A NEW COMPUTATION OF THE Σ-ORDINAL OF $\mathsf{KP}\omega$

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Abstract. We define a functional interpretation of $\mathsf{KP}\omega$ using Howard's primitive recursive tree functionals of finite type and associated terms. We prove that the Σ-ordinal of $\mathsf{KP}\omega$ is the least ordinal not given by a closed term of the ground type of the trees (the Bachmann-Howard ordinal). We also extend $\mathsf{KP}\omega$ to a second-order theory with Δ_1-comprehension and strict-Π^1_1 reflection and show that the Σ-ordinal of this theory is still the Bachmann-Howard ordinal. It is also argued that the second-order theory is Σ_1-conservative over $\mathsf{KP}\omega$.

§1. Introduction. Admissible proof theory made its debut with the 1979 doctoral dissertation of Gerhard Jäger [17]. Systems of analysis, both predicative and impredicative, can be embedded in theories of admissible sets and variations thereof. A particularly central and perspicuous system is Kripke-Platek set theory with infinity, denoted by $\mathsf{KP}\omega$, whose intendend models are the pure admissible sets with ω (see Barwise's opus [4] for notation and results used in this paper). The impredicative theory ID_1 of (non-iterated) arithmetical monotone inductive definitions can be embedded in $\mathsf{KP}\omega$ and shares with it the same proof-theoretical ordinal, viz the Bachmann-Howard ordinal. The main aim of this paper is to give a new computation of the Σ-ordinal of $\mathsf{KP}\omega$. The dominant method for computing this ordinal relies on infinitary cut-elimination for semi-formal systems of uncountably infinitary derivations. Our computation relies instead on Gödel's method of functional interpretation and is based on a finite-type system of tree terms. In his recent account [8] of the proof-theory of inductive definitions, Solomon Feferman insists – more than once – on conceptually clear and perspicuous solutions to the problems. We hope that our present attempt is one more step in this direction.

Our functional interpretation is a "bounded functional interpretation." These interpretations are, in the words of Gilda Ferreira and Paulo Oliva in [12], variants of functional interpretations where bounds (rather than precise witnesses) are extracted from proofs. They enjoy the further crucial property, proeminent in the interpretation given in the present article, that bounded quantifications are treated as computationally empty (this feature is briefly mentioned in the closing section of [12]). The specific form of the functional interpretation that we

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present here is very much indebted to the particular analysis of ID_1 by Jeremy Avigad and Henry Towsner in [2].

The theory $\mathsf{KP}\omega$ is framed in the language of set theory. Its axioms consist of extensionality, pair, union, infinity, the scheme of Δ_0-separation and the schemes of foundation and of Δ_0-collection. See [4] for precise formulations. Nevertheless, let me make some comments. Bounded or Δ_0-formulas are formulas obtained from the atomic formulas using propositional connectives and bounded quantifications. It is convenient to work with a language of set theory which includes a primitive syntactic apparatus for bounded quantifications: $\forall x \in z\, \phi$ is part of the primitive syntax and not an abbreviation of $\forall x\, (x \in z \to \phi)$. The axiom of infinity is stated as the existence of a limit ordinal: $\exists x\, Lim(x)$. The notion of limit ordinal can be defined by saying that x is a non-zero ordinal and that, whenever $y \in x$, then $y \cup \{y\} \in x$. Being an ordinal is, in turn, defined as being an hereditarily transitive set. The point is that $Lim(x)$ is a Δ_0-formula. The scheme of foundation is the following scheme, which applies to *every* formula $\phi(x)$, possibly with parameters:

$$\forall x\, (\forall y \in x\, \phi(y) \to \phi(x)) \to \forall x\, \phi(x).$$

The scheme of Δ_0-collection applies to Δ_0-formulas $\phi(x, y)$, possibly with parameters:

$$\forall x \in z \exists y\, \phi(x, y) \to \exists w \forall x \in z \exists y \in w\, \phi(x, y).$$

A Σ_1-formula ϕ is a formula of the form $\exists z\, \psi(z)$, where $\psi(z)$ is a Δ_0-formula (possibly with parameters). We have defined this notion with only one existential quantifier because the presence of the pairing axiom permits us to reduce the tuple case to the single variable case. In the literature, it is usual to introduce a more general form of Σ_1-formula: the Σ-formulas. However, in the presence of Δ_0-collection, both notions coincide and hence, in this paper, we will not distinguish them.

Gödel's hierarchy of constructible sets L_α, for ordinals α, plays an important role in the model theory of $\mathsf{KP}\omega$ as well as in admissible proof theory. The least ordinal α for which L_α is a model of $\mathsf{KP}\omega$ is the first non-recursive ordinal: the so-called Church-Kleene ordinal ω_1^{CK}. By general recursion-theoretic considerations (spelled out in [19]), the least ordinal α such that L_α is a model of all Σ_1-consequences of $\mathsf{KP}\omega$ is smaller than ω_1^{CK}. We can define the Σ-ordinal of $\mathsf{KP}\omega$, denoted by $||\mathsf{KP}\omega||_\Sigma$, as follows:

$$||\mathsf{KP}\omega||_\Sigma := \min\{\alpha : L_\alpha \models \psi \text{ for all } \Sigma_1 \text{ sentences } \psi \text{ such that } \mathsf{KP}\omega \vdash \psi\}.$$

Analogous definitions can be made for the other theories of this paper. Wolfram Pohlers discusses the importance of this recursive ordinal in [19] and argues that it is an upper bound for the order types of (primitive recursive) well-orderings for which the scheme of transfinite induction is provable in $\mathsf{KP}\omega$.

