall e\, \psi_{\mathrm{B}}(d,e)) \leftrightarrow \exists B, D \forall a, e\, (\forall n\, \phi_{\mathrm{B}}(a, (Bae)\langle n\rangle) \to \psi_{\mathrm{B}}(Da, e)).$$

Note that $\forall b\, \phi_{\mathrm{B}}(a,b)$ is equivalent to $\forall b \forall n\, \phi_{\mathrm{B}}(a, b\langle n\rangle)$. With this in mind, an approriate (partial) prenexification of the left-hand side of the equivalence yields

$$\forall a \exists d \forall e \exists b\, (\forall n\, \phi_{\mathrm{B}}(a, b\langle n\rangle) \to \psi_{\mathrm{B}}(d,e)).$$

The equivalence follows using two applications of the axiom of choice in the structure (S_{ρ}). □

Proposition 5. *If* $\mathsf{IKP}\omega + \Delta_0\text{-}\mathsf{LEM} + \mathsf{MP} + \mathsf{bIP}_{\Pi_1} + \mathsf{Coll} + \Delta_0\text{-}\mathsf{CColl} \vdash \forall x \exists y\, \phi(x,y)$, *where ϕ is an arbitrary formula of the language of set theory (x and y are the only free variables), then there is a closed term t of type $\Omega \to \Omega$ such that*

$$\forall c \in W \forall x \in L_{|c|} \exists y \in L_{|t(c)|}\ L_{\omega_1} \models \phi(x,y).$$

Proof. Let $\phi(x,y)^{\mathrm{B}}$ be $\exists a^{\tau} \forall b^{\rho} \phi_{\mathrm{B}}(a,b,x,y)$. It is easy to see that $(\forall x \exists y\, \phi(x,y))^{\mathrm{B}}$ is

$$\exists A, D \forall c^{\Omega}, b \forall x \in L_c \exists y \in L_{Dc} \forall n\, \phi_{\mathrm{B}}(Ac, b\langle n\rangle, x, y).$$

By the Soundness theorem, there are closed terms t and q such that

$$\forall c^{\Omega} \forall x \in L_c \forall b\, \exists y \in L_{tc} \forall n\, \phi_{\mathrm{B}}(qc, b\langle n\rangle, x, y)$$

holds in $L^{\mathrm{mix}}_{\omega_1}$. By the contrapositive of the Fact of the previous lemma, we get:

$$\forall c^{\Omega}\, \forall x \in L_c \exists y \in L_{tc} \forall b\, \phi_{\mathrm{B}}(qc, b, x, y).$$

In particular, $\forall c \forall x \in L_c \exists y \in L_{tc} \exists a \forall b\, \phi_{\mathrm{B}}(a, b, x, y)$, i.e., $\forall c \forall x \in L_c \exists y \in L_{tc}\, \phi^{\mathrm{B}}(x, y)$. By the previous lemma, we are done. □

6 Adding strict-Π^1_1 reflection

Admissibility is the playground where finiteness, recursive enumerability and other recursion-theoretic notions find a fertile ground for generalization (see, for instance, the preface of [18]). Weak Kőnig's lemma is an important (second-order) principle in recursion theory (viz., in relation with low degrees and compactness) as well as in subsystems of second-order arithmetic and reverse mathematics (cf. [20]). The principle of strict-Π^1_1 reflection, introduced by Barwise in [3] and [4], is a natural generalization of weak Kőnig's lemma from the arithmetical setting to the admissible setting. Strict-Π^1_1 reflection can be stated as follows:

$$\forall X \exists x\, \phi(x, X) \to \exists z \forall X \exists x \in z\, \phi(x, X),$$

where ϕ is a bounded formula (for details see chapter VIII of [5]). In the second part of [10], we extended the Σ-ordinal analysis of $\mathsf{KP}\omega$ to a second-order theory with the principle of strict-Π^1_1 reflection. We proved the novel result that the Σ-ordinal of this second-order theory is still the Bachmann-Howard ordinal. The main idea for this analysis was an extension of the functional interpretation in which second-order quantifications are treated as bounded quantifications. It may sound surprising at first that this treatment works because the transformations of formulas underlying the functional intepretation of the second-order quantifiers are not truth-preserving in our semantics (more about this later). However, on second thought – for a person familiar with the bounded functional interpretation of [11] – the idea is compelling.

In this paper we are interested in analysing theories based on intuitionistic logic. Some phenomena absent in the classical setting emerge in the intuitionistic theories. As we saw, the bounded collection scheme of $\mathsf{KP}\omega$ is inflated into unrestricted collection Coll (while the non-intuitionistic classical contrapositive of bounded collection Δ_0-CColl is kept). A similar sort of thing happens with strict-Π^1_1 reflection (see Definitions 12 and 13). The inflation also extends to the notion of bounded formula, now that second-order quantifications are regarded as bounded. There is no apparent reason to stick any longer to the usual notion of bounded formula.

Let us set up the basic second-order theories $\mathsf{IKP}\omega_2\!\restriction$ and $\mathsf{IKP}\omega_2$ (the latter is discussed in the next section). The language of second-order set theory is the enlargement of the language of set theory (as described in Section 2) with monadic second-order quantification (we have both universal and existential quantifiers). We use capital letters X, Y, Z, ... for the monadic predicates and call them *classes*. As it is common usage, we write '$x \in X$' instead of the (syntactically correct) '$X(x)$'. This is an abuse of notation because the membership sign in the expression '$x \in X$' is not the membership sign of the language of (first-order) set theory. It is just a harmless and felicitous notational device, and we read '$x \in X$' as saying that x is a member of (the class) X.

Definition 8. *The class of* $\Delta_0^{\mathcal{C}}$-formulas *of the language of second-order set theory is the smallest class of formulas that contains the atomic formulas* $x \in y$, $x = y$, $x \in X$, *the absurdity, and which is closed under propositional connectives, the bounded quantifiers* $\forall x \in y$ *and* $\exists x \in y$, *and the second-order quantifiers* $\forall X$ *and* $\exists X$.

In [19], Salipante uses the notation Δ_0^{C}, with a roman letter 'C', for a more restricted version of bounded formula, namely for Δ_0-formulas in which second-order parameters are permitted (second-order quantifications are *not* allowed). In order to distinguish our notion from Salipante's, we use a caligraphic '$\mathcal{C}$' instead. This caligraphic notation has also the advantage of cohering with a notation of Rathjen in [17] that we will be needing in the next section.

We denote the law of excluded middle restricted to $\Delta_0^{\mathcal{C}}$-formulas by $\Delta_0^{\mathcal{C}}\text{-LEM}$.

Definition 9. *The second-order theory* $\mathsf{IKP}\omega_2\!\upharpoonright$ *is the intuitionistic theory of the language of second-order set theory that contains* $\mathsf{IKP}\omega$ *and extends the scheme of foundation in order to permit* all *the formulas of the new language.*

Some comments are in order. In the above, we did not change the original schemata of separation and bounded collection. They remain exactly as in $\mathsf{IKP}\omega$. We could have opted for allowing in the collection scheme the wider class of $\Delta_0^{\mathcal{C}}$-formulas and, also, second-order parameters. This would be more in line with the definition of $\mathsf{KP}\omega_2\!\upharpoonright$ in [10]. However, this enlarged collection scheme is a particular case of the unrestricted collection scheme $\mathsf{Coll}^{\mathcal{C}}$ given in (d) of Definition 11, and our functional interpretation is able to realize it. The *point of attention* is really the separation scheme of Definition 9. It is the *original* formulation of the separation scheme, without second-order parameters and with the original bounded (i.e., Δ_0) formulas. That explains the restriction sign in the acronym of the theory. As we will see, the Σ-ordinal of the restricted theory together with the $\Delta_0^{\mathcal{C}}\text{-LEM}$, some comprehension for second-order class formation and strict-Π_1^1 reflection is still the Bachmann-Howard ordinal. On the other hand, Salipante showed that if *second-order parameters* are allowed in the separation scheme (never mind allowing for $\Delta_0^{\mathcal{C}}$-matrices) then, in the presence of suitable comprehension and strict-Π_1^1 reflection, one is able to prove the powerset axiom. This will be discussed in the next section (see Theorem 4). Antecipating the results of that section, we may add that with the help of other interpretable principles Salipante's result gives rise to a very strong theory, namely to (an intuitionistic version of) the so-called power Kripke-Platek set theory $\mathsf{KP}\omega(\mathcal{P})$, as described in [17].

