Spector's proof of the consistency of analysis

Fernando Ferreira
Departamento de Matemática
Faculdade de Ciências
Universidade de Lisboa
fjferreira@fc.ul.pt

1 Introduction

The editors of this volume asked me to present and discuss Clifford Spector's proof of the consistency of analysis. It is only fitting that, in a volume dedicated to Gerhard Gentzen, known for his epoch-making consistency proof of Peano arithmetic PA, Spector's proof of consistency of analysis is discussed. Gentzen's approach to consistency proofs has been systematically developed and generalized by the German school of proof theory (Schütte, Pohlers, Buchholz, Jäger, Rathjen, etc.) and others. For all its successes (and there were many), the approach is still very far from providing a proof of the consistency of full second-order arithmetic PA_2 (analysis). There are quite serious difficulties in analysing systems above Π^1_2-comprehension. In the words of Michael Rathjen in [32], the more advanced analyses "tend to be at the limit of human tolerance." How is it, then, that Spector was able to provide a proof of the consistency of analysis? What kind of proof is it? Spector's proof follows quite a different blueprint from Gentzen's. It does not reduce PA_2 to finististic arithmetic together with the postulation of the well-ordering of a sufficiently long primitive recursive ordinal notation system. Instead, it reduces (in a finitary manner) the consistency of analysis to the consistency of a certain quantifier-free finite-type theory. The epistemological gain, if there is one, rests in the evidence for the consistency of Spector's quantifier-free theory.

The proof of Spector was published posthumously in 1962 (Spector died young of acute leukemia). It is a descendant of Gödel's interpretation of PA in 1958, in which it was shown that PA is interpretable in Gödel's quantifier-free finite-type theory T. In

*I would like to thank Reinhard Kahle and Michael Rathjen, editors of this volume, for inviting me to contribute to this book dedicated to Gerhard Gentzen. The comments of an anonymous referee and an interesting series of discussions with Ulrich Kohlenbach concerning the contents of this paper have certainly improved it. I have also benefited from correspondence with Solomon Feferman and William Howard. The latter urged me to mention some computability issues, especially since the termination of effective processes was a fundamental issue for Gentzen. I gratefully acknowledge these comments and discussions but, of course, the responsability for the choice of material and for the writing of the paper is fully mine. The financial support of FCT-Fundação para a Ciência e a Tecnologia [PEst-OE/MAT/UI0209/2011, PTDC/MAT/104716/2008 and PTDC/FIL-FCI/109991/2009] was, of course, very welcomed. Finally, I want to thank Faculdade de Ciências da Universidade de Lisboa for conceding me a sabbatical leave in the academic year of 2012/2013.

the last paragraph of his paper [11], Gödel writes that "it is clear that, starting from the same basic idea, one can also construct systems that are much stronger than T, for example by admitting transfinite types or the sort of inference that Brouwer used in proving the 'fan theorem'." Spector took up the latter suggestion. The Brouwerian kind of inference that Gödel is presumably referring to is the bar theorem (a corollary of which is the 'fan theorem'). Brouwer's justification of the bar theorem is object of controversy (see [3] for Brouwer's own rendition and [37] for a modern defense and references) and not really formulated in a workable form. Following Stephen Kleene's enunciation in [21], the bar theorem is nowadays admitted in intuitionistic mathematics in the form of an axiom scheme known as bar induction. Spector follows this approach and advances two moves: he generalizes bar induction to finite types and, in a bold stroke, introduces a corresponding principle of definition known as (Spector's) bar recursion. In his own words, "bar recursion is a principle of definition and bar induction a corresponding principle of proof." One cannot but think of a parallel with ordinary recursion and ordinary induction. Spector's quantifier-free finite-type theory adjoins to Gödel's T new constants for the bar recursors and accepts the pertinent equations that characterize them. He is then able to show that analysis is interpretable in this extension of Gödel's T.

This paper is organized as follows. In the next section, we review Gödel's *dialectica* interpretation of 1958. We describe a direct interpretation of PA into T, instead of Gödel's own which relies on the interpretation of Heyting arithmetic accompanied by a double negation translation of classical logic into intuitionistic logic. The direct interpretation is very simple and was first described by Joseph Shoenfield in his well-known textbook [34]. Sections 3 and 5 introduce bar recursion. In the first of these sections, we discuss bar recursion from the set theoretic point of view. As opposed to standard treatments of bar recursion, we take some time doing this. We have in mind the reader unfamiliar with bar recursion but comfortable with the basics of set theory. One of the aims of this paper is to explain Spector's proof to a logician not trained in proof theory or constructive mathematics. Two set-theoretic discussions are made. The first focus on well-founded trees and their ordinal heights. The second has the advantage of immediately drawing attention to the principle of dependent choices, a principle which plays an important role in the discussions of bar recursion. Armed with the set-theoretic understanding, in Section 5 we finally discuss bar recursion from an intuitionistic point of view.

The interim Section 4 introduces Spector's quantifier-free theory with the bar recursive functionals of finite type. It also briefly mentions the two main models of this theory. Sections 6 and 7 are the heart of the paper. They present the interpretation of analysis into Spector's theory. The original proof is based on the interpretation of the so-called classical principle of numerical double negation shift (principle F in Spector's paper), and this is sufficient to interpret full second-order comprehension. The technical matter boils down to solving a certain system of equations in finite-type theory, and the bar recursive functionals permit the construction of a solution. The solution of these equations is *ad hoc* (Paulo Oliva was, nevertheless, able to find a nice motivation for it in [31]). The interpretation of bar induction is more natural because bar induction and bar recursion go hand in hand, in a way similar to that of induction and recursion in Gödel's *dialectica* interpretation (see the discussion in [7]). Moreover, it provides additional information. We owe to William Howard in [14] the interpretation of bar induction into Spector's theory. Our paper develops Howard's strategy *directly* for the classical setting.

The paper includes a short appendix. It discusses a sort of perplexity caused by the existence of the *term model* of Spector's theory, a structure whose infinite numerical sequences are all recursive. How can bar recursion hold in such a classical structure when models of bar recursion are usually associated with producing non-recursive objects? The answer lies in the failure of quantifier-free choice in the term model and reveals a little of the subtlety of Spector's interpretation.

The main body of the paper finishes with an epilogue in which Spector's consistency proof is briefly assessed. We hope that this writing is able to convey to the uninitiated a little of the depth and beauty of Spector's proof of consistency, and also that the expert finds some interest in the paper.

2 Gödel's *dialectica* interpretation of 1958

David Hilbert did not precisely define what finitary mathematics is, but a very influential thesis of William Tait [39] identifies finitism with the quantifier-free system of primitive recursive arithmetic. This theory concerns only one sort of objects: the natural numbers. These are, in the Hilbertian terms as exposed by Gödel in his 1958 paper, "in the last analysis spatiotemporal arrangements of elements whose characteristics other than their identity or nonidentity are irrelevant." Gödel considers an *extension* of finitism (the work concerns, as its title says, "a hitherto unutilized extension of the finitary standpoint"), viz. a certain quantifier-free, *many-sorted*, theory. Its "axioms (...) are formally almost the same as those of primitive recursive number theory, the only exception being that the variables (other than those on which induction is carried out), as well as the defined constants, can be of any finite type over the natural numbers" (quoted from [11]). The variables are supposed to range over the so-called computable functionals of finite type (a primitive notion for Gödel). This is the crux of the extension: the requirement that the value of the variables be concrete ("spatiotemporal arrangements") is dropped, and certain *abstracta* are accepted.

The current literature has some very clear descriptions and explanations of Gödel's theory T. Easily available sources are Avigad and Feferman's survey in [1] and Kohlenbach's monograph [25]. The latter source includes a detailed treatment of Spector's bar recursive interpretation (different from the one presented here). In the present section, we briefly highlight the main features of T but the reader is referred to the above sources for details and pointers to the literature. The quantifier-free language $\mathcal{T}$ of T has infinitely many sorts (variable ranges), one for each finite type over the natural numbers. These types are syntactic expressions defined inductively: N (the base type) is a finite type; if τ and σ are finite types, then $\tau \to \sigma$ is a finite type. These are all the types there are. It is useful to have the following (set-theoretic) interpretation in mind: the base type N is the type constituted by the natural numbers $\mathbb{N}$, whereas $\tau \to \sigma$ is the type of all (total) set-theoretic functions of objects of type τ to objects of type σ. To ease reading, we often omit brackets and associate the arrows to the right. E.g., $\mathrm{N} \to \mathrm{N} \to \mathrm{N}$ means $\mathrm{N} \to (\mathrm{N} \to \mathrm{N})$. $\mathcal{T}$ has a denumerable set of variables x^σ, y^σ, z^σ, etc. for each type σ. When convenient, we omit the type scripts. There are two kinds of constants:

(a) *Logical constants* or *combinators*. For each pair of types σ, τ there is a logical constant $\Pi_{\sigma,\tau}$ of type $\sigma \to \tau \to \sigma$. For each triple of types δ, σ, τ there is a logical constant $\Sigma_{\delta,\sigma,\tau}$ of type $(\delta \to \sigma \to \tau) \to (\delta \to \sigma) \to (\delta \to \tau)$.

(b) *Arithmetical constants*. The constant 0 of type N. The *successor* constant S of type $\mathrm{N} \to \mathrm{N}$. For each type σ, there is a *recursor* constant R_σ of type $\mathrm{N} \to \sigma \to (\sigma \to \mathrm{N} \to \sigma) \to \sigma$.