This paper is organized as follows. In the next section, we introduce Howard's primitive recursive tree functionals and define the Bachmann-Howard ordinal as the least ordinal not given by the ordinal height of a closed term of the ground type of the trees. The main section of the paper follows. In it, we define the new functional interpretation and prove a corresponding soundness theorem. As a corollary, the Bachmann-Howard ordinal is shown to be an upper bound for

$||\mathsf{KP}\omega||_\Sigma$. Section 4 is an interlude section, with brief comments on alternative functional interpretations. In Section 5, we define a second-order variant of $\mathsf{KP}\omega$ with the scheme of strict Π^1_1-reflection and extend the functional interpretation to this theory. The following section internalizes the functional interpretations within $\mathsf{KP}\omega$. As a consequence, it is shown that the Bachmann-Howard ordinal is a lower bound for $||\mathsf{KP}\omega||_\Sigma$ and it is also proved a conservation result concerning strict Π^1_1-reflection.

§2. Howard's primitive recursive tree functionals. In this section, we describe the language $\mathcal{L}_\Omega$ of the primitive recursive tree functionals of finite type and its set-theoretical interpretation. $\mathcal{L}_\Omega$ is a language consisting only of terms (there are no formulas) which is an extension of the (term part of the) language of Gödel's finite-type theory T. We expand the language of T with a new ground type Ω, for the countable constructive tree ordinals. We denote the ground type of the natural numbers by N. The complex types are obtained from the ground types by closing under arrow. We use the letters ρ, τ, σ, ... to denote the types. The *pure* Ω*-types* are the types obtained (via the arrow) from the ground type Ω only. The language has a denumerable stock of variables a, b, c, ... for each type. When we want to make explicit the type of the variables, we use the notation a^ρ, b^τ, c^σ, ... We follow the usual conventions concerning omission of parentheses (as in [1]). The language consists of the following constants:

(a) *Logical constants* or *combinators.* For each pair of types ρ, τ there is a combinator of type $\rho \to \tau \to \rho$ denoted by $\Pi_{\rho,\tau}$. For each triple of types δ, ρ, τ there is a combinator of type
$$(\delta \to \rho \to \tau) \to (\delta \to \rho) \to (\delta \to \tau)$$
denoted by $\Sigma_{\delta,\rho,\tau}$.

(b) *Arithmetical constants.* The constant 0_N of type N. The *successor* constant S of type $N \to N$. For each type ρ, a *(number) recursor* constant of type
$$N \to \rho \to (N \to \rho \to \rho) \to \rho$$
denoted by R^N_ρ.

(c) *Tree constants.* The constant 0_Ω of type Ω. The *supremum* constant Sup of type $(N \to \Omega) \to \Omega$. For each type ρ, a *tree recursor* constant of type
$$\Omega \to \rho \to ((N \to \Omega) \to (N \to \rho) \to \rho) \to \rho$$
denoted by R^Ω_ρ.

The terms of $\mathcal{L}_\Omega$ are obtained from the variables and the constants by successively using the operation of application: If t is a term of type $\rho \to \tau$ and q is a term of type ρ then $App(t, q)$ is a term of type τ. We write (tq) instead of $App(t, q)$, as usual. There are no other terms in $\mathcal{L}_\Omega$. As it is well-known, the presence of the combinators allows the introduction of lambda terms via the usual definitions. The original variant of this term language was defined by William Howard in [16]. Our exposition is a streamlined version of the language described in section 9.1 of [1] (without the so-called μ-operator).

Before proceeding with further syntactic discussions, let me give the intended set-theoretic interpretation of this language (the interpretation can be carried in Zermelo-Fraenkel set theory ZF). The variables of each type ρ range over a set S_ρ defined according to the following clauses:

1. $S_N = \mathbb{N}$
2. S_Ω is the smallest set W which contains 0 and is such that, whenever f is a function that maps ω into W, then $(1, f) \in W$.
3. $S_{\rho\to\tau} = \{f : f \text{ is a function that maps } S_\rho \text{ into } S_\tau\}$

We skip the interpretations of the combinators and the arithmetical constants (they are as usual). The constant 0_Ω is interpreted by 0. The constant *Sup* is interpreted by the function which, on input $f \in S_{N\to\Omega}$, outputs the element $(1, f)$ of W. We simply write $Sup(f) = (1, f)$, instead of $Sup^I(f) = (1, f)$, where the superscript I stands for the interpretation function. In the sequel, we omit the superscript and use the same notation for a symbol and its (intended) interpretation. This ambiguity (and abuse of language) will be systematic, but harmless.