Next, we introduce the principles of class comprehension that will be of our interest (notice the analogy with similar principles in subsystems of second-order arithmetic). A $\Sigma_1^{\mathcal{C}}$-formula is a formula of the form $\exists x \varphi(x)$, where $\varphi(x)$ is a $\Delta_0^{\mathcal{C}}$-formula. The notion of $\Pi_1^{\mathcal{C}}$-formula is defined dually.

Definition 10. *The following schemata are defined in the second-order language of set theory (first and second-order parameters are allowed):*

I. *The scheme* $\Delta_0^{\mathcal{C}}\text{-CA}$ *is* $\exists X \forall x\, (x \in X \leftrightarrow \phi(x))$, *where* $\phi(x)$ *is a* $\Delta_0^{\mathcal{C}}$*-formula (X is a fresh variable).*

II. *The scheme* $\Delta_1^{\mathcal{C}}\text{-CA}$ *is* $\forall x\, (\phi(x) \leftrightarrow \psi(x)) \to \exists X \forall x\, (x \in X \leftrightarrow \phi(x))$, *where* $\phi(x)$ *is a* $\Sigma_1^{\mathcal{C}}$*-formula and* $\psi(x)$ *is a* $\Pi_1^{\mathcal{C}}$*-formula (X is a fresh variable).*

We now list some principles which our functional interpretation is able to realize:

Definition 11. *The following schemata are defined in the second-order language of set theory (in all the schemata below, both first and second-order parameters are allowed):*

(a) Markov's principle MP^C *is the scheme* $\neg\forall x\,\phi(x) \to \exists x\,\neg\phi(x)$*, for* ϕ *a* Δ_0^C*-formula.*

(b) *The* independence of premises principle $\mathsf{bIP}_{\Pi_1^C}$ *is the scheme*

$$(\forall x\,\phi(x) \to \exists y\,\psi(y)) \to \exists y\,(\forall x\,\phi(x) \to \exists z \in y\,\psi(z)),$$

where ϕ *is a* Δ_0^C*-formula and* ψ *is* any *formula of the second-order language.*

(c) *The principle of* bounded contra-collection Δ_0^C-CColl *is the scheme*

$$\forall z \exists y \in w \forall x \in z\,\phi(x,z) \to \exists y \in w \forall x \phi(x,y),$$

where ϕ *is a* Δ_0^C*-formula.*

(d) *The principle of (unrestricted)* collection Coll^C *is the scheme*

$$\forall y \in w \exists x\,\phi(x,y) \to \exists z \forall y \in w \exists x \in z\,\phi(x,y),$$

where ϕ *is* any *formula of the second-order language.*

Some of the results of Section 2 adapt to the new setting. The following result is analogous to the law of excluded middle of Corollary 1. The proof of that corollary used a separation result that we do not have in our present setting, due to the restrictions of the separation scheme discussed above. However a direct proof is forthcoming (moreover, Markov's principle is not needed, as observed by Fujiwara):

Proposition 6. *The theory* $\mathsf{IKP}\omega_2\!\restriction + \Delta_0^C$-$\mathsf{LEM} + \mathsf{bIP}_{\Pi_1^C}$ *proves the* Δ_1^C *law of excluded middle, i.e., it proves* $(\forall u\,\phi(u) \leftrightarrow \exists v\,\psi(v)) \to (\forall u\,\phi(u) \vee \neg\forall u\,\phi(u))$*, for* Δ_0^C*-formulas* ϕ *and* ψ *(possibly with first and second-order parameters).*

Proof. Suppose that $\forall u\,\phi(u) \leftrightarrow \exists v\,\psi(v)$. Applying $\mathsf{bIP}_{\Pi_1^C}$ to the left-to-right direction of the equivalence, there is v_0 such that $\forall u\,\phi(u) \to \exists v \in v_0\,\psi(v)$. If $\exists v \in v_0\,\psi(v)$, our supposition entails $\forall u\,\phi(u)$. If $\neg\exists v \in v_0\,\psi(v)$, we directly conclude that $\neg\forall u\,\phi(u)$. □

A version of the lesser limited principle of omniscience also holds in the present setting (first and second-order parameters are allowed in the following, of course). The proof is analogous to the proof of Corollary 2.

Proposition 7. *If* ϕ *and* ψ *are* Δ_0^C*-formulas, then*

$$\mathsf{IKP}\omega_2\!\restriction +\ \Delta_0^C\text{-}\mathsf{LEM} + \Delta_0^C\text{-}\mathsf{CColl} \vdash \forall u, v\,(\phi(u) \vee \psi(v)) \to \forall u\,\phi(u) \vee \forall v\,\psi(v).$$

The appropriate version of the strict-Π_1^1 reflection scheme in our intuitionistic setting comes in two installments. The reader will notice that they are like collection schemes. The first one is a version of the contrapositive of strict-Π_1^1 reflection. The second one is a vast generalization, only possible because we are in an intuitionistic setting. Of course, each installment entails strict-Π_1^1 reflection (the first one classically, the second intuitionistically).

Definition 12. *The principle of* bounded class contra-collection Δ_0^C-CColl_2 *is the scheme*

$$\forall z \exists X \forall x \in z\, \phi(x, X) \to \exists X \forall x\, \phi(x, X),$$

where ϕ is a Δ_0^C-bounded formula (possibly with first and second-order parameters).

It was argued in [10] that with the aid of strict-Π_1^1 reflection, the bounded comprehension scheme upgrades to a Δ_1-comprehension scheme. A similar result holds in the present setting.

Proposition 8. $\mathsf{IKP}\omega_2\!\upharpoonright + \Delta_0^C$-$\mathsf{LEM} + \Delta_0^C$-$\mathsf{CA} + \Delta_0^C$-$\mathsf{CColl}_2 \vdash \Delta_1^C$-$\mathsf{CA}$.

Proof. The proof is *mutatis mutandis* the argument for lemma 5.3 of [10]. Suppose that $\forall u\, (\exists y\, \phi(u, y) \leftrightarrow \forall z\, \psi(u, z))$, where ϕ and ψ are Δ_0^C-formulas. Then,

$$\forall w \exists X \forall x \in w \forall u, y, z \in x\, ((\phi(u, y) \to u \in X) \wedge (u \in X \to \psi(u, v))).$$

It is easy to argue this. Given w, take $\tilde{w}$ its transitive closure. Clearly, we can take X to be $\{u : \exists y \in \tilde{w}\, \phi(u, y)\}$. Note that this class exists by Δ_0^C-CA.

By Δ_0^C-CColl, we get $\exists X \forall x \forall u, y, z \in x\, ((\phi(u, y) \to u \in X) \wedge (u \in X \to \psi(u, z)))$. Clearly, this X is formed by the elements u that satisfy $\exists y\, \phi(u, y)$. □

We have said that Δ_0^C-CColl_2 is like a collection scheme. In fact, it generalizes Δ_0^C-CColl:

Proposition 9. $\mathsf{IKP}\omega_2\!\upharpoonright + \Delta_0^C$-$\mathsf{LEM} + \Delta_0^C$-$\mathsf{CA} + \Delta_0^C$-$\mathsf{CColl}_2 \vdash \Delta_0^C$-$\mathsf{CColl}$.