Constants and variables of type σ are terms of type σ. If t is a term of type $\sigma \to \tau$ and q is a term of type σ then one can form a new term, denoted by $App(t, q)$, of type τ (t is said to be applied to q). These are all the terms there are. We write tq or $t(q)$ for $App(t, q)$. We also write $t(q, r)$ instead of $(t(q))(r)$. In general, $t(q, r, \ldots, s)$ stands for $(\ldots((t(q))(r))\ldots)(s)$.

The intended meaning of these constants are given by certain identities. There are the identities for the combinators: $\Pi(x, y)$ is x and $\Sigma(x, y, z)$ is $x(z, yz)$. The identities for the combinators make possible the definition of *lambda terms* within Gödel's T: given a term t^σ and a variable x^τ, there is a term $q^{\tau\to\sigma}$ (denoted by the lambda notation $\lambda x.t$) whose variables are all those of t other than x, such that, for every term s of type τ, one has the identity between qs and $t[s/x]$ (the notation '$[s/x]$' indicates the substitution of the variable x by the term s in the relevant expression). For the recursors, we have the following identities: $R(0, y, z)$ is y and $R(Sx, y, z)$ is $z(R(x, y, z), x)$. These identities formulate definitions by recursion.

We have been speaking loosely about identities because there are subtle issues concerning the treatment of equality in functional interpretations: consult [40] and [1] for discussions. (These issues surface because extensional equality suffers from a serious shortcoming with respect to the *dialectica* interpretation, viz: the axiom of extensionality, i.e., the postulation that extensional equality enjoys substitution *salva veritate* fails to be interpretable. This was shown by Howard in [15].) We adopt the following minimal treatment: there is only the symbol for equality between terms of the base type N, and the formulas of $\mathcal{T}$ are defined as Boolean combinations of equalities of the form $t = q$, where t and q are terms of type N. How are the identities for the combinators and recursors to be formulated within this framework? They give rise to certain axiom schemes. For instance, the axioms for the recursors are given by the equivalences $A[R(0, y, z)/w] \leftrightarrow A[y/w]$ and $A[R(Sx, y, z)/w] \leftrightarrow A[z(R(x, y, z), x)/w]$, where A is any formula of $\mathcal{T}$ with a distinguished variable w.

The axioms of T are the axioms of *classical* propositional calculus, the axioms of equality $x = x$ and $x = y \wedge A[x/w] \to A[y/w]$ (A is any formula of $\mathcal{T}$, and x, y and w are of type N, of course), the schemata coming from the identities of combinators and recursors and, finally, the usual arithmetical axioms for the constants 0 and S, namely: $Sx \neq 0$ and $Sx = Sy \to x = y$. There are also two rules. The rule of substitution that allows to infer $A[s^\sigma/x]$ from A and the rule of induction that, from $A(0)$ and $A(x^{\mathrm{N}}) \to A(Sx)$ allows the inference of $A(x)$ (in both rules, A can be any formula of $\mathcal{T}$).

We have described Gödel's quantifier-free, many-sorted, system T. Gödel showed that it is possible to interpret Heyting arithmetic (and, hence, Peano arithmetic) into T in a finitistic way. This result entails that the consistency of PA is finitistically reducible to the consistency of a natural extension of finitism. In the sequel, we describe Gödel's result. We formulate a direct interpretation of an extension (to finite types) of PA into T. The extension, which we denote by PA^ω, is a quantifier version of T. Its language $\mathcal{L}^\omega$ is obtained from $\mathcal{T}$ by adding quantifiers for each type. Formulas of $\mathcal{L}^\omega$ can now be constructed in the usual way by means of quantification. Note that the quantifier-free fragment of $\mathcal{L}^\omega$ is constituted exactly by the formulas of $\mathcal{T}$. PA^ω is formulated in classical logic. Its axioms consist of the universal closures of the axioms of T and the induction scheme constituted by the universal closures of

$$A(0) \wedge \forall x^{\mathrm{N}}(A(x) \to A(Sx)) \to \forall x A(x)$$

where A can be any formula of $\mathcal{L}^\omega$. There is (now) no (need for the) substitution rule nor (the) induction rule. PA^ω can be considered an extension of first-order arithmetic PA because both sum and product can be defined using the recursors. As an aside, it is now possible to *define* equality $x =_\sigma y$ in higher types by $\forall F^{\sigma\to 0}(Fx = Fy)$. With this Leibnizian definition, we have the usual properties of equality (reflexivity, symmetry, transitivity and substitution *salva veritate*, but *not* that it coincides with extensional equality).

We are now ready to define an interpretation of PA^ω into T. As noted in the introduction, this interpretation is due to Shoenfield in [34]. Like all functional interpretations, it consists of a trade-off between quantifier complexity and higher types. Since the logic is classical, we may assume that the primitive logical connectives are disjunction, negation and universal quantifications.

Definition. *To each formula A of the language $\mathcal{L}^\omega$ we assign formulas A^{S} and A_{S} so that A^{S} is of the form $\forall \underline{x} \exists \underline{y} A_{\mathrm{S}}(\underline{x}, \underline{y})$, with $A_{\mathrm{S}}(\underline{x}, \underline{y})$ a quantifier-free formula of $\mathcal{L}^\omega$, according to the following clauses:*

1. *A^{S} and A_{S} are simply A, for atomic formulas A*

If we have already interpretations of A and B given by $\forall \underline{x} \exists \underline{y} A_{\mathrm{S}}(\underline{x}, \underline{y})$ and $\forall \underline{z} \exists \underline{w} B_{\mathrm{S}}(\underline{z}, \underline{w})$ (respectively) then we define:

2. *$(A \vee B)^{\mathrm{S}}$ is $\forall \underline{x}, \underline{z} \exists \underline{y}, \underline{w}(A_{\mathrm{S}}(\underline{x}, \underline{y}) \vee B_{\mathrm{S}}(\underline{z}, \underline{w}))$*
3. *$(\neg A)^{\mathrm{S}}$ is $\forall \underline{f} \exists \underline{x} \neg A_{\mathrm{S}}(\underline{x}, \underline{f}\underline{x})$*
4. *$(\forall u A(u))^{\mathrm{S}}$ is $\forall u \forall \underline{x} \exists \underline{y} A_{\mathrm{S}}(\underline{x}, \underline{y}, u)$*

In the above, the underlined variables denote tuples of variables (possibly empty). In the sequel, we omit the underlining. E.g., $(\neg A)^{\mathrm{S}}$ is written as $\forall f \exists x \neg A_{\mathrm{S}}(x, fx)$. The formulas A_{S} are the matrices of A^{S}. For instance, $(\neg A)_{\mathrm{S}}$ is $\neg A_{\mathrm{S}}(x, fx)$. There is a principle of choice that plays a fundamental role in Shoenfield's interpretation. It is the quantifier-free axiom of choice in all finite types, denoted by $\mathsf{AC}^\omega_{\mathrm{qf}}$:

$$\forall x^\sigma \exists y^\tau A_{\mathrm{qf}}(x,y) \to \exists f^{\sigma\to\tau} \forall x A_{\mathrm{qf}}(x, fx)$$

where σ and τ are any types and A_{qf} is a quantifier-free formula. This principle is called the *characteristic principle* of Shoenfield's interpretation because of the following result:

Proposition (Characterization of Shoenfield's interpretation)**.** *For any formula A of $\mathcal{L}^\omega$, the theory $\mathsf{PA}^\omega + \mathsf{AC}^\omega_{\mathrm{qf}}$ proves the equivalence $A \leftrightarrow A^{\mathrm{S}}$.*

The proposition is easy to prove by induction on the complexity of A. All the clauses of Shoenfield's translation, with the exception of negation, give rise to classically equivalent formulas. The choice principle $\mathsf{AC}^\omega_{\mathrm{qf}}$ is exactly what is needed to deal with the negation clause. We are now ready to state Gödel's result of 1958 in the form that is most convenient for us:

Theorem (after Gödel and Shoenfield)**.** *Let A be a sentence ot $\mathcal{L}^\omega$. If $\mathsf{PA}^\omega + \mathsf{AC}^\omega_{\mathrm{qf}} \vdash A$, then there are closed terms t (of appropriate types) of $\mathcal{T}$ such that $\mathsf{T} \vdash A_{\mathrm{S}}(x, tx)$.*

The proof is not difficult, but it is delicate at some points. One works with a suitable axiomatization of classical logic (the one given by Shoenfield in [34] is specially convenient) and with the usual axioms of arithmetic (it is simpler to work with an induction rule instead). It can be shown that the axioms are interpretable and that the rules of inference preserve the interpretation. Roughly, the logical part of the calculus is dealt by the combinators whereas the recursors are used to interpret induction. The quantifier-free axiom of choice is interpretable (essentially) because of the way that the clause of negation is defined. The remaining axioms are universal and, therefore, trivially interpretable. This is an obviously finitistic proof.

3 What is bar recursion? Set-theoretic considerations

Let C and D be non-empty sets, and let $F : C^{<\mathbb{N}} \mapsto D$, $G : C^{<\mathbb{N}} \times D^C \mapsto D$ and $Y : C^{\mathbb{N}} \mapsto \mathbb{N}$ be given functions (here, $C^{<\mathbb{N}}$ denotes the set of all finite sequences of elements of C). We introduce some notation. First, we distinguish an element 0_C of C. Given $s \in C^{<\mathbb{N}}$, denote by $|s|$ the length of $s = \langle s_0, s_1, \ldots, s_{|s|-1}\rangle$; if $s, t \in C^{<\mathbb{N}}$, $s * t$ is the concatenation of s with t. For $i \leq |s|$, let $s|_i$ be the sequence $\langle s_0, \ldots, s_{i-1}\rangle$. To each finite sequence $s \in C^{<\mathbb{N}}$, we denote by $\hat{s}$ the infinite sequence of $C^{\mathbb{N}}$ which prolongs s by zeroes. More precisely: $\hat{s}(i) = s_i$, for $i < |s|$; $\hat{s}(i) = 0_C$, for $i \geq |s|$. Finally, for $x \in C^{\mathbb{N}}$ and i a natural number, $\overline{x}(i)$ is the finite sequence $\langle x(0), \ldots, x(i-1)\rangle$.