To each element c of W, we can associate a countable set-theoretical ordinal $|c|$ so that $|0| = 0$ and, for $f : \mathbb{N} \to W$, $|Sup(f)| = \sup\{|f(n)| + 1 : n \in \mathbb{N}\}$. Observe that $|f(n)| < |Sup(f)|$, for each natural number n. Hence, we can define by classical ordinal recursion the interpretations of the tree recursors R^Ω_ρ so that:

$$R^\Omega_\rho(0_\Omega, a, F) = a \text{ and } R^\Omega_\rho(Sup(f), a, F) = F(f, \lambda x^N.R^\Omega_\rho(f(x), a, F)),$$

for all $a \in S_\rho$ and F a function that maps $S_{N\to\Omega}$ into $S_\rho^{S_{N\to\rho}}$.

Finally, the application operation is interpreted simply by function application. We have given the set-theoretic interpretation of $\mathcal{L}_\Omega$. As a consequence, each closed term t of $\mathcal{L}_\Omega$ of type ρ has an interpretation, which we denote by t again. We can see t as an element of S_ρ. In particular, if ρ is the arrow type $\tau \to \sigma$ then t is a function from S_τ into S_σ and it makes sense to write $t(a)$, for $a \in S_\tau$. If t is a closed term of ground type Ω, then its interpretation is an element of W. Therefore, it has an associated ordinal $|t|$. The supremum of all these ordinals is the *Bachmann-Howard* ordinal. An anonymous referee pointed out that this is not the original definition of the Bachmann-Howard ordinal. The referee is right, of course. This ordinal was first defined in 1950 by Heinz Bachmann through a primitive recursive ordinal notation system (cf. [3]). Only in his groundbreaking paper [16] of 1972, did Howard prove that this ordinal coincides with the supremum of the ordinal heights of the closed terms of (an inessential variant of) $\mathcal{L}_\Omega$.

In this paragraph, we define some important terms and describe their interpretations. We let $c+1 :\equiv Sup(\lambda x^N.c)$. Of course, $|c+1| = |c|+1$. It is easy to define (using the number recursor) a closed term t such that, for all $a, b \in W$ and $n \in \mathbb{N}$, $t(a, b, 0) = a$ and $t(a, b, Sn) = b$. We consider the term $Sup(\lambda x^N.t(a, b, x))$, which we denote by $\max(a, b)+1$ (this notation should be viewed syncategorematically, of course). Clearly, $|\max(a, b) + 1| = \max(|a|, |b|) + 1$. By number recursion, we can define a closed term $q^{N\to\Omega}$ such that $q(0) = 0_\Omega$ and $q(Sn) = Sup(\lambda x^N.q(n))$. We write n_Ω instead of $q(n)$. Clearly, $|n_\Omega| = n$. Let $\omega_\Omega := Sup(\lambda x^N.x_\Omega)$. Obviously, $|\omega_\Omega| = \omega$. By tree recursion, take the functional Sup^{-1} of type $\Omega \to (N \to \Omega)$ such that $Sup^{-1}(0_\Omega) = \lambda x^N.0_\Omega$ and $Sup^{-1}(Supf) = f$. We abbreviate $Sup^{-1}(a)(n)$ by $a\langle n\rangle$. With this notation, $(Sup(f))\langle n\rangle = f(n)$ and, therefore, $|a\langle n\rangle| < |a|$, for every $n \in \mathbb{N}$ and $a \in W \setminus \{0_\Omega\}$. With the definitions just introduced, it is clear that $(a+1)\langle n\rangle = a$, $(\max(a,b)+1)\langle 0\rangle = a$ and $(\max(a,b)+1)\langle n+1\rangle = b$.

§3. The functional interpretation. In the previous section, we described the term language $\mathcal{L}_\Omega$ and its intended set-theoretical interpretation. Let us come back to syntactic matters again. We consider a mixed language that is the union of the term language $\mathcal{L}_\Omega$ and of the first-order language of set theory together with a unary function symbol $L(a)$, which we write L_a, acting on terms of type Ω and giving out a set-theoretical term (in the intended interpretation, L_a stands for the set $L_{|a|}$ of Gödel's constructible hierarchy of sets). The *atomic formulas* of the mixed language are the formulas of the form $x = y$, $x \in y$ or $x \in L_t$, for x and y set-theoretical variables and t a term of $\mathcal{L}_\Omega$ of type Ω. Finally, we also allow in the mixed language quantifications of the form $\exists n^N \phi$. The set-theoretical interpretation of this quantification is the obvious one. We classify this kind of quantification as a *bounded* quantification (intuitively, a numerical quantification corresponds to a bounded quantification in $\mathsf{KP}\omega$). In our functional interpretation, we are going to associate to each formula of the language of set theory (as described above) a set-theoretical predicate. We describe these predicates via formulas of the mixed language.

DEFINITION 1. *The class of* bounded mixed formulas *is the smallest class of formulas of the mixed language that contains the atomic formulas and is closed under Boolean connectives and bounded quantifications.*

From the discussion above, the intended set-theoretic interpretation of the bounded mixed formulas is clear.