Proof. Let ϕ be a Δ_0^C-formula and suppose that $\forall z \exists y \in w \forall x \in z\, \phi(x, y)$. We claim that $\forall z \exists X \forall x \in z\, [\exists y \in w\, (y \in X \wedge \forall u \in w\, (u \in X \to u = y) \wedge \phi(x, y))]$. Given z, by the supposition there is $y_0 \in w$ such that $\forall x \in z\, \phi(x, y_0)$. We just have to take X to be the singleton class formed by y_0 (it exists by Δ_0^C-CA). Since the formula between square parenthesis is a Δ_0^C-formula, we can apply Δ_0-CColl_2 in order to get

$$\exists X \forall x\, [\exists y \in w\, (y \in X \wedge \forall u \in w\, (u \in X \to u = y) \wedge \phi(x, y))].$$

Clearly, $X \cap w$ must be a singleton (i.e., this *class* has only one element). Let y_0 be the only element of this class. We get $y_0 \in w$ and $\forall x\, \phi(x, y_0)$. □

The second installment of strict-Π_1^1 reflection is the following:

Definition 13. *The principle of (unrestricted)* class collection Coll_2^C *is the scheme*

$$\forall X \exists x\, \phi(x, X) \to \exists z \forall X \exists x \in z\, \phi(x, X),$$

where ϕ is any *formula (possibly with first and second-order parameters).*

At this point, the following proposition should not be surprising:

Proposition 10. $\mathsf{IKP}\omega_2\!\upharpoonright + \Delta_0^C$-$\mathsf{LEM} + \Delta_0^C$-$\mathsf{CA} + \mathsf{Coll}_2^C \vdash \mathsf{Coll}^C$.

Proof. Let ϕ be a Δ_0^C-formula and suppose that we have $\forall y \in w \exists x\, \phi(x, y)$. We claim that $\forall X \exists x\, [\forall y \in w\, (y \in X \wedge \forall u \in w\, (u \in X \to u = y) \to \phi(x, y))]$. Given X, either there is $y \in w$ such that $y \in X \wedge \forall u \in w\, (u \in X \to u = y)$ or not (the previous formula is bounded and, hence, we can apply Δ_0^C-LEM). In the first case, by supposition, there is x such that $\phi(x, y)$ and we are done. If not, the assertion is trivially true.

By Coll_2^C, there is z_0 such that

$$\forall X \exists x \in z_0\, [\forall y \in w\, (y \in X \wedge \forall u \in w\, (u \in X \to u = y) \to \phi(x, y))].$$

Given $y \in w$, take X to be the singleton class formed by y. Clearly, $\exists x \in z_0\, \phi(x, y)$. □

The theory $\mathsf{IKP}\omega_2\!\upharpoonright + \Delta_0^C\text{-}\mathsf{LEM} + \Delta_0^C\text{-}\mathsf{CA} + \mathsf{MP}^C + \mathsf{bIP}_{\Pi_1^C} + \Delta_0^C\text{-}\mathsf{CColl}_2 + \mathsf{Coll}_2^C$ (which, as we saw, includes Δ_0^C-CColl and Coll^C) is very robust. For instance, modulo this theory, the Σ_1^C-formulas and the Π_1^C-formulas enjoy strong closure properties and this permits the smooth introduction of Δ_1^C-relation symbols and of Σ_1^C-function symbols. In effect, modulo the above theory, the Σ_1^C-formulas are closed under conjunctions, disjunctions, bounded quantifications, second-order quantifications and existential (first-order) quantifications. The closure under conjunctions, disjunctions and bounded, second-order and unbounded existential quantifications is clear. The closure under bounded and second-order universal quantifications follows from Coll^C and Coll_2^C, respectively. Dually, the Π_1^C-formulas are closed under conjunctions, disjunctions, bounded quantifications, second-order quantifications and universal (first-order) quantifications. The closure under conjunctions, bounded, second-order and unbounded universal quantifications is clear. The closure under disjunction is a consequence of Proposition 7. The closure under bounded and second-order existential quantifications follows from Δ_0^C-CColl and Δ_0^C-CColl_2, respectively. The introduction of the powerset operation in the next section uses these facts crucially. They are essential for the interpretation of the power Kripke-Platek set theory in a theory based on $\mathsf{IKP}\omega_2$.

The functional interpretation given in Definition 7 is extended to the second-order language by the following two clauses.

9. $(\forall X\, \phi(X))^{\mathrm{B}}$ is $\exists a \forall b\, [\forall X\, \phi_{\mathrm{B}}(a, b, X)]$,

10. $(\exists X\, \phi(X))^{\mathrm{B}}$ is $\exists a \forall b\, [\exists X \forall n\, \phi_{\mathrm{B}}(a, b\langle n \rangle, X)]$.

Notice that now the lower B-translations of formulas include second-order quantifications. The notion of bounded mixed formula of Section 4 has to be generalized to the notion of *second-order bounded mixed formula* in which closure under second-order quantifications is also allowed. The formulas of the second-order mixed language $\mathcal{L}_\Omega^{\mathrm{mix}^C}$ are defined accordingly, as those that are generated from the second-order bounded formulas by means of propositional connectives and quantifications of the form $\forall a^\rho$, where a is a term variable (of a certain type ρ) of the term language $\mathcal{L}_\Omega$. Notice that, as before, we only need a set of classical connectives (our verifying semantics – described in the next paragraph – is classical) and that unbounded set-theoretic quantifiers are not present in the language $\mathcal{L}_\Omega^{\mathrm{mix}^C}$.

As with the soundness theorem of Section 2, the soundness theorem of this section is verified semantically. There are several ways of extending the structures $L_{\omega_1}^{\mathrm{mix}}$ and

$V^{\text{mix}}_{\omega_1}$ to the second-order setting. The first semantics that we consider is $(L^{\text{mix}^C}_{\omega_1}, \mathcal{P}(L_{\omega_1})\restriction)$. In this semantics, the second-order variables range over $\mathcal{P}(L_{\omega_1})\restriction$, i.e., over the sets of the form $x \cap L_\alpha$, where $x \subseteq L_{\omega_1}$ and $\alpha < \omega_1$. There is a subtlety here. The intuitive second-order semantics is $\mathcal{P}(L_{\omega_1})$, not the subsets of L_{ω_1} truncated at a certain level α ($\alpha < \omega_1$) of the construtible hierarchy. However, the truncated semantics is enough. A second semantics that we will briefly consider is $(L^{\text{mix}^C}_{\omega_1}, L_{\omega_1})$. In this semantics the values of second-order variables range over *elements* of L_{ω_1}, i.e., over the sets in L_{ω_1}. This semantics is even subtler because both first-order set variables and second-order class variables range over the same domain, viz. L_{ω_1}. It was, in fact, the semantics used in [10]. In the next section we will consider the structure $(V^{\text{mix}^C}_{\omega_1}, V_{\omega_1})$. In this structure, the terms t of $\mathcal{L}_\Omega$ of type Ω index the (countable) stages V_t of the cumulative hierarchy. Of course, by this is meant that V_t is interpreted as $V_{|t|}$, as discussed at the end of Section 4. On the other hand, the second-order class variables range over elements of V_{ω_1} (the same range as the first-order set variables). Note that this range can also be described as being constituted by the sets of the form $x \cap V_\alpha$, where $x \subseteq V_{\omega_1}$ and $\alpha < \omega_1$.