A function B from $C^{<\mathbb{N}}$ to D is defined by *bar recursion* from F, G and Y if it satisfies the following equality:

$$B(s) = \begin{cases} F(s) & \text{if } \exists i \leq |s| \, (Y(\widehat{s|_i}) \leq i) \\ G(s, \lambda w. B(s * \langle w \rangle)) & \text{otherwise} \end{cases}$$

(The knowledgeable reader will notice that the above definition is slightly different from Spector's definition. The present definition has the advantage of having the functional Y directly related to a certain *tree* – as will be discussed below.) The above specification does not always define a total function. Take, for instance, the functional $Y : \mathbb{N}^{\mathbb{N}} \mapsto \mathbb{N}$ given by

$$Y(x) = \begin{cases} 0 & \text{if } \forall k\,(x(k) \neq 0) \\ i+1 & \text{if } x(i) = 0 \wedge \forall k < i\,(x(k) \neq 0) \end{cases}$$

Then, with appropriate F and G, we could consider

$$B(s^{\mathbb{N}^{<\mathbb{N}}}) = \begin{cases} 0 & \text{if } \exists i \leq |s|\,(Y(\widehat{s|_i}) \leq i) \\ 1 + B(s * \langle 1 \rangle) & \text{otherwise} \end{cases}$$

but it is easy to argue that B is not defined on the empty sequence $\langle\rangle$.

There is a simple condition on the function Y whose validity ensures that B is always defined. Consider the *tree*

$$T_Y := \{s \in C^{<\mathbb{N}} : \forall i \leq |s|(Y(\widehat{s|_i}) > i)\}$$

(This set is a tree because whenever $s * t \in T_Y$ then $s \in T_Y$.) We say that the tree T_Y is *well-founded* if $\forall x \in C^{\mathbb{N}} \exists i\,(\overline{x}(i) \notin T_Y)$, i.e., $\forall x \in C^{\mathbb{N}} \exists i\,(Y(\widehat{\overline{x}(i)}) \leq i)$. We call the latter condition, *Spector's condition* for Y.

Theorem. *Let $Y : C^{\mathbb{N}} \mapsto \mathbb{N}$ be given. The function Y satisfies Spector's condition if, and only if, there is a map hgt from T_Y into the ordinals such that, whenever s is a strict subsequence of t, then $hgt(t) < hgt(s)$.*

Proof. Suppose that there is an order inverting map hgt as above. Let $x \in C^{\mathbb{N}}$ be given and assume, in view of a contradiction, that $\forall i\,(\overline{x}(i) \in T_Y)$. Since, for each natural number i, $\overline{x}(i)$ is a strict subsequence of $\overline{x}(i+1)$, then $hgt(\overline{x}(i+1)) < hgt(\overline{x}(i))$. This gives an infinite descending sequence of ordinals, a contradiction.

Now, let us assume that Y satisfies Spector's condition, i.e., that the tree T_Y is well-founded. Suppose, in order to reach a contradiction, that there is no order inverting map from T_Y into the ordinals. This assumption implies that $T_Y \neq \emptyset$. Given $s \in T_Y$, let T_Y/s be $\{t \in C^{<\mathbb{N}} : s * t \in T_Y\}$. Note that T_Y/s is a non-empty tree. Consider the subset $T \subseteq T_Y$ constituted by the finite sequences $s \in T_Y$ such that T_Y/s is a tree for which there is no order inverting map to the ordinals. Since $\langle\rangle \in T_Y$ and $T_Y = T_Y/\langle\rangle$, we have $\langle\rangle \in T$. Moreover, it is clear that T is a *subtree* of T_Y. We claim that T has no endnodes, i.e., we show that if $s \in T$ then there is $w \in C$ such that $s * \langle w \rangle \in T$. Suppose not. Then there is $s \in T$ such that, for each $w \in C$, we can find an ordinal α_w and an order inverting map h_w from $T_Y/(s * \langle w \rangle)$ into α_w. Let $\alpha = \sup_{w \in C}(\alpha_w + 1)$ (the axiom of replacement of Zermelo-Fraenkel set theory is being used in this argument), and define

$$h(t) := \begin{cases} \alpha & \text{if } t = \langle\rangle \\ h_w(q) & \text{if } t = \langle w \rangle * q \end{cases}$$

for all $t \in T_Y/s$. By construction, h is an order inverting function from T_Y/s into the ordinals. This is a contradiction.

Now, since T is a non-empty tree without endnodes, then T has an infinite path, i.e., there is a function $x : N \mapsto T$ such that, for all natural numbers i, $\overline{x}(i) \in T$. This path is actually also a path throught T_Y, contradicting Spector's condition for Y. $\square$

If Y satisfies Spector's condition, the above theorem permits to justify bar recursive definitions by *transfinite recursion*. In order to see this, note that if $s \in T_Y$, $w \in C$ and $s * \langle w \rangle \in T_Y$, then we have $hgt(s * \langle w \rangle) < hgt(s)$. So $B(s)$ is defined by $G(s, \lambda w.B(s*\langle w \rangle))$, an operation that only uses values of B at points of T_Y of smaller ordinal height than s (the points outside T_Y pose no problem).

There are two important conditions that easily ensure Spector's condition for the function Y. One is the *continuity condition*:

$$\forall x \in C^{\mathbb{N}} \exists k \in \mathbb{N} \forall y \in C^{\mathbb{N}} \, (\overline{y}(k) = \overline{x}(k) \to Y(x) = Y(y)).$$

The other is the (weaker) *bounding condition*:

$$\forall x \in C^{\mathbb{N}} \exists n \in \mathbb{N} \forall i \in \mathbb{N} \, Y(\widehat{\overline{x}(i)}) < n.$$

As we will briefly discuss in the next section, these two conditions are related to important structures for bar recursion. However, the bounding conditon seems to be more fundamental (see [8]).

We saw that bar recursion is a form of definition by transfinite recursion on well-founded trees. We used set-theoretic arguments at will. The existence of the bar recursive functionals for well-founded Y is, nevertheless, amenable to a more elementary set theoretic treatment. It is sufficient to be able to form certain subsets of $Z \subseteq C^{<\mathbb{N}} \times D$ and to use the following principle of dependent choices:

$$\forall s \in C^{<\mathbb{N}} \exists w \in C \, A(s, s * \langle w \rangle) \to \exists x \in C^{\mathbb{N}} \forall i \in \mathbb{N} \, A(\overline{x}(i), \overline{x}(i+1))$$

for suitable predicates A. (Dependent choices ensures that there are "enough" infinite sequences around.) Let us briefly see why this is so.

Suppose that Y satisfies Spector's condition. A set $Z \subseteq C^{<\mathbb{N}} \times D$ is a *partial bar function*, and we write $\mathbb{P}(Z)$, if Z is a partial function (i.e., whenever $(s, d) \in Z$ and $(s, d') \in Z$ then $d = d'$) and, for all $s \in C^{<\mathbb{N}}$ with $s \in \mathrm{dom}(Z)$, either $s \notin T_Y \wedge Z(s) = F(s)$ or

$$s \in T_Y \wedge \forall w \in C \, (s * \langle w \rangle \in \mathrm{dom}(Z) \wedge Z(s) = G(s, \lambda w.Z(s * \langle w \rangle)))$$

We claim that if $\mathbb{P}(Z)$ and $\mathbb{P}(W)$ then $\mathbb{P}(Z \cup W)$. First, observe that it is easy to argue that if $s \in \mathrm{dom}(Z) \cap \mathrm{dom}(W)$ and $Z(s) \neq W(s)$ then there exists $w \in C$ such that $s * \langle w \rangle \in \mathrm{dom}(Z) \cap \mathrm{dom}(W)$ and $Z(s * \langle w \rangle) \neq W(s * \langle w \rangle)$. Now, if Z and W are not compatible at a certain given sequence $s \in C^{<\mathbb{N}}$, then (by the above observation) there must exist an infinite path $x \in C^N$ such that $\overline{x}(|s|) = s$ and, for all natural numbers $i \geq |s|$, $\overline{x}(i) \in \mathrm{dom}(Z) \cap \mathrm{dom}(W)$ and $Z(\overline{x}(i)) \neq W(\overline{x}(i))$. Of course, the existence of this path requires the principle of dependent choices. Clearly, we have $\forall i \, (\overline{x}(i) \in T_Y)$ and this contradicts the well-foundedness of T_Y.

Let $U := \bigcup \{Z : \mathbb{P}(Z)\}$. By the discussion above, it is clear that $\mathbb{P}(U)$. Note, also, that $C^{<\mathbb{N}} \setminus T_Y \subseteq \mathrm{dom}(U)$. If we show that U is a total function, i.e., defined for every $s \in C^{<\mathbb{N}}$, then U is the bar functional that we want. The following fact is easy to prove: if $s \in C^{<\mathbb{N}}$ and, for all $w \in C$, $s * \langle w \rangle \in \mathrm{dom}(U)$, then $s \in \mathrm{dom}(U)$. If $s \notin T_Y$, there is nothing to prove. If $s \in T_Y$, then $\mathbb{P}(U \cup \{(s, G(s, \lambda w.U(s * \langle w \rangle)))\})$.