In the following, we are going to associate to each formula $\phi(x_1, \ldots, x_n)$ of the language of set theory (free variables as shown) a bounded mixed formula ϕ_S of the form

$$\phi_S(a_1, \ldots, a_k, b_1, \ldots, b_m, x_1, \ldots, x_n),$$

with free variables as shown (the a's and the b's are variables of $\mathcal{L}_\Omega$ of pure Ω-type, determined by the definition below). Note that k or m or both can be zero, in which case these variables do not occur. For notational simplicity, we often omit tuples of variables and simply write $\phi(x)$ and $\phi_S(a, b, x)$ (as noted, the variables may be absent).

For a crucial clause in the definition below, we need to extend the Sup^{-1} from the ground type Ω to all pure Ω-types. The extension is defined pointwise. Let t be a term of pure Ω-type $\tau_1 \to \tau_2 \to \ldots \tau_n \to \Omega$ (pure Ω types are necessarily of this form). We define $t\langle n\rangle :\equiv \lambda \underline{x}.((t\underline{x})\langle n\rangle)$, where $\underline{x}$ is a n-tuple of variables of appropriate types.

The functional interpretation in Definition 2 is given for classical logic directly. It is based on the direct interpretation of Peano arithmetic by Joseph Shoenfield in his textbook [22]. The logical primitives of Shoenfield's calculus are just $\neg$, $\vee$ and $\forall$. The other logical symbols are defined as usual. In particular, we will use systematically the definitions $\phi \to \psi :\equiv \neg\phi \vee \psi$ and $\exists x\, \phi :\equiv \neg\forall x \neg\phi$. As discussed, we also have the primitive bounded quantifier $\forall x \in z$ and define

$\exists x \in z\, \phi :\equiv \neg \forall x \in z \neg\, \phi$. We are now ready to define a functional interpretation for $\mathsf{KP}\omega$.

Definition 2. *To each formula ϕ of the language of set theory, we assign formulas ϕ^{S} and ϕ_{S} so that ϕ^{S} is of the form $\forall a \exists b\, \phi_{\mathrm{S}}(a,b)$, with $\phi_{\mathrm{S}}(a,b)$ a bounded mixed formula, according to the following clauses:*

1. *$(\phi)^{\mathrm{S}}$ and ϕ_{S} are simply ϕ, for bounded formulas ϕ of the language of set theory.*

For the remaining *cases, if we have already interpretations for ϕ and ψ given by $\forall a \exists b\, \phi_{\mathrm{S}}(a,b)$ and $\forall d \exists e\, \psi_{\mathrm{S}}(d,e)$ (respectively), then we define:*

2. *$(\phi \vee \psi)^{\mathrm{S}}$ is $\forall a, d \exists b, e\, [\phi_{\mathrm{S}}(a,b) \vee \psi_{\mathrm{S}}(d,e)]$,*
3. *$(\forall x \phi(x))^{\mathrm{S}}$ is $\forall a, c^{\Omega} \exists b\, [\forall x \in L_c\, \phi_{\mathrm{S}}(a,b,x)]$,*
4. *$(\forall x \in z\, \phi(x,z))^{\mathrm{S}}$ is $\forall a \exists b\, [\forall x \in z\, \phi_{\mathrm{S}}(a,b,x,z)]$,*
5. *$(\neg\phi)^{\mathrm{S}}$ is $\forall B \exists a\, [\exists n^{N} \neg \phi_{\mathrm{S}}(a\langle n\rangle, B(a\langle n\rangle))]$.*

In the above definition of ϕ^{S}, we are using *quantifications* on variables of $\mathcal{L}_{\Omega}$. This is incidental, as it is costumary in functional interpretations, and one can see them as a mere aid for defining the bounded mixed formulas ϕ_{S} (alternatively, we can be literal and extend the language with these quantifiers). We have written the lower case S-translations inside square parentheses. For instance, $(\forall x \phi(x))_{\mathrm{S}}$ is, by definition, the bounded mixed formula $\forall x \in L_c\, \phi_{\mathrm{S}}(a,b,x)$. Let me make a final clarification regarding tuples. In clause (5) above, if ϕ^{S} is of the form $\forall a_1, a_2 \exists b\, \phi_{\mathrm{S}}(a_1,a_2,b)$, then $(\neg\phi)^{\mathrm{S}}$ is

$$\forall B \exists a_1 \exists a_2\, [\exists n^{N}, m^{N}\, \neg \phi_{\mathrm{S}}(a_1\langle n\rangle, a_2\langle m\rangle, B(a_1\langle n\rangle, a_2\langle m\rangle))].$$

The general case follows this pattern.