Theorem 3 (Second-order soundness theorem I)**.** *Let ϕ be a sentence of the language of second-order set theory. Suppose that*

$$\mathsf{IKP}\omega_2\restriction + \Delta^C_0\text{-}\mathsf{LEM} + \Delta^C_0\text{-}\mathsf{CA} + \mathsf{MP}^C + \mathsf{bIP}_{\Pi^C_1} + \Delta^C_0\text{-}\mathsf{CColl}_2 + \mathsf{Coll}^C_2 \vdash \phi.$$

Then there are closed terms t of $\mathcal{L}_\Omega$ such that, for appropriate types ρ,

$$(L^{\text{mix}^C}_{\omega_1}, \mathcal{P}(L_{\omega_1})\restriction) \models \forall b^\rho\, \phi_B(t, b).$$

Proof. The proof is by induction on the length of the derivation. We show that if a formula $\phi(w, W)$ is provable in the theory of the theorem, then there are closed terms t of $\mathcal{L}_\Omega$ such that, for appropiate types ρ, we have

$$L^{\text{mix}}_{\omega_1} \models \forall c^\Omega \forall b^\rho\, [\forall W \forall w \in L_c\, \phi_{\text{B}}(tc, b, w, W)],$$

where $\phi(w, W)^{\text{B}}$ is $\exists a \forall b\, \phi_{\text{B}}(a, b, w, W)$.

The various verifications are, *mutatis mutandis*, the ones given in the proof of Theorem 2. We only need to complement the verifications with the study of the logical rules for the second-order quantifiers and the principles Δ^C_0-CColl_2, Coll^C_2 and Δ^C_0-CA. We use the same layout as in the proof of Theorem 2. Let us start with the four new axioms and rules for second-order quantifiers.

13. $\phi \to \psi(W) \Rightarrow \phi \to \forall W\, \psi(W)$, where W is not free in ϕ. By induction hypothesis, there are terms q and r such that

$$\forall a, e\, [\forall W\, (\forall n\, \phi_{\text{B}}(a, (qae)\langle n\rangle) \to \psi_{\text{B}}(ra, e, W))]$$

holds in $(L^{\text{mix}^C}_{\omega_1}, \mathcal{P}(L_{\omega_1})\restriction)$. We must obtain terms t and s such that

$$\forall a, e\, [\forall n\, \phi_{\text{B}}(a, (sae)\langle n\rangle) \to \forall W\, \psi_{\text{B}}(ta, e, W)]$$

holds in $(L^{\text{mix}^C}_{\omega_1}, \mathcal{P}(L_{\omega_1})\restriction)$. Just take $t :\equiv r$ and $s :\equiv q$.

14. $\forall X\,\phi(X) \to \phi(W)$. A computation of the upper-B translation of this formula shows that we must find terms q and r such that

$$\forall a, b' \,[\forall W\,(\forall n \forall X\,\phi_{\mathrm{B}}(a,(qab')\langle n\rangle, X) \to \phi_{\mathrm{B}}(ra, b', W))]$$

holds in $(L_{\omega_1}^{\mathrm{mix}^C}, \mathcal{P}(L_{\omega_1})\!\upharpoonright)$. Just put $q :\equiv \lambda a, b'.(b'+1)$ and $r :\equiv \lambda a.a$.

15. $\phi(W) \to \exists X\,\phi(X)$. A computation of the upper-B translation of this formula shows that we must find terms q and r such that

$$\forall a, b' \,[\forall W\,(\forall n\,\phi_{\mathrm{B}}(a,(qab')\langle n\rangle, W) \to \exists X \forall n\,\phi_{\mathrm{B}}(ra, b'\langle n\rangle, X))]$$

holds in $(L_{\omega_1}^{\mathrm{mix}^C}, \mathcal{P}(L_{\omega_1})\!\upharpoonright)$. Just put $q :\equiv \lambda a, b'.b'$ and $r :\equiv \lambda a.a$.

16. $\phi(W) \to \psi \;\Rightarrow\; \exists W\,\phi(W) \to \psi$, where W is not free in ψ. By induction hypothesis, there are terms q and r such that

$$\forall a, e[\forall W\,(\forall n\,\phi_{\mathrm{B}}(a,(qae)\langle n\rangle, W) \to \psi_{\mathrm{B}}(ra, e))]$$

holds in $(L_{\omega_1}^{\mathrm{mix}^C}, \mathcal{P}(L_{\omega_1})\!\upharpoonright)$. We must obtain terms t and s such that

$$\forall a, e\,[\forall n \exists W \forall k\,\phi_{\mathrm{B}}(a,(sae)\langle n\rangle\langle k\rangle, W) \to \psi_{\mathrm{B}}(ta, e)]$$

holds in $(L_{\omega_1}^{\mathrm{mix}^C}, \mathcal{P}(L_{\omega_1})\!\upharpoonright)$. Just take $t :\equiv r$ and $s :\equiv \lambda a, e.((qae)+1)$.

Let us now discuss the principle Δ_0^C-CColl_2. According to its upper-B translation, we must find a term q such that

$$\forall c^{\Omega}\,[\forall n \forall z \in L_{(qc)\langle n\rangle} \exists X \forall x \in z\,\phi(x, X) \to \exists X \forall n \forall x \in L_{c\langle n\rangle}\,\phi(x, X)]$$

holds in in $(L_{\omega_1}^{\mathrm{mix}^C}, \mathcal{P}(L_{\omega_1})\!\upharpoonright)$. Well, it does hold with $q :\equiv \lambda c.(c+2)$. To see this, note that the hypotehsis above entails $\forall z \in L_{c+1} \exists X \forall x \in z\,\phi_{\mathrm{B}}(x, X)$. In particular, this holds with z particularized as L_c, and we get what we want.

In order to discuss the principle Coll_2^C, we compute the upper-B translations of its antecedent and consequent. They are, $\exists c^{\Omega}, a \forall b\,[\forall X \exists x \in L_c \forall n\,\phi_{\mathrm{B}}(a, b\langle n\rangle, x, X)$ and $\exists c^{\Omega}, a \forall b\,[\exists z \in L_c \forall k \forall X \exists x \in z \forall n\,\phi_{\mathrm{B}}(a, b\langle k\rangle\langle n\rangle, x, X)]$, respectively. Therefore, we need term q, r and t such that

$$\forall c^{\Omega}, a, b\,[\forall k \forall X \exists x \in L_c \forall n\,\phi_{\mathrm{B}}(a,(qcab)\langle k\rangle\langle n\rangle, x, X) \to$$

$$\exists z \in L_{rca} \forall k \forall X \exists x \in z \forall n\,\phi_{\mathrm{B}}(tca, b\langle k\rangle\langle n\rangle, x, X)]$$

holds in $(L_{\omega_1}^{\mathrm{mix}^C}, \mathcal{P}(L_{\omega_1})\!\upharpoonright)$. Take $q :\equiv \lambda c, a, b.b$, $r :\equiv \lambda c, a.(c+1)$ and $t :\equiv \lambda c, a.a$. With these data, the above holds (just let z be L_c).

Finally, we study the comprehension principle Δ_0^C-CA. An instance of this principle has the form $\forall W, w \exists X \forall x\,(x \in X \leftrightarrow \phi(x, w, W))$, where ϕ is a Δ_0^C-formula in which X does not occur. The upper B-translation of this instance is:

$$\forall d^{\Omega}, b^{\Omega}[\forall W \forall w \in L_d \exists X \forall n^N \forall x \in L_{b\langle n\rangle}(x \in X \leftrightarrow \phi(x, w, W))].$$

This statement holds with the set $X :\equiv \{x \in L_\alpha : \phi(x, w, W)\}$, where $\alpha = |b|$. $\square$

The proof of the previous theorem goes through in $(L^{\text{mix}^C}_{\omega_1}, L_{\omega_1})$ except for one single step. It is in the verification of the scheme Δ^C_0-CA. If instead of Δ^C_0-CA one had the scheme $\exists X \forall x\, (x \in X \leftrightarrow \phi(x, w, W))$, where $\phi(x)$ is restricted to be a Δ_0-formula, then the set $\{x \in L_\alpha : \phi(x, w, W)\}$ is in L_{ω_1}. In effect, the parameter W (and, also, w) of the structure $(L^{\text{mix}^C}_{\omega_1}, L_{\omega_1})$ takes a value in a certain L_β, for a certain $\beta < \omega_1$. Therefore, the previous set is an element of L_γ, for $\alpha, \beta < \gamma$. We think that this is worth remarking (specially because this was the strategy adopted in [10]).