By the maximality of U, $s \in \mathrm{dom}(U)$. Now, to see that U is a total function assume, in order to get a contradiction, that there is a sequence $s \in C^{<\mathbb{N}}$ such that $s \notin \mathrm{dom}(U)$. Using the above fact and dependent choices, it is easy to obtain $x \in C^{\mathbb{N}}$ such that, for all natural numbers i, if $i \geq |s|$, then $\overline{x}(i) \notin \mathrm{dom}(U)$. This entails $\forall i\, (\overline{x}(i) \in T_Y)$, contradicting the well-foundedness of T_Y.

4 Spector's quantifier-free theory for bar recursion

In [36], Spector introduces a logic-free theory of computable functionals of finite type (called Σ_4 in Spector' paper). In this section, we describe a quantifier-free variant of Σ_4 building on Gödel's quantifier-free theory T described in Section 2. The terms of the language of Spector's theory include the terms of $\mathcal{T}$ together with those obtained by term application from new constants $B_{\sigma,\tau}$ of type

$$(\sigma^{<\mathrm{N}} \to \tau) \to (\sigma^{<\mathrm{N}} \to \tau^{\sigma} \to \tau) \to ((\mathrm{N} \to \sigma) \to \mathrm{N}) \to \sigma^{<\mathrm{N}} \to \tau,$$

one for each pair of types σ, τ. We are casually using the type $\sigma^{<\mathrm{N}}$ of finite sequences of elements of type σ even though this is not a primitive type of our language. It is nevertheless possible to deal with finite sequences via a pair consisting of an infinite sequence and a natural number (whose intended meaning is to signal the truncation of the infinite sequence at the length of the given natural number). We will not worry about these technical issues in here. Let us denote the extended quantifier-free language by $\mathcal{T}_{\mathsf{BR}}$. Its formulas are built as in Gödel's T, only now with new terms coming from the bar constants. The theory $\mathsf{T} + \mathsf{BR}$ includes the rule of induction (and substitution) for the new formulas and the quantifier-free bar axioms (naturally) associated with the following equality:

$$B_{\sigma,\tau}(F, G, Y)(s) = \begin{cases} F(s) & \text{if } \exists i \leq |s| (Y(\widehat{s|_i}) \leq i) \\ G(s, \lambda w. B_{\sigma,\tau}(F, G, Y)(s * \langle w \rangle)) & \text{otherwise} \end{cases}$$

where, of course, F, G, Y and s are variables of types $\sigma^{<\mathrm{N}} \to \tau$, $\sigma^{<\mathrm{N}} \to \tau^{\sigma} \to \tau$, $(\mathrm{N} \to \sigma) \to \mathrm{N}$ and $\sigma^{<\mathrm{N}}$, respectively. As noted in the previous section, the above definition of bar recursion is not quite the same as Spector's. It is easy to see that our theory is included in Spector's theory. We do not know if they are the same theory.

We saw that the set-theoretical structure is not a model of $\mathsf{T} + \mathsf{BR}$. This is one of the main differences between Gödel's T and Spector's $\mathsf{T} + \mathsf{BR}$: the former, but not the latter, has the usual set-theoretic interpretation. This is due to the fact that definitions by ordinary number recursion are always available set-theoretically whereas definitions by bar-recursion depend on a certain well-foundedness condition (Spector's condition). Spector's theory enjoys the astonishing property that *every* functional whose type is of the form $(\mathrm{N} \to \sigma) \to \mathrm{N}$ automatically satisfies Spector's condition (cf. Kreisel's trick in Section 6). The property of well-foundedness is *unconditionally* associated with certain functionals, unlike in ordinary settings where one must always explicitly hypothesize conditions for it. This feature is related to certain intuitionistic ideas according to which uniform continuity automatically holds for real-valued functions defined on a closed bounded interval.

The first rigorous proofs that certain structures are models of Spector's $\mathsf{T}+\mathsf{BR}$ only appeared in the early seventies. If we put aside the term model (see [42] or [30]), Bruno Scarpellini's proof in [33] that the structure of all sequentially continuous functionals is a model of $\mathsf{T} + \mathsf{BR}$ is, to my knowledge, the first such rigorous proof. Spector's condition of the pertinent functionals is assured by the continuity condition mentioned in the previous section. Together with the fact that *all* functions with domain the (discrete topological) space of the natural numbers are sequentially continuous, it ensures – as we saw – that bar recursive functionals can be defined (one has also to check that they are sequentially continuous). Anne Troelstra shows in [42] that the structures ICF^ω and ECF^ω of the intensional (respectively, extensional) continuous functionals are models of $\mathsf{T} + \mathsf{BR}$ (it can be proven that the extensional structure is isomorphic to Scarpellini's model – cf. [19]). These structures, based on continuity assumptions, are natural to consider because they flow from the very intuitionistic ideas that were at the source of Spector's interpretation (see the next section). In 1985, Marc Bezem presents a quite different model. Bezem's structure [2] uses the so-called strongly majorizable functionals and admits discontinuous functionals. Spector's condition of the pertinent functionals is assured by the bounding condition mentioned in the previous section. Since *all* infinite sequences of (strongly) majorizable functionals are, themselves, strongly majorizable (i.e., are in Bezem's model), bar recursive functionals can be defined (of course, one must also check that the functionals so obtained are strongly majorizable). All rigorous proofs that some structures are models of Spector's theory appeared quite some years after 1962. This fact is a source of amazement for me, and it tells much about the spell of intuitionism among some logicians at the time.

As in the case of Gödel's T, we can extend the quantifier-free language of the theory $\mathsf{T} + \mathsf{BR}$ to a quantificational language $\mathcal{L}^\omega_{\mathsf{BR}}$ and consider the corresponding quantificational theory $\mathsf{PA}^\omega + \mathsf{AC}^\omega_{\mathrm{qf}} + \mathsf{BR}$. This theory consists of $\mathsf{PA}^\omega + \mathsf{AC}^\omega_{\mathrm{qf}}$, allowing now for the new formulas in the schemata of induction and quantifier-free choice, together with the new bar axioms.

Theorem (Soundness theorem for bar recursion)**.** *Let A be a sentence of the language of $\mathcal{L}^\omega_{\mathsf{BR}}$. If $\mathsf{PA}^\omega + \mathsf{AC}^\omega_{\mathrm{qf}} + \mathsf{BR} \vdash A$, then there are closed terms (of appropriate types) of $\mathcal{T}_{\mathsf{BR}}$ such that $\mathsf{T} + \mathsf{BR} \vdash A_{\mathrm{S}}(x, tx)$.*

Proof. The proof of the soundness of Shoenfield's interpretation needs hardly any additional work because the new bar axioms are universal closures of quantifier-free formulas and, hence, are automatically interpreted (by themselves). ☐

5 What is bar recursion? Brouwerian considerations

Spector's paper is entitled "Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics." According to Georg Kreisel in p. 161 of [29], the long title incorporates contributions by Spector, Gödel and Kreisel himself. Be that as it may, the catchword 'extension' is common to the title of Gödel's paper of 1958. Spector's paper, like Gödel's, tries to reduce the consistency of a classical theory to the acceptance of an *extension* of a certain foundational framework: Hilbert's finitism in Gödel's paper, Brouwer's intuitionism in Spector's case. Furthermore, both extensions share a similar pattern: they follow Gödel's cherished idea of "gain(ing) knowledge abstractly by means of notions of higher type" (quoted from Gödel's [11]).

A form of bar induction commonly accepted in intuitionistic mathematics is *monotone bar induction*. In the following, we formulate this principle in the language of finite-type arithmetic $\mathcal{L}^\omega$. The type σ and the formulas P and Q below are unrestricted (note that, in Brouwerian intuitionism, σ must be the type N of the natural numbers): If

Hyp1. $\forall x^{\mathrm{N}\to\sigma} \exists k^{\mathrm{N}} P(\overline{x}(k))$

Hyp2. $\forall s^{\sigma^{<\mathrm{N}}} \forall i \leq |s| (P(s|_i) \to P(s))$

Hyp3. $\forall s^{\sigma^{<\mathrm{N}}} (P(s) \to Q(s))$

Hyp4. $\forall s^{\sigma^{<\mathrm{N}}} (\forall w^\sigma Q(s * \langle w \rangle) \to Q(s))$

then $Q(\langle\rangle)$.

It is easy to argue that this principle is set-theoretically true (contrast this fact with bar recursion). Suppose that $Q(\langle\rangle)$ is false. Then, by Hyp4, there is w_0 such that $\neg Q(\langle w_0 \rangle)$. By Hyp 4 again, there is w_1 with $\neg Q(\langle w_0, w_1 \rangle)$. We can continue this process and get a sequence w of elements of type σ such that $\forall k \in \mathbb{N} \neg Q(\overline{w}(k))$. Of course, a form of dependent choices is needed to arrive at this conclusion. By Hyp3, $\forall k \in \mathbb{N} \neg P(\overline{w}(k))$. This contradicts Hyp1. Notice that the monotonicity condition Hyp2 was not used in the argument. Even though Hyp2 is not needed to justify *classically* the principle of bar induction, without Hyp2 the principle is not intuitionistically acceptable (because it would entail the lesser principle of omniscience, a form of excluded middle rejected by the intuitionists: cf. exercise 4.8.11 in [41]).