Lemma 1. *Let ϕ be a Δ_0-formula. Then:*

(i) *$(\exists x\, \phi(x))^{\mathrm{S}}$ is $\exists b^{\Omega} [\exists n^{N} \exists x \in L_{b\langle n\rangle} \phi(x)]$.*
(ii) *$(\forall x \exists y\, \phi(x,y))^{\mathrm{S}}$ is $\forall a^{\Omega} \exists b^{\Omega}\, [\forall x \in L_a \exists n^{N} \exists y \in L_{b\langle n\rangle} \phi(x,y)]$.*

Proof. The proof is easy. Let us check (i). We must compute $(\neg\forall x \neg\phi(x))^{\mathrm{S}}$. Well, $(\forall x \neg\phi(x))^{\mathrm{S}}$ is $\forall b^{\Omega} \forall x \in L_b \neg\phi(x)$. The result follows. ⊣

A more natural definition of $(\neg\phi)^{\mathrm{S}}$ would, of course, be $\forall B \exists a\, [\neg\phi_{\mathrm{S}}(a, B(a))]$. Observe that $\exists a\, [\neg\phi_{\mathrm{S}}(a,B(a))]$ is equivalent to $\exists a\, [\exists n^{N} \neg\phi_{\mathrm{S}}(a\langle n\rangle, B(a\langle n\rangle))]$. However, as it is typical of bounded functional interpretations, the definition proposed does not work. Even though the upper S-translations are equivalent, the lower S-translations are not. In order for the proof of the interpretation theorem to go through, it is *crucial* to have a monotonicity property regarding the existential quantification. Following [2], we define $b \sqsubseteq c$ as $\forall n \in \mathbb{N} \exists m \in \mathbb{N}\, (b\langle n\rangle = c\langle m\rangle)$, for b and c of the same pure Ω-type. We also use the notation $b \sqsubseteq c$ for b and c tuples of the same length and same corresponding pure Ω-types. In this case, the notation means that each entry of the first tuple is below (in the sense of $\sqsubseteq$) the corresponding entry of the second tuple.

Lemma 2 (Monotonicity Property). *Let $\phi(x)$ a formula of the language of set theory. If $b \sqsubseteq c$ and $\phi_{\mathrm{S}}(a,b,x)$, then $\phi_{\mathrm{S}}(a,c,x)$.*

PROOF. The proof is by induction on the complexity of ϕ. It is clear that clauses (2), (3) and (4) of the functional interpretation preserve the monotonicity property. The final clause (5) is designed to preserve monotonicity. ⊣

Fix p a pairing term of type $N \to (N \to N)$ with inverse functions l and r, both of type $N \to N$. Hence, $p(l(n), r(n)) = n$, $l(p(m,k)) = m$ and $r(p(m,k)) = k$, for all natural numbers n, m, k. Given t a term of type $N \to \Omega$, we let $\tilde{t} :\equiv \lambda z^N.t(l(z))\langle r(z)\rangle$ and define $\bigsqcup t :\equiv \mathit{Sup}\, \tilde{t}$. It follows easily from the definitions that, for every natural number k, $t(k) \sqsubseteq \bigsqcup t$. It is clear that we can also define (pointwise) $\bigsqcup t$ for t of type $N \to \rho$, with ρ a pure Ω-type. If ρ is of the form $\tau_1 \to \tau_2 \to \ldots \tau_n \to \Omega$, we let $\bigsqcup t :\equiv \lambda \underline{x}.\mathit{Sup}\, \widetilde{\lambda z^N.tz\underline{x}}$ (here, $\underline{x}$ is a n-tuple of variables of appropriate types). Of course, $t(k) \sqsubseteq \bigsqcup t$ also holds in this case for all natural numbers k. It is also easy to define $t \sqcup q$, for terms t and q of type $N \to \rho$ (with ρ of pure Ω-type), so that $t \sqsubseteq t \sqcup s$ and $s \sqsubseteq t \sqcup s$.

THEOREM 1 (Soundness Theorem). *Let ϕ be a sentence of the language of set theory. Suppose that* $\mathsf{KP}\omega \vdash \phi$. *Then there are closed terms t of $\mathcal{L}_\Omega$ such that, for appropriate types ρ, $\forall a \in S_\rho\, \phi_S(a, t(a))$.*

PROOF. The proof is by induction on the number of lines of the derivation. We show that if the formula $\phi(x)$ is provable in $\mathsf{KP}\omega$, then there are closed terms t of $\mathcal{L}_\Omega$ such that, for appropriate types ρ, we have

$$\forall c \in W \forall a \in S_\rho \forall x \in L_c\, \phi_S(a, t(a,c), x).$$

For ease of reading, we ignore parameters that do not play an important role in the functional interpretation.

For the logical part of the calculus, we rely on the complete axiomatization of classical logic described in sections 2.6 and 8.3 of [22]. Shoenfield's axiomatization consists of three types of axioms and five rules. The axioms are:

- Excluded middle: $\neg\phi \vee \phi$
- Substitution: $\forall x \phi(x) \to \phi(w)$
- Equality axioms: $x = x$ and $x = y \to (w = z \to (x \in w \to y \in z))$

The rules are:

- Expansion: from ϕ infer $\phi \vee \psi$
- Contraction: from $\phi \vee \phi$ infer ϕ
- Associativity: from $\phi \vee (\psi \vee \theta)$ infer $(\phi \vee \psi) \vee \theta$
- Cut: from $\phi \vee \psi$ and $\neg\phi \vee \theta$ infer $\psi \vee \theta$
- $\forall$-introduction: from $\phi(x) \vee \psi$ infer $\forall x \phi(x) \vee \psi$, provided that x does not occur free in ψ

The verification of the logical axioms and rules is formally similar to the verifications in [9] and, specially, in [2] (this is particularly the case for the cut rule where the argument has an extra complication, absent in [9]). In the following, we always take ϕ^{S} as $\forall a^\rho \exists b\, \phi_{\mathrm{S}}(a,b)$, ψ^{S} as $\forall d^\eta \exists e\, \psi_{\mathrm{S}}(d,e)$ and θ^{S} as $\forall u \exists v\, \theta_{\mathrm{S}}(u,v)$.