Proposition 11. *If* $\mathsf{IKP}\omega_2\!\upharpoonright + \Delta^C_0\text{-}\mathsf{LEM} + \Delta^C_0\text{-}\mathsf{CA} + \mathsf{MP}^C + \mathsf{bIP}_{\Pi^C_1} + \Delta^C_0\text{-}\mathsf{CColl}_2 + \mathsf{Coll}^C_2$ *proves* $\forall x \exists y\, \phi(x, y)$*, where* ϕ *is a* Δ_0*-formula (x and y are the only free variables), then there is a closed term t of type* $\Omega \to \Omega$ *such that*

$$L^{\text{mix}}_{\omega_1} \models \forall c^\Omega \forall x \in L_c \exists y \in L_{tc}\, \phi(x, y).$$

Moreover, $L_{\mathrm{BH}} \models \forall x \exists y\, \phi(x, y)$*, where* BH *is the Bachmann-Howard ordinal.*

The above proposition is an immediate consequence of Theorem 3. Note that the formulas ϕ are restricted to Δ_0-formulas.

Corollary 4. *If* $\mathsf{IKP}\omega_2\!\upharpoonright + \Delta^C_0\text{-}\mathsf{LEM} + \Delta^C_0\text{-}\mathsf{CA} + \mathsf{MP}^C + \mathsf{bIP}_{\Pi^C_1} + \Delta^C_0\text{-}\mathsf{CColl}_2 + \mathsf{Coll}^C_2$ *proves* $\exists x\, \phi(x)$*, where* ϕ *is a* Δ_0*-formula (x is the only free variable), then there is an ordinal* α *smaller than the Bachmann-Howard ordinal so that* $L_\alpha \models \exists x\, \phi(x)$.

Lemma 9 does not generalize to the functional interpretation extended to the second-order language, neither when the semantics is $(L^{\text{mix}^C}_{\omega_1}, \mathcal{P}(L_{\omega_1})\!\upharpoonright)$, nor when the semantics is $(L^{\text{mix}^C}_{\omega_1}, L_{\omega_1})$. For instance, consider the following instance of Coll^C:

$$\forall X \exists x \forall z\, (X = \{z\} \to Ord(x) \wedge z \in L_x) \to \exists w \forall X \exists x \in w \forall z\, (X = \{z\} \to Ord(x) \wedge z \in L_x).$$

It is clear that this sentence is false in both structures above. The reason why the proof of Lemma 9 does not generalize is, of course, the fact that the transformations (9) and (10) of the definition of the functional interpretation are not truth preserving (in the said structures). However, as it was shown in Lemma 9, the remaining transformations are truth preserving. Hence, as long as we restrict ourselves to first-order formulas ϕ, we have the following:

Proposition 12. *If* $\mathsf{IKP}\omega_2\!\upharpoonright + \Delta^C_0\text{-}\mathsf{LEM} + \Delta^C_0\text{-}\mathsf{CA} + \mathsf{MP}^C + \mathsf{bIP}_{\Pi^C_1} + \Delta^C_0\text{-}\mathsf{CColl}_2 + \mathsf{Coll}^C_2$ *proves* $\forall x \exists y\, \phi(x, y)$*, where* ϕ *is any formula (x and y are the only free variables)* without *second-order quantifications, then there is a closed term t of type* $\Omega \to \Omega$ *such that*

$$\forall c \in W \forall x \in L_{|c|} \exists y \in L_{|t(c)|}\ L_{\omega_1} \models \phi(x, y).$$

The proof is like the one of Proposition 5.

7 Salipante's result and power Kripke-Platek set theory

The theory $\mathsf{IKP}\omega_2$ is like the theory $\mathsf{IKP}\omega_2\!\upharpoonright$ of Definition 9 except that we now permit Δ^C_0-formulas in the separation scheme *with (first and) second-order parameters*. In

[19], Salipante observed the following (he worked in a classical theory, but the argument is the same):

Theorem 4 (Salipante)**.** *The theory* $\mathsf{IKP}\omega_2 + \Delta_0\text{-}\mathsf{LEM} + \Delta_0\text{-}\mathsf{CA} + \mathsf{s}\Pi^1_1\text{-}\mathsf{ref}$ *proves the powerset axiom, i.e., it proves the sentence* $\forall y \exists z \forall x\,(x \in z \leftrightarrow x \subseteq y)$.

Proof. Let y be given. The theory $\mathsf{IKP}\omega_2$ proves $\forall X \exists w\,(w = X \cap y)$, where $X \cap y$ abbreviates the *set* $\{u \in y : u \in X\}$. This set exists by separation (using the second-order parameter X). By $\mathsf{s}\Pi^1_1$-ref, $\exists z \forall X \exists w \in z\,(w = X \cap y)$. Let z_0 be such a set. We claim that $\forall x\,(x \subseteq y \to x \in z_0)$. To see this, take x a subset of y. By Δ_0-CA, let $X_0 = \{u : u \in x\}$. By the choice of z_0, there is $w \in z_0$ such that $w = X_0 \cap y$. Since $X_0 \cap y = x$, we conclude that $x \in z_0$, as wanted. The powerset of y can now be obtained from z_0 by ordinary separation. □

The above proof also holds if the separation scheme applies only to Δ_0-formulas (that is how Salipante stated his theorem). The *crucial thing* is to allow second-order parameters in the separation scheme.

In [17] Rathjen introduced the theory $\mathsf{KP}\omega\,(\mathcal{P})$ of power Kripke-Platek set theory. In order to formulate this theory we need the following definition:

Definition 14. *The class of* $\Delta^{\mathcal{P}}_0$*-formulas is the smallest class of formulas of the language of set theory containing the atomic formulas (including* $\perp$*) and closed under* $\wedge$*,* $\vee$*,* $\to$ *and the quantifications*

$$\forall x \in z,\ \exists x \in z,\ \forall x \subseteq z,\ \exists x \subseteq z,$$

where the last two quantifications abbreviate $\forall x\,(x \subseteq z \to \ldots)$ *and* $\exists x\,(x \subseteq z \wedge \ldots)$*, respectively.* $\Sigma^{\mathcal{P}}_1$*-formulas are formulas of the form* $\exists s\, \psi(z)$*, where* $\psi(z)$ *is a* $\Delta^{\mathcal{P}}_0$*-formula (possibly with parameters).* $\Pi^{\mathcal{P}}_1$*-formulas are defined dually. A* $\Pi^{\mathcal{P}}_2$*-formula is a formula of the form* $\forall w\, \phi(w)$*, where* $\phi(w)$ *is a* $\Sigma^{\mathcal{P}}_1$*-formula.*

The theory $\mathsf{KP}\omega\,(\mathcal{P})$ is a classical theory in the language of set theory with the following axioms: extensionality, pairing, union, infinity, powerset, $\Delta^{\mathcal{P}}_0$-separation, $\Delta^{\mathcal{P}}_0$-collection and unrestricted foundation. The transitive models of $\mathsf{KP}\omega\,(\mathcal{P})$ are the *power admissible* sets introduced by Harvey Friedman in [12]. As Rathjen observes, the theory $\mathsf{KP}\omega\,(\mathcal{P})$ can also be described as the theory $\mathsf{KP}\omega$ framed in the language of set theory extended with a new primitive unary function symbol $\mathcal{P}$ for the powerset operation, the axiom $\forall x\,(x \in \mathcal{P}(y) \leftrightarrow x \subseteq y)$, and the schemata of Δ_0-separation and Δ_0-collection extended to the Δ_0-formulas of this new language. It should be noticed, as Rathjen warns us, that the theory $\mathsf{KP}\omega\,(\mathcal{P})$ is *not* the same theory as $\mathsf{KP}\omega$ with the powerset axiom. This latter theory is much weaker than $\mathsf{KP}\omega\,(\mathcal{P})$, as Rathjen discusses in [17].