Together with a continuity argument, the above principle of bar induction proves (intuitionistically) the existence of the bar recursive functionals. Take F, G and Y as in the previous section. Let $P(s^{\sigma^{<\mathrm{N}}})$ be $\exists i \leq |s| (Y(\widehat{s|_i}) \leq i)$ and define $Q(s^{\sigma^{<\mathrm{N}}})$ by:

$$\exists B \forall t^{\sigma^{<\mathrm{N}}} [(P(s*t) \wedge B(s*t) = F(s*t)) \vee \\ (\neg P(s*t) \wedge B(s*t) = G(s*t, \lambda w^\sigma . B(s*t*\langle w \rangle)))]$$

where the variables have appropriate types. It is clear that Hyp2 and Hyp3 hold. The verification of Hyp4 uses an intuitionistically admissible form of choice. Hyp1 is true by appealing to the continuity of the functional Y (this is the only place in the argument which is not set-theoretically sound). Therefore, we can conclude $Q(\langle\rangle)$, i.e., that there exists the bar functional $B(F, G, Y)$. It is also not difficult to prove (by bar induction) that this functional is unique.

We have shown that bar recursion of type σ reduces intuitionistically to bar induction of the same type (with the aid of a principle of continuity). Can bar recursion of finite type be constructively justified? The matter was taken up in a seminar on the foundations of analysis led by Kreisel at Stanford in the summer of 1963, and a report

[27] circulated. The answer was that "for the precise formulation in this report of constructive principles implicit in known intuitionistic mathematics, the answer is negative by a wide margin (...)." What are these principles? They "concern primarily functionals of finite and transfinite types, free choice sequences, and generalized inductive definitions." The story of the accomplishments of the seminar and of the ensuing work over the next years is long-winded. To cut through the fog, I believe that it is fair to say that the proof-theoretic strength of the principles of intuitionistic mathematics considered by Kreisel lies at the level of the theory ID_1 of non-iterated monotone inductive definitions. They are enough to justify bar recursion of type $\mathbb{N}$ (i.e., when σ is $\mathbb{N}$) and perhaps (by slightly stronger theories) also of type $\mathbb{N} \to \mathbb{N}$ (see section 7 of [18]), but not more. The results were certainly disillusioning. Kreisel confides in [28] that "when I originally considered the extension of [Spector] to analysis I believed that the *particular* notion of functional of finite type there described could be proved by intuitionistic methods to satisfy [the functional interpretation of analysis]. Put differently, I thought that the *existing* intuitionistic theory of free choice sequences, especially if one uses the formally powerful continuity axioms, was of essentially the same proof theoretic strength as full classical analysis!" (italics as in the original). As we now know, ID_1 has the proof-theoretic strength of Π^1_1-comprehension without set parameters and it is a far cry from full second-order comprehension.

Note, however, that the answer is negative as measured against *existing* intuitionistic theory. The title of Spector's paper explicitly mentions an *extension* of principles formulated in current intuitionistic mathematics. We believe that the benefits of Spector's consistency proof have to be judged on its own terms: against the intuitionistic plausibility of the extension proposed. In the Stanford report, Kreisel writes that, according to Gödel, "if one finds Brouwer's argument for the bar theorem conclusive then one should accept the generalization in Spector's paper." (Interestingly, it is added that "nothing much was intended to follow from this because [Gödel] does not find Brouwer's argument conclusive." In the same vein, Spector says in his paper that the bar theorem is itself questionable and in need of a suitable foundation.) I see in Gödel's opinion the implication that a conclusive argument for the bar theorem would generalize to bar induction in finite types. It would be expedient if the specialists who are convinced by Brouwer's argument could give their assessement of its possible generalization to higher types. There is also the other leg of the argument, the one regarding the continuity condition. In an intuitionistic setting, continuity is a consequence of Brouwer's doctrine about choice sequences. Their analogue in Spector's framework are sequences of higher type functionals (vis-à-vis sequences of concrete natural numbers). Do choice sequences of higher-order *abstracta* make sense for the 'creating subject'?

6 Bar recursion entails bar induction (in the presence of quantifier-free choice)

We prove a result that is preparatory for interpretating analysis in Spector's $\mathsf{T}+\mathsf{BR}$, viz. that a certain simplified form of bar induction is a consequence of $\mathsf{PA}^\omega + \mathsf{AC}^\omega_{\mathrm{qf}} + \mathsf{BR}$.

The next proposition is instrumental in showing this. It says that, in the presence of BR, functionals of type $Y^{(\mathrm{N}\to\sigma)\to\mathrm{N}}$ automatically satisfy Spector's condition:

Proposition (Kreisel's trick [26])**.** *For any type σ, the theory* $\mathsf{PA}^\omega + \mathsf{BR}$ *proves the sentence* $\forall Y^{(\mathrm{N}\to\sigma)\to\mathrm{N}} \forall x^{\mathrm{N}\to\sigma} \exists i^{\mathrm{N}}\, (Y(\widehat{\overline{x}(i)}) \leq i)$.

Proof. Fix Y and x. Define $W : \sigma^{<\mathrm{N}} \to \mathrm{N}$ by bar recursion in the following way:

$$W(s) := \begin{cases} 0 & \text{if } \exists i \leq |s|\, (Y(\widehat{s|_i}) \leq i) \\ 1 + W(s * \langle x(|s|)\rangle) & \text{otherwise} \end{cases}$$

and let $h(k) := W(\overline{x}(k))$. By definition, it is clear that

$$h(k) = \begin{cases} 0 & \text{if } \exists i \leq k\, (Y(\widehat{\overline{x}(i)}) \leq i) \\ 1 + h(k+1) & \text{otherwise} \end{cases}$$

Let k be given. Clearly, if $h(k) \neq 0$ and $i \leq k$, then $h(0) = i + h(i)$. In particular, if $h(k) \neq 0$, $h(0) = k + h(k)$. Instantiating k by $h(0)$, we can conclude that if $h(h(0)) \neq 0$ then $h(0) = h(0) + h(h(0))$. Therefore, $h(h(0)) = 0$. By definition of h, we conclude that $\exists i \leq h(0)\, (Y(\widehat{\overline{x}(i)}) \leq i)$. $\square$

Given a type σ and an existential formula $P(s)$ with a distinguished variable s of type $\sigma^{<\mathrm{N}}$, we consider the following simplified version of monotone bar induction, denoted by $\mathsf{BI}_\exists^-$: From the three hypotheses

H1. $\forall x^{\mathrm{N}\to\sigma} \exists k^N P(\overline{x}(k))$

H2. $\forall s^{\sigma^{<\mathrm{N}}} \forall i \leq |s|\, (P(s|_i) \to P(s))$

H3. $\forall s^{\sigma^{<\mathrm{N}}} (\forall w^\sigma P(s * \langle w\rangle) \to P(s))$

one can conclude $P(\langle\rangle)$.

Theorem (after Howard)**.** *The theory* $\mathsf{PA}^\omega + \mathsf{AC}^\omega_{\mathrm{qf}} + \mathsf{BR}$ *proves* $\mathsf{BI}^-_\exists$.

Proof. Let $P(s)$ be the existential statement $\exists a^\tau P_{\mathrm{qf}}(s, a)$, where s has type $\sigma^{<\mathrm{N}}$ and P_{qf} is a quantifier-free formula. Assume the hypotheses of bar-induction. By the first hypothesis, $\forall x \exists k, a P_{\mathrm{qf}}(\overline{x}(k), a)$. By $\mathsf{AC}^\omega_{\mathrm{qf}}$, there are functionals $Y : (\mathrm{N} \to \sigma) \to \mathrm{N}$ and $H : (\mathrm{N} \to \sigma) \to \tau$ such that

$\tilde{\mathrm{H}}1.\quad \forall x\, P_{\mathrm{qf}}(\overline{x}(Yx), Hx)$

By the second hypothesis, $\forall s \forall i \leq |s| \forall a \exists b (P_{\mathrm{qf}}(s|_i, a) \to P_{\mathrm{qf}}(s, b))$. Hence, by $\mathsf{AC}^\omega_{\mathrm{qf}}$, there is a functional $F : \sigma^{<\mathrm{N}} \to \mathrm{N} \to \tau \to \tau$ such that

$\tilde{\mathrm{H}}2.\quad \forall s \forall i \leq |s| \forall a\, (P_{\mathrm{qf}}(s|_i, a) \to P_{\mathrm{qf}}(s, F(s, i, a)))$

By $\mathsf{AC}^\omega_{\mathrm{qf}}$, it is easy to see that $\forall s, f^{\sigma\to\tau} \exists w, b\, (P_{\mathrm{qf}}(s * \langle w\rangle, fw) \to P_{\mathrm{qf}}(s, b))$ is equivalent to the last hypothesis of bar induction. Using $\mathsf{AC}^\omega_{\mathrm{qf}}$ to witness w and b and then disregarding the witness of w, there is $G : \sigma^{<\mathrm{N}} \to (\sigma \to \tau) \to \tau$ such that

H̃3. $\forall s, f\,(\forall w P_{\mathrm{qf}}(s * \langle w \rangle, fw) \to P_{\mathrm{qf}}(s, G(s,f)))$

Lat us define, by bar-recursion, the following functional:

$$B(s) := \begin{cases} F(s, Y(\widehat{s|_{i_0}}), H(\widehat{s|_{i_0}})) & \text{if } \exists i \le |s|\,(Y(\widehat{s|_i}) \le i) \\ G(s, \lambda w.B(s * \langle w \rangle)) & \text{otherwise} \end{cases}$$

where i_0 is the least number i such that $Y(\widehat{s|_i}) \le i$.