The upper S-translation of excluded middle is

$$\forall B, a \exists \tilde{a}, b\, [\forall n^N \phi_{\mathrm{S}}(\tilde{a}\langle n\rangle, B(\tilde{a}\langle n\rangle)) \to \phi_{\mathrm{S}}(a,b)].$$

Clearly $\tilde{a} := a+1$ and $b := B(a)$ work. The upper S-translation of substitution is

$$\forall d, B, a \exists \tilde{a}, c, b\, (\forall n^N, m^N \forall x \in L_{c\langle m\rangle}\, \phi_{\mathrm{S}}(\tilde{a}\langle n\rangle, B(\tilde{a}\langle n\rangle), x) \to \forall w \in L_d\, \phi_{\mathrm{S}}(a, b, w)).$$

Clearly, $\tilde{a} := a+1$, $c := d+1$ and $b := B(a)$ do the job. The equality axioms pose no problem because they are universal.

Let us consider the rules now. Expansion is easy. By induction, there is a term t such that $\forall a \in S_\rho\, \phi_{\mathrm{S}}(a, t(a))$. We need to find witnesses for the existential claims of the upper S-translation of the conclusion of the rule:

$$\forall a, d \exists b, e^\sigma\, (\phi_{\mathrm{S}}(a, b) \vee \psi_{\mathrm{S}}(d, e)).$$

Just take $b := ta$ and $e := 0^\sigma_\Omega$. Here, 0^σ_Ω is just 0_Ω in case σ is the ground type Ω. Otherwise, σ is of the form $\tau \to \Omega$, for a certain pure Ω-type τ and we let 0^σ_Ω be $\lambda x^\tau.0_\Omega$.

Contraction is usually a delicate affair in ordinary functional interpretations, but is rather easy for bounded interpretations. By induction hypothesis, there are terms t and q such that

$$\forall a, \tilde{a} \in S_\rho\, (\phi_{\mathrm{S}}(a, t(a, \tilde{a})) \vee \phi_{\mathrm{S}}(\tilde{a}, q(a, \tilde{a})).$$

By monotonicity, if we let $r(a) := t(a, a) \sqcup q(a, a)$, we get $\forall a \in S_\rho\, \phi_{\mathrm{S}}(a, r(a))$, as wanted.

Associativity is trivial. Let us consider the cut rule. By induction, there are terms t, q, r and s such that

$$(\$) \qquad \forall a^\rho, d\, (\phi_{\mathrm{S}}(a, t(a, d)) \vee \psi_{\mathrm{S}}(d, q(a, d))) \quad \text{and}$$

$$(\$\$) \qquad \forall B, u\, (\exists n^N \neg \phi_{\mathrm{S}}(r(B, u)\langle n\rangle, B(r(B, u)\langle n\rangle)) \vee \theta_{\mathrm{S}}(u, s(B, u)))$$

hold in the set-theoretic interpretation. We need now to define terms k and l so that $\forall d, u\, (\psi_{\mathrm{S}}(d, k(d, u)) \vee \theta_{\mathrm{S}}(u, l(d, u)))$ also holds in the set-theoretic interpretation. Given d and u, consider $B_d := \lambda x^\rho.t(x, d)$ and $\tilde{a}(d, u) := \lambda n^N.r(B_d, u)\langle n\rangle$. Finally, let $\tilde{q}(d, u) := \lambda n^N.q(a_n, d)$, where a_n denotes $\tilde{a}(d, u)(n)$, i.e, $r(B_d, u)\langle n\rangle$. We can now define k and l in the following way:

$$k(d, u) := \bigsqcup \tilde{q}(d, u) \quad \text{and} \quad l(d, u) := s(B_d, u).$$

By (\$), we have that $\forall n^N\, (\phi_{\mathrm{S}}(a_n, t(a_n, d)) \vee \psi_{\mathrm{S}}(d, q(a_n, d)))$ holds in the set-theoretic interpretation. Since $q(a_n, d) \sqsubseteq k(d, u)$, by monotonicity we conclude that

$$\forall n^N \phi_{\mathrm{S}}(a_n, t(a_n, d)) \vee \psi_{\mathrm{S}}(d, k(d, u))$$

holds in the interpretation. On the other hand, instantiating (\$\$) with $B = B_d$ we get $\exists n^N \neg \phi_{\mathrm{S}}(a_n, t(a_n, d)) \vee \theta_{\mathrm{S}}(u, l(d, u))$, and $\psi_{\mathrm{S}}(d, k(d, u)) \vee \theta_{\mathrm{S}}(u, l(d, u))$ follows.