In this paper we are interested in semi-constructive theories. The natural theories to consider are the theories of Section 2 in which the Δ_0-formulas are replaced by the wider class of $\Delta^{\mathcal{P}}_0$-formulas. We are naturally led to consider the theory

$$\mathsf{IKP}\omega\,(\mathcal{P}) + \Delta^{\mathcal{P}}_0\text{-}\mathsf{LEM} + \mathsf{MP}^{\mathcal{P}} + \mathsf{bIP}_{\Pi^{\mathcal{P}}_1} + \Delta^{\mathcal{P}}_0\text{-}\mathsf{CColl} + \mathsf{Coll}^{\mathcal{P}},$$

where it should be clear what the acronyms above stand for. This is also a very robust theory, and it is clear that we have the analogue of Proposition 1:

Proposition 13. *The theory* $\mathsf{KP}\omega\,(\mathcal{P})$ *is* $\Pi_2^{\mathcal{P}}$*-conservative over* $\mathsf{IKP}\omega\,(\mathcal{P}) + \Delta_0^{\mathcal{P}}\text{-}\mathsf{LEM} + \mathsf{MP}^{\mathcal{P}}$.

We could adapt the analysis that we have made of the semi-constructive theory $\mathsf{IKP}\omega + \Delta_0\text{-}\mathsf{LEM} + \mathsf{MP} + \mathsf{bIP}_{\Pi_1} + \Delta_0\text{-}\mathsf{CColl} + \mathsf{Coll}$ to the powerset version mentioned above (this adaptation requires the cumulative hierarchy instead of the constructible hierarchy). The next theorem provides an illuminating alternative. We need a lemma first:

Lemma 10. *The theory* $\mathsf{IKP}\omega_2 + \Delta_0^{C}\text{-}\mathsf{LEM} + \Delta_0^{C}\text{-}\mathsf{CA} + \mathsf{MP}^{C} + \mathsf{bIP}_{\Pi_1^C} + \Delta_0^{C}\text{-}\mathsf{CColl}_2 + \mathsf{Coll}_2^{C}$ *proves* Δ_1^{C}*-separation, i.e., it proves*

$$\forall x\,(\forall u\,\phi(u,x) \leftrightarrow \exists v\,\psi(v,x)) \rightarrow \forall z\exists y\forall x\,(x \in y \leftrightarrow \ (x \in z \wedge \exists v\,\psi(v,x))),$$

for Δ_0^{C}*-formulas* ϕ *and* ψ *(possibly with first and second-order parameters).*

The above lemma is proven like Proposition 2. The proof is possible because we now permit Δ_0^{C}-formulas and parameters (first and second-order) in the separation scheme.

Theorem 5. *The theory* $\mathsf{IKP}\omega\,(\mathcal{P}) + \Delta_0^{\mathcal{P}}\text{-}\mathsf{LEM} + \mathsf{MP}^{\mathcal{P}} + \mathsf{bIP}_{\Pi_1^{\mathcal{P}}} + \Delta_0^{\mathcal{P}}\text{-}\mathsf{CColl} + \mathsf{Coll}^{\mathcal{P}}$ *is a subtheory of the second-order theory*

$$\mathsf{IKP}\omega_2 + \Delta_0^{C}\text{-}\mathsf{LEM} + \Delta_0^{C}\text{-}\mathsf{CA} + \mathsf{MP}^{C} + \mathsf{bIP}_{\Pi_1^C} + \Delta_0^{C}\text{-}\mathsf{CColl}_2 + \mathsf{Coll}_2^{C}.$$

Proof. This result hinges on two facts. The first is that the relation $z = \mathcal{P}(y)$ is given by the Δ_0^{C}-formula $\forall x \in z\,(x \subseteq y) \wedge \forall X\,(X \cap y \in z)$. Without the abbreviation '$X \cap y \in z$', it reads

$$\forall x \in z\,(x \subseteq y) \wedge \forall X \exists x \in z \forall w\,(w \in x \leftrightarrow w \in X \wedge w \in y).$$

Let us denote this Δ_0^{C}-formula by $P(y,z)$. By Theorem 4 (and its argument), the second-order theory of the theorem proves $\forall y,z\,(P(y,z) \leftrightarrow \forall x\,(x \in z \leftrightarrow x \subseteq y))$ and $\forall y \exists^1 z\, P(y,z)$. Therefore, it proves the powerset axiom. The second important fact is that the second-order theory of the theorem has a good theory for introducing Σ_1^{C}-function symbols (see the comments after Definition 13). In particular, Δ_0^{C}-formulas in the new language with the extra function symbols translate into Δ_1^{C}-formulas of the original language (the equivalence between the corresponding pair of Σ_1^{C}-formulas and Π_1^{C}-formulas is proven in the second-order theory of the theorem, of course). Therefore, we can introduce a Σ_1^{C}-function symbol that satisfies the defining axiom of the powerset operation and, as a consequence, $\Delta_0^{\mathcal{P}}$-formulas are rendered by Δ_1^{C}-formulas. The above lemma entails that $\Delta_0^{\mathcal{P}}$-separation is provable in the second-order theory. The theorem should now be clear. □

The structure $(V_{\omega_1}^{\mathrm{mix}^C}, V_{\omega_1})$ for the second-order mixed language $\mathcal{L}_{\Omega}^{\mathrm{mix}^C}$ was introduced just before Theorem 3. Remember that in this structure both the first-order set variables and second-order (class) variables of $\mathcal{L}_{\Omega}^{\mathrm{mix}^C}$ take values in V_{ω_1}.

Theorem 6 (Second-order soundness theorem II). *Let ϕ be a sentence of the language of second-order set theory. Suppose that*

$$\mathsf{IKP}\omega_2 + \Delta_0^{\mathcal{C}}\text{-}\mathsf{LEM} + \Delta_0^{\mathcal{C}}\text{-}\mathsf{CA} + \mathsf{MP}^{\mathcal{C}} + \mathsf{bIP}_{\Pi_1^{\mathcal{C}}} + \Delta_0^{\mathcal{C}}\text{-}\mathsf{CColl}_2 + \mathsf{Coll}_2^{\mathcal{C}} \vdash \phi.$$

Then there are closed terms t of $\mathcal{L}_\Omega$ such that, for appropriate types ρ,

$$(V_{\omega_1}^{\mathrm{mix}^{\mathcal{C}}}, V_{\omega_1}) \models \forall b^\rho\, \phi_B(t, b).$$

Proof. The proof is *mutatis mutandis* the proof of Theorem 3 (see also the note after that proof), but one must replace systematically the construtible hierarchy by the cumulative hierarchy. We need the cumulative hierarchy because the separation axiom of $\mathsf{IKP}\omega_2$ has second-order parameters. Let us see in detail why this is so. The separation axiom with (first and) second-order parameters is

$$\forall w, W \forall y \exists z \forall x\, (x \in z \leftrightarrow x \in y \wedge \phi(x, w, W)),$$

where ϕ is a $\Delta_0^{\mathcal{C}}$-formula in which the variable z does not occur. As before, notice that the inner universal statement can be considered $\Delta_0^{\mathcal{C}}$. Hence, the upper B-translation of this formula shows that we need a closed term t of type $\Omega \to (\Omega \to \Omega)$ such that

$$\forall c^\Omega, e^\Omega\, [\forall y \in V_c \forall w \in V_e \forall W \exists z \in V_{tce} \forall x\, (x \in z \leftrightarrow x \in y \wedge \phi(x, w, W))]$$

holds in $(V_{\omega_1}^{\mathrm{mix}^{\mathcal{C}}}, V_{\omega_1})$. It is clear that the term $t :\equiv \lambda c, e.(c+1)$ does the job. To see this, just take z to be $\{x \in V_\alpha : x \in y \wedge \phi(x, w, W)\}$, where $\alpha = |c|$.