We claim that, for all s of type $\sigma^{<\mathrm{N}}$, if $\exists i \le |s|\, Y(\widehat{s|_i}) \le i$, then $P_{\mathrm{qf}}(s, Bs)$. In fact, by H̃1, we have $P_{\mathrm{qf}}(\overline{\widehat{s|_{i_0}}}(Y(\widehat{s|_{i_0}})), H(\widehat{s|_{i_0}}))$. Since $Y(\widehat{s|_{i_0}}) \le i_0 \le |s|$, the finite sequence $\overline{\widehat{s|_{i_0}}}(Y(\widehat{s|_{i_0}}))$ is actually the sequence $s|_{Y(\widehat{s|_{i_0}})}$. Hence, $P_{\mathrm{qf}}(s|_{Y(\widehat{s|_{i_0}})}, H(\widehat{s|_{i_0}}))$. Using H̃2, we get $P_{\mathrm{qf}}(s, F(s, Y(\widehat{s|_{i_0}}), H(\widehat{s|_{i_0}})))$, that is, $P_{\mathrm{qf}}(s, Bs)$.

Secondly, we claim that, for all s of type $\sigma^{<\mathrm{N}}$, if $\forall i \le |s|\, Y(\widehat{s|_i}) > i$, then

$$\forall w P_{\mathrm{qf}}(s * \langle w \rangle, B(s * \langle w \rangle)) \to P_{\mathrm{qf}}(s, Bs).$$

To see this, suppose that $\forall w P_{\mathrm{qf}}(s * \langle w \rangle, B(s * \langle w \rangle))$. Let $f := \lambda w.B(s * \langle w \rangle)$. With this notation, we have $\forall w P_{\mathrm{qf}}(s * \langle w \rangle, fw)$. By H̃3, we conclude that $P_{\mathrm{qf}}(s, G(s,f))$, that is, $P_{\mathrm{qf}}(s, Bs)$.

Of course, the above two claims entail that, for every s of type $\sigma^{<\mathrm{N}}$,

$$\forall w P_{\mathrm{qf}}(s * \langle w \rangle, B(s * \langle w \rangle)) \to P_{\mathrm{qf}}(s, Bs).$$

Therefore,

$$\forall s\,[\neg P_{\mathrm{qf}}(s, Bs) \to \exists w \neg P_{\mathrm{qf}}(s * \langle w \rangle, B(s * \langle w \rangle))].$$

By $\mathsf{AC}^{\omega}_{\mathrm{qf}}$, there is a functional $T : \sigma^{<\mathrm{N}} \to \sigma$ such that

$$\forall s\,[\neg P_{\mathrm{qf}}(s, Bs) \to \neg P_{\mathrm{qf}}(s * \langle Ts \rangle, B(s * \langle Ts \rangle))].$$

We now define, by recursion, a functional z of type $N \to \sigma$ according to the following clause: $z(k) = T(\langle z(0), z(1), \ldots, z(k-1) \rangle)$. Note that $z(0) = T(\langle\rangle)$. By construction, we have

$$\neg P_{\mathrm{qf}}(\overline{z}(k), B(\overline{z}(k))) \to \neg P_{\mathrm{qf}}(\overline{z}(k+1), B(\overline{z}(k+1))),$$

for every natural number k.

Suppose, in order to reach a contradiction, that the conclusion of bar-induction fails, i.e., that $\neg P(\langle\rangle)$. Therefore, $\forall a \neg P_{\mathrm{qf}}(\langle\rangle, a)$. In particular, $\neg P_{\mathrm{qf}}(\overline{z}(0), B(\overline{z}(0)))$. By induction, we get $\neg P_{\mathrm{qf}}(\overline{z}(k), B(\overline{z}(k)))$, for all k^{N}. This contradicts the fact that, by Kreisel's trick, there is i^{N} such that $Y(\widehat{\overline{z}(i)}) \le i$ and, hence, by the first claim above, that $P_{\mathrm{qf}}(\overline{z}(i), B(\overline{z}(i)))$. □

We finish this section with a discussion concerning equality. At a certain point of the previous argument, we apparently used the axiom of extensionality. The finite sequences $\widehat{\overline{s|_{i_0}}(Y(\widehat{s|_{i_0}}))}$ and $s|_{Y(\widehat{s|_{i_0}})}$ are extensionally equal. As a result, we used the implication

$$P_{\mathrm{qf}}(\widehat{\overline{s|_{i_0}}(Y(\widehat{s|_{i_0}}))}, H(\widehat{s|_{i_0}})) \to P_{\mathrm{qf}}(s|_{Y(\widehat{s|_{i_0}})}, H(\widehat{s|_{i_0}})),$$

which amounts to a substitution *salva veritate*. Actually, by carefully defining the notion of finite sequence, this implication can be justified without any extensionality assumptions, and the theorem above is correct as it stands (i.e., based on the *minimal* theory PA^ω). There are alternatives, though: one way out is not to worry about the precise definition of finite sequences and simply admit in our theory the universal statements $j \leq i \leq |s| \wedge \Phi(\widehat{\overline{s|_i}(j)}) \to \Phi(s|_j)$. Granted, this is an *ad hoc* maneuver (but quite an admissible one). A more systematic way of getting the desired universal statements is to include in our base theory a so-called weak extensionality rule. This is the choice of Spector in his original paper. In [25], Kohlenbach follows this route and the reader is directed to this reference for a thorough discussion of this rule.

7 The interpretation of analysis

Analysis, a.k.a. full second-order arithmetic PA_2, is the extension of first-order arithmetic PA to a language $\mathcal{L}^2$ with a new sort of (second-order) variables for sets of natural numbers, a new kind of atomic fomulas taking the form '$t \in X$', where t is a first-order term and X is a second-order variable, and whose axioms include the full comprehension scheme:

$$\exists X \forall x\, (x \in X \leftrightarrow A(x))$$

where A is any formula of $\mathcal{L}^2$ (first and second-order parameters are allowed). Induction in PA_2 can be stated by the single axiom

$$\forall X(0 \in X \wedge \forall x(x \in X \to Sx \in X) \to \forall x(x \in X))$$

Given that we have full comprehension, induction actually applies to every formula of the second-order language. The language $\mathcal{L}^2$ can be embedded into $\mathcal{L}^\omega$ by letting the number variables run over arguments of type N, letting the set variables run over variables of type $\mathrm{N} \to \mathrm{N}$ subjected to a process of normalization (so that they take values in $\{0, 1\}$), and by interpreting $t \in X$ by $X(t) = 0$.

We need three definitions within $\mathcal{L}^\omega$:

Definition. *The principle of full numerical comprehension* CA^{N} *is the following scheme:*

$$\exists f^{\mathrm{N}\to\mathrm{N}} \forall x^{\mathrm{N}} (fx = 0 \leftrightarrow A(x))$$

where A is any formula.

Definition. *The principle of dependent choices* DC^ω *is the following scheme:*

$$\forall x^\sigma \exists y^\sigma A(x,y) \to \forall u^\sigma \exists f^{\mathrm{N}\to\sigma}(f0 = u \wedge \forall k A(fk, f(k+1))),$$

where σ *is any type and* A *is any formula. The restriction of the above principle to universal formulas* A *is denoted by* $\mathsf{DC}^\omega_\forall$*.*

Definition. *The principle of numerical choice* $\mathsf{AC}^{\mathrm{N},\omega}$ *is the following scheme:*

$$\forall k \exists x^\sigma A(k,x) \to \exists f^{\mathrm{N}\to\sigma} \forall k\, A(k, fk),$$

where σ *is any type and* A *is any formula. The restriction of this principle when* σ *is the type of natural numbers* N *is denoted by* $\mathsf{AC}^{\mathrm{N},\mathrm{N}}$*.*

Proposition (Easy facts)**.**

1. $\mathsf{PA}^\omega + \mathsf{BI}^-_\exists \vdash \mathsf{DC}^\omega_\forall$
2. $\mathsf{PA}^\omega + \mathsf{AC}^\omega_{\mathrm{qf}} + \mathsf{DC}^\omega_\forall \vdash \mathsf{DC}^\omega$
3. $\mathsf{PA}^\omega + \mathsf{DC}^\omega \vdash \mathsf{AC}^{\mathrm{N},\omega}$
4. $\mathsf{PA}^\omega + \mathsf{AC}^{\mathrm{N},\mathrm{N}} \vdash \mathsf{CA}^{\mathrm{N}}$

Proof. Let $A(x^\sigma, y^\sigma)$ be a universal formula such that $\forall x \exists y A(x,y)$. Fix u^σ. We must show that there is $f : \mathrm{N} \to \sigma$ such that $f0 = u$ and $\forall k A(fk, f(k+1))$. Take v^σ such that $A(u,v)$ and define the following existential formula $P(s^{\sigma<\mathrm{N}})$:

$$P(s) := \exists i \leq |s| \neg A((\langle u,v\rangle * s)_i, (\langle u,v\rangle * s)_{i+1})$$

By the choice of v, it is clear that $\neg P(\langle\rangle)$. We claim that the hypotheses H2 and H3 of $\mathsf{BI}^-_\exists$ hold for P. This is straightforward for H2 and not so difficult to verify for H3. Supose that, for a given $s^{\sigma<\mathrm{N}}$ one has $\forall w P(s * \langle w\rangle)$. This means that, for all w^σ, either

$$\exists i \leq |s| \neg A((\langle u,v\rangle * s * \langle w\rangle)_i, (\langle u,v\rangle * s * \langle w\rangle)_{i+1})$$

or $\neg A((\langle u,v\rangle * s * \langle w\rangle)_{|s|+1}, (\langle u,v\rangle * s * \langle w\rangle)_{|s|+2})$. The first disjunct is equivalent to $P(s)$ whereas the second is equivalent to $\neg A(z,w)$, where $z = v$ if $|s| = 0$ and $z = s_{|s|-1}$ otherwise. By the arbitrariness of w, one has $P(s) \vee \forall w \neg A(z,w)$. This entails $P(s)$ and, therefore, the verification of H3 is finished.