Let us turn to the last logical rule: $\forall$-introduction. By induction hypothesis, there are terms t and q such that

$$\forall c \in W \forall a \in S_\rho \forall d \in S_\eta\, (\forall x \in L_c\, (\phi_{\mathrm{S}}(a, t(c, a, d), x) \vee \psi_{\mathrm{S}}(d, q(c, a, d)))).$$

This provides the verification of the conclusion of the rule (with the very same terms).

Before considering the proper axioms of KPω, we need to check that the interpretation also works for the scheme that regulates the primitive apparatus of the bounded quantifiers:

$$\forall y\, (\forall x \in y\, \phi(x, y) \leftrightarrow \forall x\, (x \in y \to \phi(x, y))),$$

where ϕ is any formula. We see the above equivalence as two general conditional statements. Let us analyze the left-to-right general conditional. Given d^Ω, B, c and a we need to produce $\tilde{a}$ and b such that

$$\forall y \in L_d\, (\forall n^N \forall x \in y\, \phi_{\mathrm{S}}(\tilde{a}\langle n\rangle, B(\tilde{a}\langle n\rangle), x, y) \to \forall x \in L_c\, (x \in y \to \phi_{\mathrm{S}}(a, b, x, y))).$$

Clearly, $\tilde{a} := a + 1$ and $b := B(a)$ work. Let us now analyze the right-to-left general conditional. Given d^Ω, B and a, we need to produce c, $\tilde{a}$ and b such that, for all $y \in L_d$,

$$\forall m^N, n^N \forall x \in L_{c\langle m\rangle}\, (x \in y \to \phi_{\mathrm{S}}(\tilde{a}\langle n\rangle, B(\tilde{a}\langle n\rangle), x, y)) \to \forall x \in y\, \phi_{\mathrm{S}}(a, b, x, y).$$

It is clear that $c := d + 1$, $\tilde{a} := a + 1$ and $b := B(a)$ do the job.

In the remaining part of the proof, we check the axioms of KPω. Extensionality poses no problem because the formula $x = y \leftrightarrow \forall w\, (w \in x \leftrightarrow w \in y)$ is (equivalent to) a bounded formula. The upper S-translation of the pairing axiom is

$$\forall a^\Omega, b^\Omega \exists c^\Omega [\forall x \in L_a \forall y \in L_b \exists n^N \exists z \in L_{c\langle n\rangle}\, (x \in z \wedge y \in z)].$$

It is clear that $c := \max(a, b) + 2$ does the job (note the abuse of language). The upper S-translation of the union axiom is

$$\forall a^\Omega \exists c^\Omega\, [\forall x \in L_a \exists n^N \exists z \in L_{c\langle n\rangle} \forall y \in x \forall w \in y\, (w \in z)].$$

It is clear that $c := a + 1$ does the job. We now turn to the infinity axiom. It says that $\exists x\, Lim(x)$. Its upper S-translation is $\exists b^\Omega \exists n^N \exists x \in L_{b\langle n\rangle}\, Lim(x)$. The closed term $\omega_\Omega + 2$ does the witnessing.

Let us consider Δ_0-separation: $\forall w \forall y \exists z \forall x\, (x \in z \leftrightarrow x \in y \wedge \phi(x, w))$, for ϕ a bounded formula in which the variable z does not occur. Note that the formula $\forall x\, (x \in z \leftrightarrow x \in y \wedge \phi(x, w))$ is (equivalent to) a bounded formula. Having this in mind, it is easy to check that the upper S-translation of bounded separation is

$$\forall d^\Omega, a^\Omega \exists c^\Omega [\forall w \in L_d \forall y \in L_a \exists n^N \exists z \in L_{c\langle n\rangle} \forall x\, (x \in z \leftrightarrow x \in y \wedge \phi(x, w))].$$

It is clear that $c := \max(a, d) + 2$ does the job.

We now check the foundation scheme. It is easier to consider the foundation rule: from $\forall x\, (\forall y \in x\, \phi(y) \to \phi(x))$ infer $\forall x\, \phi(x)$. The rule is equivalent to the foundation scheme because there are no restrictions on the formula ϕ. If one computes the functional translation of the premise of the rule and applies the induction hypothesis, we know that there are terms $t = t(B, a, d)$ and $q = q(B, a, d)$ such that

$$(*) \qquad \forall B, a, d^\Omega [\forall x \in L_d\, (\forall n^N \forall y \in x\, \phi_{\mathrm{S}}(t\langle n\rangle, B(t\langle n\rangle), y)) \to \phi_{\mathrm{S}}(a, q, x))]$$

holds in the set-theoretic interpretation.