[Note that the proof does not go through in the constructible hierarchy because the term t is only allowed to depend on y and w, via c and e (respectively), but *not* on W.] □

As usual, we can draw some consequences regarding $\Pi_2^{\mathcal{C}}$ and $\Sigma_1^{\mathcal{C}}$ consequences of the second-order theory of the theorem. We are going to present them in a particlular fashion, with an eye to their application to power Kripke-Platek set theories.

We have the following absoluteness property:

Lemma 11. *If α and β are ordinals and $\phi(x_1, \ldots, x_n)$ is a $\Delta_0^{\mathcal{C}}$-formula with its free variables as shown (they are all first-order), then*

$$x_1, \ldots, x_n \in V_\alpha \wedge \alpha < \beta \to [(V_\alpha, V_{\alpha+1}) \models \phi(x_1, \ldots, x_n) \leftrightarrow (V_\beta, V_\beta) \models \phi(x_1, \ldots, x_n)].$$

Observation 2. *In the above, in a structure of the form (V, W), with $W \subseteq \mathcal{P}(V)$, the first-order variables take values in V and the second-order variables take values in W.*

Proof. We show a bit more in order to get an induction on $\Delta_0^{\mathcal{C}}$-formulas going. We prove by induction on $\Delta_0^{\mathcal{C}}$-formulas $\phi(x_1, \ldots, x_n, X_1, \ldots, X_k)$ that for all $x_1, \ldots, x_n \in V_\alpha$ and $X_1, \ldots, X_k \in V_\beta$ we have

$$(V_\alpha, V_{\alpha+1}) \models \phi(x_1, \ldots, x_n, X_1 \cap V_\alpha, \ldots, X_k \cap V_\alpha) \leftrightarrow (V_\beta, V_\beta) \models \phi(x_1, \ldots, x_n, X_1, \ldots, X_k)$$

Note that we are abusing notation by confusing variables with the sets that take their values. For ease of reading, we will also omit tuples. The proof by induction is straightforward except for the case of second-order quantifications. We study the universal second-order quantifier (the case of the existential second-order quantifier follows immediately because $\Delta_0^{\mathcal{C}}$-formulas are closed under negation).

Consider the formula $\forall W\, \phi(x, X, W)$, with ϕ a $\Delta_0^{\mathcal{C}}$-formula. Let $x \in V_\alpha$, $X \in V_\beta$ and assume that $(V_\alpha, V_{\alpha+1}) \models \forall W\, \phi(x, X \cap V_\alpha, W)$. Let Y be an arbitrary element of V_β. Since $Y \cap V_\alpha \in V_{\alpha+1}$, we have $(V_\alpha, V_{\alpha+1}) \models \phi(x, X \cap V_\alpha, Y \cap V_\alpha)$. By induction hypothesis, we get $(V_\beta, V_\beta) \models \phi(x, X, Y)$. By the arbitrariness of Y, we conclude $(V_\beta, V_\beta) \models \forall W\, \phi(x, X, W)$. To prove the converse, let $x \in V_\alpha$, $X \in V_\beta$ and assume that $(V_\beta, V_\beta) \models \forall W\, \phi(x, X, W)$. Let $Y \in V_{\alpha+1}$ be arbitrary. In particular, $Y \in V_\beta$. Hence, $(V_\beta, V_\beta) \models \phi(x, X, Y)$. By induction hypothesis, we get $(V_\alpha, V_{\alpha+1}) \models \phi(x, X \cap V_\alpha, Y \cap V_\alpha)$. Since $Y \cap V_\alpha = Y$, we have $(V_\alpha, V_{\alpha+1}) \models \phi(x, X \cap V_\alpha, Y)$. Therefore, by the arbitrariness of Y, we conclude that $(V_\alpha, V_{\alpha+1}) \models \forall W\, \phi(x, X \cap V_\alpha, W)$. □

Proposition 14. *If* $\mathsf{IKP}\omega_2 + \Delta_0^{\mathcal{C}}\text{-}\mathsf{LEM} + \Delta_0^{\mathcal{C}}\text{-}\mathsf{CA} + \mathsf{MP}^{\mathcal{C}} + \mathsf{bIP}_{\Pi_1^{\mathcal{C}}} + \Delta_0^{\mathcal{C}}\text{-}\mathsf{CColl}_2 + \mathsf{Coll}_2^{\mathcal{C}}$ *proves* $\forall x \exists y\, \phi(x, y)$*, where* ϕ *is a* $\Delta_0^{\mathcal{C}}$*-formula (*x *and* y *are the only free variables), then there is a closed term* t *of type* $\Omega \to \Omega$ *such that*

$$(V_{\omega_1}^{\mathrm{mix}^{\mathcal{C}}}, V_{\omega_1}) \models \forall c^\Omega \forall x \in V_c \exists y \in V_{tc}\, \phi(x, y).$$

Moreover, $(V_{\mathrm{BH}}, V_{\mathrm{BH}}) \models \forall x \exists y\, \phi(x, y)$*, where* BH *is the Bachmann-Howard ordinal.*

The above proposition is an immediate consequence of Theorem 6. The final conclusion follows from two applications of Lemma 11. With this lemma, we also get:

Corollary 5. *If* $\mathsf{IKP}\omega_2 + \Delta_0^{\mathcal{C}}\text{-}\mathsf{LEM} + \Delta_0^{\mathcal{C}}\text{-}\mathsf{CA} + \mathsf{MP}^{\mathcal{C}} + \mathsf{bIP}_{\Pi_1^{\mathcal{C}}} + \Delta_0^{\mathcal{C}}\text{-}\mathsf{CColl}_2 + \mathsf{Coll}_2^{\mathcal{C}} \vdash \exists x\, \phi(x)$*, where* $\phi(x)$ *is a* $\Delta_0^{\mathcal{C}}$*-formula (*x *is its only free variable), then there is an ordinal* α *smaller than the Bachmann-Howard ordinal such that* $(V_\alpha, V_{\alpha+1}) \models \exists x\, \phi(x)$*.*

An analysis of the theory $\mathsf{IKP}\omega\,(\mathcal{P}) + \Delta_0^{\mathcal{P}}\text{-}\mathsf{LEM} + \mathsf{MP}^{\mathcal{P}} + \mathsf{bIP}_{\Pi_1^{\mathcal{P}}} + \Delta_0^{\mathcal{P}}\text{-}\mathsf{CColl} + \mathsf{Coll}^{\mathcal{P}}$ is now forthcoming. The strategy is clear: Use Theorem 5 to reduce the analysis of this theory to the analysis of the second-order theory of that proposition. In the following lemma, $P(y, z)$ is the $\Delta_0^{\mathcal{C}}$-formula of the proof of Theorem 5:

Lemma 12. *Let* $\phi(x_1, \ldots, x_n)$ *be a* $\Delta_0^{\mathcal{P}}$*, with its free variables as shown. Then there is a* Δ_0*-formula* $\phi^*(x_1, \ldots, x_n, z)$ *such that the second-order theory*

$$\mathsf{IKP}\omega_2 + \Delta_0^{\mathcal{C}}\text{-}\mathsf{LEM} + \Delta_0^{\mathcal{C}}\text{-}\mathsf{CA} + \mathsf{MP}^{\mathcal{C}} + \mathsf{bIP}_{\Pi_1^{\mathcal{C}}} + \Delta_0^{\mathcal{C}}\text{-}\mathsf{CColl}_2 + \mathsf{Coll}_2^{\mathcal{C}}$$

proves the equivalence

$$\phi(x_1, \ldots, x_n) \leftrightarrow \exists z\, (P(\mathrm{tc}(x_1 \cup \ldots \cup x_n), z) \wedge \phi^*(x_1, \ldots, x_n, z)).$$

Moreover, for all $x_1, \ldots, x_n \in V_{\omega_1}$*,*

$$V_{\omega_1} \models \phi(x_1, \ldots, x_n) \quad \textit{if, and only if,} \quad V_{\omega_1} \models \phi^*(x_1, \ldots, x_n, \mathcal{P}(\mathrm{tc}(x_1 \cup \ldots \cup x_n))).$$

Here, $\mathrm{tc}(w)$ *stands for the transitive closure of* w*.*

Proof. The proof is by induction on the complexity of ϕ. We will only study negation and the universal quantifications $\forall w \subseteq x\,(\ldots)$ and $\forall w \in x\,(\ldots)$. Negation is clear because, using the induction hypothesis, the equivalence

$$\neg\phi(x_1, \ldots, x_n) \leftrightarrow \exists z\,(P(\mathrm{tc}(x_1 \cup \ldots \cup x_n), z) \wedge \neg\phi^*(x_1, \ldots, x_n, z))$$

is provable in the second-order theory of the lemma. Of course, $(\neg\phi)^*$ is defined as being $\neg(\phi^*)$. Let us now consider the formula $\forall w \subseteq x\,\phi(w, x, x_1, \ldots, x_n)$, with $\phi \in \Delta_0^{\mathcal{P}}$. By induction hypothesis, the second-order theory of the lemma proves the equivalence of the above formula with $\forall w \subseteq x \exists z\,(P(\mathrm{tc}(w \cup x \cup x_1 \cup \ldots \cup x_n), z) \wedge \phi^*(w, x, x_1, \ldots, x_n, z))$. This is equivalent to

$$\exists z\,(P(\mathrm{tc}(x \cup x_1 \cup \ldots \cup x_n), z) \wedge \forall w \in z\,(w \subseteq x \to \phi^*(w, x, x_1, \ldots, x_n, z))).$$

This is due to the fact that $w \cup x = x$ (and the uniqueness of the z). The argument for the second part of the lemma is similar. The situation is clear now.

The treatment of the usual bounded quantification $\forall w \in x\,(\ldots)$ is analogous. Here one takes notice that $\mathrm{tc}(w \cup x \cup x_1 \cup \ldots \cup x_n) = \mathrm{tc}(x \cup x_1 \cup \ldots \cup x_n)$ when $w \in x$. □

We are ready to prove the following proposition and corollary:

Proposition 15. *If* $\mathsf{IKP}\omega\,(\mathcal{P}) + \Delta_0^{\mathcal{P}}\text{-}\mathsf{LEM} + \mathsf{MP}^{\mathcal{P}} + \mathsf{bIP}_{\Pi_1^{\mathcal{P}}} + \Delta_0^{\mathcal{P}}\text{-}\mathsf{CColl} + \mathsf{Coll}^{\mathcal{P}} \vdash \forall x \exists y\,\phi(x, y)$, *where* ϕ *is a* $\Delta_0^{\mathcal{P}}$*-formula (x and y are the only free variables), then there is a closed term t of type* $\Omega \to \Omega$ *such that*

$$\forall c \in W \forall x \in V_{|c|} \exists y \in V_{|t(c)|}\ V_{\omega_1} \models \phi(x, y).$$

Moreover, $V_{\mathrm{BH}} \models \forall x \exists y\,\phi(x, y)$*, where* BH *is the Bachmann-Howard ordinal.*

Proof. Suppose that the theory $\mathsf{IKP}\omega\,(\mathcal{P}) + \Delta_0^{\mathcal{P}}\text{-}\mathsf{LEM} + \mathsf{MP}^{\mathcal{P}} + \mathsf{bIP}_{\Pi_1^{\mathcal{P}}} + \Delta_0^{\mathcal{P}}\text{-}\mathsf{CColl} + \mathsf{Coll}^{\mathcal{P}}$ proves $\forall x \exists y\,\phi(x, y)$. By Theorem 5, so does the theory of Proposition 14. By the previous lemma, this second-order theory proves $\forall x \exists y \exists z\,(P(\mathrm{tc}(x \cup y), z) \wedge \phi^*(x, y, z))$. Using Proposition 14, it can be argued that there is a closed term t such that

$$\forall c \in W \forall x \in V_{|c|} \exists y \in V_{|t(c)|}\ (V_{\omega_1}, V_{\omega_1}) \models \exists z\,(P(\mathrm{tc}(x \cup y, z)) \wedge \phi^*(x, y, z)).$$

Since $(V_{\omega_1}, V_{\omega_1}) \models P(\mathrm{tc}(x \cup y), z) \leftrightarrow z = \mathcal{P}(\mathrm{tc}(x \cup y))$, we obtain the desired conclusion using the second part of previous lemma.

It also follows that $V_{\mathrm{BH}} \models \forall x \exists y\,\phi(x, y)$ because $\Delta_0^{\mathcal{P}}$-formulas are absolute between the various levels of the cumulative hierarchy. □

Corollary 6. *If* $\mathsf{IKP}\omega\,(\mathcal{P}) + \Delta_0^{\mathcal{P}}\text{-}\mathsf{LEM} + \mathsf{MP}^{\mathcal{P}} + \mathsf{bIP}_{\Pi_1^{\mathcal{P}}} + \Delta_0^{\mathcal{P}}\text{-}\mathsf{CColl} + \mathsf{Coll}^{\mathcal{P}} \vdash \exists x\,\phi(x)$, *where* $\phi(x)$ *is a* $\Delta_0^{\mathcal{P}}$*-formula (x is its only free variable), then there is an ordinal* α *smaller than the Bachmann-Howard ordinal such that* $V_\alpha \models \exists x\,\phi(x)$*.*

Using Proposition 13, we can give a Σ-ordinal analysis (in the relativized sense of [17]) of the *classical* theory $\mathsf{KP}\omega\,(\mathcal{P})$:

Proposition 16. *If* $\mathsf{KP}\omega(\mathcal{P}) \vdash \forall x \exists y\, \phi(x, y)$*, where* ϕ *is a* $\Delta_0^{\mathcal{P}}$*-formula (*x *and* y *are the only free variables), then there is a closed term* t *of type* $\Omega \to \Omega$ *such that*

$$\forall c \in W \forall x \in V_{|c|} \exists y \in V_{|t(c)|}\ V_{\omega_1} \models \phi(x, y).$$

Moreover, $V_{\mathrm{BH}} \models \forall x \exists y\, \phi(x, y)$*, where* BH *is the Bachmann-Howard ordinal.*

Corollary 7. *If* $\mathsf{KP}\omega(\mathcal{P}) \vdash \exists x\, \phi(x)$*, where* $\phi(x)$ *is a* $\Delta_0^{\mathcal{P}}$*-formula (*x *is its only free variable), then there is an ordinal* α *smaller than the Bachmann-Howard ordinal such that* $V_\alpha \models \exists x\, \phi(x)$*.*

These two results are due to Rathjen in [17]. We obtained them in a very roundabout way, via second-order semi-constructive theories. A direct way, using our kind of functional interpretations, would be just to adapt – replacing in a straightforward manner the constructible hierarchy by the cumulative hierarchy – the analysis of $\mathsf{KP}\omega$ provided in [10].

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