By $\mathsf{BI}^-_\exists$, we must conclude that H1 fails for P. Therefore,

$$\exists x^{\mathrm{N}\to\sigma} \forall k \forall i \leq k\, A((\langle u,v\rangle * \overline{x}(k))_i, (\langle u,v\rangle * \overline{x}(k))_{i+1})$$

It is now clear that the function

$$f(k) := \begin{cases} u & \text{if } k = 0 \\ v & \text{if } k = 1 \\ x(k-1) & \text{if } k \geq 2 \end{cases}$$

satisfies $f0 = u$ and $\forall k A(fk, f(k+1))$.

We have just proved the first easy fact. (This argument is basically in the appendix of [14].) In order to prove the second fact, take $A(x^\sigma, y^\sigma)$ any formula such that $\forall x \exists y A(x, y)$ and fix u^σ. By Proposition 2, the formula $A(x, y)$ is equivalent to $\forall w \exists z A_{\mathrm{S}}(w, z, x, y)$ for w and z of appropriate types. Using $\mathsf{AC}^\omega_{\mathrm{qf}}$, $A(x, y)$ is equivalent to $\exists h \forall w A_{\mathrm{S}}(w, hw, x, y)$. By hypothesis, and inserting a dummy variable g (of the same type as h), we have

$$\forall x, g \exists y, h \forall w A_{\mathrm{S}}(w, hw, x, y)$$

Since A_{S} is quantifier-free, we are in the conditions of application of $\mathsf{DC}^\omega_\forall$. Therefore, there are $f^{\mathrm{N}\to\sigma}$ and appropriate l such that $f(0) = u$ and

$$\forall k \forall w A_{\mathrm{S}}(w, l(k+1)w, fk, f(k+1))$$

We conclude that $\forall k A(fk, f(k+1))$.

Let us now consider the third fact. Suppose that $\forall k \exists x^\sigma A(k, x)$. Fix u such that $A(0, u)$. Clearly, $\forall k, x \exists n, y\, (n = k+1 \wedge A(n, y))$. By DC^ω, there are $f^{\mathrm{N}\to\sigma}$ and $g^{\mathrm{N}\to\mathrm{N}}$ such that $f(0) = u$, $g(0) = 0$ and $\forall k (g(k+1) = g(k) + 1 \wedge A(g(k+1), f(k+1)))$. It is clear, by induction, that $\forall k (gk = k)$. It easily follows that $\forall k A(k, fk)$.

The proof of the fourth fact is well-known. Take an arbitrary formula $A(k^{\mathrm{N}})$. Clearly, $\forall k \exists n ((n = 0 \wedge A(k)) \vee (n = 1 \wedge \neg A(k)))$. By $\mathsf{AC}^{\mathrm{N,N}}$, there is $f^{\mathrm{N}\to\mathrm{N}}$ such that, for all k^{N}, $A(k)$ if, and only if, $fk = 0$. □

Howard's theorem of the previous section, together the above facts, entails that DC^ω is a consequence of the theory $\mathsf{PA}^\omega + \mathsf{AC}^\omega_{\mathrm{qf}} + \mathsf{BR}$. We highlight the following result:

Corollary. *The theory* $\mathsf{PA}^\omega + \mathsf{AC}^\omega_{\mathrm{qf}} + \mathsf{BR}$ *proves* CA^{N}.

By the above corollary, PA_2 can be considered a subtheory of $\mathsf{PA}^\omega + \mathsf{AC}^\omega_{\mathrm{qf}} + \mathsf{BR}$. For ease of reading, in the next theorem we identify a formula of $\mathcal{L}^2$ with its translation into $\mathcal{L}^\omega$.

Theorem (after Spector)**.** *Let A be a sentence of the language of second-order arithmetic. If* $\mathsf{PA}_2 \vdash A$*, then there are closed terms t (of appropriate types) of* $\mathcal{T}_{\mathsf{BR}}$ *such that* $\mathsf{T} + \mathsf{BR} \vdash A_{\mathrm{S}}(x, tx)$.

Proof. Suppose that $\mathsf{PA}_2 \vdash A$. By the discussion above, $\mathsf{PA}^\omega + \mathsf{AC}^\omega_{\mathrm{qf}} + \mathsf{BR} \vdash A$. By the soundness theorem of Section 4, the result follows. □

Note that the above proof is finitistic. Therefore, by considering the formula A to be $0 = 1$, this theorem shows that the consistency of analysis is finistitically reducible to the consistency of Spector's quantifier-free theory $\mathsf{T} + \mathsf{BR}$.

The restriction of $\mathsf{T}+\mathsf{BR}$ to bar recursion of type N is very-well understood (see the next section), and it has played a fruitful role in the foundations of mathematics. For instance, Kohlenbach gave in [23] a particularly perspicuous analysis of arithmetical comprehension based on the bar-recursor $B_{\mathrm{N},\mathrm{N}\to\mathrm{N}}$.

8 Epilogue

Spector's consistency proof is a beautiful and sophisticated piece of work. It provides a surprising way of replacing the comprehension principles of analysis by forms of transfinite recursion. But, as a *consistency proof*, does it command any epistemological conviction? In Section 5, we defended that the most promising argument for a possible epistemological gain provided by Spector's proof is still the original intended one: to rely on an *extension* of the principles of Brouwerian intuitionism by considering the generalization of bar induction to finite types. It is nevertheless an almost universal conviction that Brouwer's argument for the bar theorem is inconclusive, let alone its possible extension to higher types. With no conclusive arguments for the extension, nothing much is attained.

Other readings of Spector's proof are possible. Spector's proof reduces the comprehension principles of analysis to the termination of some effective processes (viz, to the normalization of the closed terms of $\mathcal{T}_{\mathsf{BR}}$). This is no mean achievement. The postulation of the normalization of the closed terms of $\mathcal{T}_{\mathsf{BR}}$ is sufficient to prove (modulo some weak arithmetic) the consistency of analysis. Proofs of normalization for the terms of $\mathcal{T}_{\mathsf{BR}}$ do exist in the literature (they, in fact, guarantee the existence of the term model). The first such proof is, to my knowledge, due to Tait in [38]. Of necessity (by Gödel's second incompleteness theorem), these proofs use proof-theoretic power stronger than the power of analysis. Tait's proof is not ordinal informative. However, the situation is different for some subclasses of $\mathcal{T}_{\mathsf{BR}}$. There are proofs of the normalization of the terms of Gödel's T by the method of assigning to them ordinals less than ϵ_0 (see [40] for references), therefore providing another route to Gentzen's proof of the consistency of PA. Moreover, Helmut Vogel and Howard gave in [44], [16] and [17] a detailed ordinal analysis of bar recursion of type N. As far as we are aware, ordinal analyses of stronger forms of bar recursion have not been pursued.

The most important benefits of Spector's proof probably lie elsewhere. Not in consistency proofs but in applications to the extraction of computational information from ordinary proofs of mathematics. The methods of Kohlenbach's proof mining (conveniently reported in [25]) can be applied to full second-order arithmetic because of the work of Spector. Kohlenbach, as a matter of course, works with systems with full second-order comprehension. In more recent studies, bar recursion has also been extended to new types, used to interpret – for instance – abstract normed spaces (see [24] and [10]). Even though the uses of bar recursion have not yet shown up in an essential way in the analyses of ordinary mathematical proofs, the situation can – in principle – change. Kohlenbach's methods are also deeply interwoven with questions of uniformity (i.e., the obtaining of bounds independent from some parameters), including a set-theoretic false uniform boundedness principle. These methods are possible within analysis because of the majorizability of the bar recursive constants. Majorizability considerations have played an important role in the removal of ideal elements (conservation results). The paramount example is the elimination of weak König's lemma (fan theorem) for theories without arithmetical comprehension: see [35] and [22]. The bounded functional interpretation of Ferreira and Oliva [9] can be seen as a thorough exploitation of majorizability properties. It was first defined for arithmetic but it extends to analysis via bar recursion (cf. [5] and [4]). The relations between functional interpretations, majorizability, uniformity results, elimination of ideal elements, extraction of computational information and the role of some classically false principles constitute a fascinating topic in the foundations of mathematics. My paper [6] includes a general discussion on these issues.

9 Appendix

Let T_K be a Kleene binary tree, i.e., a primitive recursive infinite tree of finite binary sequences with no infinite recursive path (Kleene introduced such an example in [20]). The form of bar induction described in Section 6 can be used to prove that T_K has an infinite path. This can be seen by considering the existential statement

$$P(s^{\{0,1\}^{<\mathrm{N}}}) := \exists l^{\mathrm{N}}(l \geq |s| \wedge \forall t^{\{0,1\}^{<\mathrm{N}}}(|s * t| = l \to s * t \notin T_K))$$

(The universal quantification on the finite binary sequence t can be considered bounded because the length of t does not exceed l.) Both H2 and H3 hold but, since Kleene's tree is infinite, the conclusion $P(\langle\rangle)$ fails. Therefore, H1 must fail and this readily entails that there is an infinite path through T_K. By the choice of T_K, this infinite Boolean sequence is not recursive. A close inspection of the proof of Howard's theorem in Section 6 shows that the amount of quantifier-free choice needed to prove the required bar induction is just $\mathsf{AC}_{\mathsf{qf}}^{\mathrm{N}\to\{0,1\},\mathrm{N}}$ (the meaning of this notation should be clear). This amount of choice justifies the existence of a functional Y of type $(\mathrm{N} \to \{0,1\}) \to \mathrm{N}$ with the property that $\forall x^{\mathrm{N}\to\{0,1\}}(\overline{x}(Yx) \notin T_K)$, and this fact is sufficient to pull the proof through.

It is the *combination* of bar recursion and quantifier-free choice that is responsible for the introduction of non-recursive Boolean sequences in models of PA^ω (via forms of bar induction or, what is classically the same thing, via dependent choices). The lack of just one of these ingredients may result in the failure of introducing non-recursive sequences. For instance, the structure HRO^ω of the hereditarily recursive operations is a case where $\mathsf{AC}_{\mathsf{qf}}^{\mathrm{N}\to\{0,1\},\mathrm{N}}$ is available but BR fails. On the other hand, the term model is a case where BR holds but $\mathsf{AC}_{\mathsf{qf}}^{\mathrm{N}\to\{0,1\},\mathrm{N}}$ fails. Both structures only have recursive Boolean sequences.

Interestingly, the soundness theorem for bar recursion applies to the theory with the combination of quantifier-free choice and bar recursion (see the end of Section 4). Hence, bar induction is available and the theory proves the existence of non-recursive Boolean sequences (of course, only a very restricted form of bar recursion is needed for obtaining infinite paths through infinite binary recursive trees but, as we saw, unrestricted bar recursion even proves full second-order comprehension). From the soundness theorem, one easily shows that $\mathsf{PA}^\omega + \mathsf{AC}_{\mathrm{qf}}^\omega + \mathsf{BR}$ is conservative over $\mathsf{PA}^\omega + \mathsf{BR}$ with respect to sentences which, in prenex normal form, have quantifier prefix $\forall\exists$. The existence of an infinite path through Kleene's tree T_K is a statement of quantifier prefix $\exists\forall$, and the conservation result does not apply. Spector's interpretation is subtle indeed.

References

[1] J. Avigad and S. Feferman. Gödel's functional ("Dialectica") interpretation. In S. R. Buss, editor, *Handbook of Proof Theory*, volume 137 of *Studies in Logic and the Foundations of Mathematics*, pages 337–405. North Holland, Amsterdam, 1998.

[2] M. Bezem. Strongly majorizable functionals of finite type: a model for bar recursion containing discontinuous functionals. *The Journal of Symbolic Logic*, 50:652–660, 1985.

[3] L. E. J. Brouwer. Über Definitionsbereiche von Funktionen. *Mathematische Annalen*, 93:60–75, 1927. English translation in [43], pp. 457-463.

[4] P. Engrácia. The bounded functional interpretation of bar induction. *Annals of Pure and Applied Logic*, 163(9):1183 – 1195, 2012.

[5] P. Engrácia and F. Ferreira. The bounded functional interpretation of the double negation shift. *The Journal of Symbolic Logic*, 75:759–773, 2010.

[6] F. Ferreira. A most artistic package of a jumble of ideas. *dialectica*, 62:205–222, 2008. Special Issue: Gödel's *dialectica* Interpretation. Guest editor: Thomas Strahm.

[7] F. Ferreira. A short note on Spector's proof of consistency of analysis. In B. Cooper, A. Dawar, and B. Löwe, editors, *How the World Computes*, volume 7318 of *Lecture Notes in Computer Science*, pages 222–227. Springer-Verlag, 2012.

[8] F. Ferreira and U. Kohlenbach. Manuscript in preparation (2014).

[9] F. Ferreira and P. Oliva. Bounded functional interpretation. *Annals of Pure and Applied Logic*, 135:73–112, 2005.

[10] P. Gerhardy and U. Kohlenbach. General logical metatheorems for functional analysis. *Transactions of the American Mathematical Society*, 360:2615–2660, 2008.

[11] K. Gödel. Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. *dialectica*, 12:280–287, 1958. Reprinted with an English translation in [13], pp. 240-251.

[12] K. Gödel. On an extension of finitary mathematics which has not yet been used, 1972. Published in [13], pages 271–280. Revised version of [11].

[13] K. Gödel. *Collected Works, Vol. II*. S. Feferman *et al.*, eds. Oxford University Press, Oxford, 1990.

[14] W. A. Howard. Functional interpretation of bar induction by bar recursion. *Compositio Mathematica*, 20:107–124, 1968.

[15] W. A. Howard. Hereditarily majorizable functionals of finite type, 1973. Published in [42], pages 454–461.

[16] W. A. Howard. Ordinal analysis of simple cases of bar recursion. *The Journal of Symbolic Logic*, 46:17–30, 1980.

[17] W. A. Howard. Ordinal analysis of bar recursion of type zero. *Compositio Mathematica*, 42:105–119, 1981.

[18] W. A. Howard and G. Kreisel. Transfinite induction and bar induction of types zero and one, and the role of continuity in intuitionistic analysis. *The Journal of Symbolic Logic*, 31:325–358, 1966.

[19] M. Hyland. Filter spaces and continuous functionals. *Annals of Mathematical Logic*, 16:101–143, 1979.

[20] S. Kleene. Recursive functions and intuitionistic mathematics. In *Proceedings of the International Congress of Mathematicians, Cambridge 1950*, volume 1, pages 679–685. American Mathematics Society, 1952.

[21] S. Kleene and R. Vesley. *The Foundations of Intuitionistic Mathematics*. North Holland, Amsterdam, 1965.

[22] U. Kohlenbach. Effective bounds from ineffective proofs in analysis: an application of functional interpretation and majorization. *The Journal of Symbolic Logic*, 57:1239–1273, 1992.

[23] U. Kohlenbach. On the no-counterexample interpretation. *The Journal of Symbolic Logic*, 64:1491–1511, 1999.

[24] U. Kohlenbach. Some logical metatheorems with applications in functional analysis. *Transactions of the American Mathematical Society*, 357:89–128, 2005.

[25] U. Kohlenbach. *Applied Proof Theory: Proof Interpretations and their Use in Mathematics*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2008.

[26] G. Kreisel. Interpretation of analysis by means of constructive functionals of finite types. In A. Heyting, editor, *Constructivity in Mathematics*, pages 101–128. North Holland, Amsterdam, 1959.

[27] G. Kreisel. Reports of the seminar on the foundations of analysis. Technical report, Stanford University, 1963. Mimeographed report in two parts.

[28] G. Kreisel. A survey of proof theory. *The Journal of Symbolic Logic*, 3:321–388, 1968.

[29] G. Kreisel. Gödel's excursions into intuitionistic logic. In *Gödel Remembered*, pages 65–186. Bibliopolis, Napoli, 1987.

[30] H. Luckhardt. *Extensional Gödel Functional Interpretation: A Consistency Proof of Classical Analysis*, volume 306 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1973.

[31] P. Oliva. Understanding and using Spector's bar recursive interpretation of classical analysis. In B. Löwe A. Beckmann, U. Berger and J. Tucker, editors, *Logical Approaches to Computational Barriers – Proceedings of Computability in Europe 2006*, volume 3988 of *Lecture Notes in Computer Science*, pages 423–434. Springer-Verlag, New York, 2006.

[32] M. Rathjen. The realm of ordinal analysis. In S. B. Cooper and J. K. Truss, editors, *Sets and Proofs*, volume 258 of *London Mathematical Society Lecture Notes Series*, pages 219–280. Cambridge University Press, 1999.

[33] B. Scarpellini. A model for bar recursion of higher types. *Compositio Mathematica*, 23:123–153, 1971.

[34] J. R. Shoenfield. *Mathematical Logic*. Addison-Wesley Publishing Company, 1967. Republished in 2001 by AK Peters.

[35] W. Sieg. Fragments of arithmetic. *Annals of Pure and Applied Logic*, 28:33–71, 1985.

[36] C. Spector. Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles in current intuitionistic mathematics. In F. D. E. Dekker, editor, *Recursive Function Theory: Proceedings of Symposia in Pure Mathematics*, volume 5, pages 1–27. American Mathematical Society, Providence, Rhode Island, 1962.

[37] G. Sundholm and M. van Atten. The proper explanation of intuitionistic logic: on Brouwer's demonstration of the Bar Theorem. In M. van Atten, P. Boldini, M. Bourdeau, and G. Heinzmann, editors, *One Hundred Years of Intuitionism (1907–2007)*, Publications des Archives Henri Poincaré / Publications of the Henri Poincaré Archives, pages 60–77. Birkhäuser Basel, 2008.

[38] W. Tait. Normal form theorem for bar recursive functions of finite type. In *Proceedings of the 2nd Scandinavian Logic Symposium*, Studies in Logic and the Foundations of Mathematics, pages 353–367. Elsevier, 1971.

[39] W. Tait. Finitism. *Journal of Philosophy*, 78:524–546, 1981.

[40] A. S. Troelstra. Introductory note to [11] and [12], 1990. Published in [13], pages 214–241.

[41] A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics. An Introduction*, volume 121 of *Studies in Logic and the Foundations of Mathematics*. North Holland, Amsterdam, 1988.

[42] A. S. Troelstra (ed.). *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, volume 344 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1973.

[43] J. van Heijenoort (ed.). *From Frege to Gödel*. Harvard University Press, 1967.

[44] H. Vogel. Über die mit dem Bar-Rekursor vom Typ 0 definierbaren Ordinalzahlen. *Archiv für mathematische Logik und Grundlagenforschung*, 19:165–173, 1978/79.