We must find a witnessing term for the translation of the conclusion. I.e., we must find a term $s = s(a, d)$ such that, $\forall d \in W \forall a \in S_\rho \forall x \in L_d\, \phi_{\mathrm{S}}(a, s(a, d), x)$. Let us define, by tree recursion,

$$s(a, d) := q(\lambda c^\rho. \bigsqcup(\lambda i^N.s(c, d\langle i\rangle)), a, d),$$

for $d \neq 0_\Omega$, and $s(a, 0_\Omega) := 0^\sigma_\Omega$, where σ is the type of b. We prove, by transfinite induction on $|d|$, that this term s works. Fix $d \in W$. We may suppose that $d \neq 0_\Omega$. By induction hypothesis, we have:

$$(**) \qquad \forall n \in \mathbb{N} \forall a \in S_\rho \forall y \in L_{d\langle n\rangle}\, \phi_{\mathrm{S}}(a, s(a, d\langle n\rangle), y).$$

Let $a \in S_\rho$ be given and $x \in L_d$. We want to show $\phi_{\mathrm{S}}(a, s(a, d), x)$. Consider

$$B_d :\equiv \lambda c^\rho. \bigsqcup(\lambda i^N.s(c, d\langle i\rangle)).$$

By $(*)$, it is enough to show $\forall n^N \forall y \in x\, \phi_{\mathrm{S}}(t\langle n\rangle, B_d(t\langle n\rangle), y)$. Take $n \in \mathbb{N}$ and $y \in x$. Since $x \in L_d$, there is $n_0 \in \mathbb{N}$ such that $y \in L_{d\langle n_0\rangle}$. By $(**)$, we can conclude that $\forall a \in S_\rho\, \phi_{\mathrm{S}}(a, s(a, d\langle n_0\rangle), y)$. In particular, $\phi_{\mathrm{S}}(t\langle n\rangle, s(t\langle n\rangle, d\langle n_0\rangle), y)$. Notice that

$$s(t\langle n\rangle, d\langle n_0\rangle) \sqsubseteq \bigsqcup \lambda i^N.s(t\langle n\rangle, d\langle i\rangle) = B_d(t\langle n\rangle)$$

and so, by monotonicity, $\phi_{\mathrm{S}}(t\langle n\rangle, B_d(t\langle n\rangle), y)$. This is what we want.

Finally, we check Δ_0-collection. A simple computation shows that the upper S-translation of an instance of this scheme is

$$\forall a^\Omega \exists c^\Omega\, [\forall x \in z \exists n^N \exists y \in L_{a\langle n\rangle}\, \phi(x, y) \to \exists n^N \exists w \in L_{c\langle n\rangle} \forall x \in z \exists y \in w\, \phi(x, y)].$$

It is clear that $c := a + 2$ does the job. ⊣

Corollary 1. *If $\mathsf{KP}\omega \vdash \forall x \exists y\, \phi(x, y)$, where $\phi(x, y) \in \Delta_0$ (x and y are the only free variables), then there is a closed term t of type $\Omega \to \Omega$ such that*

$$\forall a \in W \forall x \in L_a \exists y \in L_{t(a)}\, \phi(x, y).$$

Proof. By (ii) of Lemma 1 and the soundness theorem, there is a closed term t of type $\Omega \to \Omega$ such that

$$\forall a \in W \forall x \in L_a \exists n \in \mathbb{N} \exists y \in L_{t(a)\langle n\rangle}\, \phi(x, y).$$

The result follows with the same witnessing term t, since $L_{t(a)\langle n\rangle} \subseteq L_{t(a)}$ for every $n \in \mathbb{N}$ and $a \in W$. ⊣

The following corollary shows that the Bachmann-Howard ordinal is an upper bound for $||\mathsf{KP}\omega||_\Sigma$.

Corollary 2. *If $\mathsf{KP}\omega \vdash \exists x\, \phi(x)$, where $\phi(x) \in \Delta_0$ (x is the only free variable), then there is an ordinal α smaller than the Bachmann-Howard ordinal such that $L_\alpha \models \exists x \phi(x)$.*

Proof. It follows easily from the soundness theorem that there is a closed term q of type Ω such that $\exists x \in L_q\, \phi(x)$. Hence, $L_\alpha \models \exists x\, \phi(x)$, where $\alpha = |q|$. ⊣

Within $\mathsf{KP}\omega$ it is possible to define the Σ_1-operation $\alpha \rightsquigarrow L_\alpha$ that, to each ordinal α, associates the αth-stage of the constructible hierarchy. Since the constructible sets provide an internal model of $\mathsf{KP}\omega$, it follows that $\mathsf{KP}\omega + \mathsf{V} = \mathsf{L}$ is Σ_1-conservative over $\mathsf{KP}\omega$ and, in particular, that the Σ-ordinal of $\mathsf{KP}\omega + \mathsf{V} = \mathsf{L}$ is still the Bachmann-Howard ordinal. This observation has applications. For instance, in his recent textbook [19] on proof theory and impredicativity, Pohlers considers a strengthening of $\mathsf{KP}\omega$: the theory Π_2-REF. This theory is the theory $\mathsf{KP}\omega$ augmented with the Π_2-reflection scheme:

$$\forall w\,(\forall x \exists y\, \phi(x,y,w) \to \exists z\,(z \neq \emptyset \wedge Trans(z) \wedge w \in z \wedge \forall x \in z \exists y \in z\, \phi(x,y,w))),$$

where $\phi(x,y,w)$ is a Δ_0-formula with its free variables as shown and $Trans(z)$ says that z is a transitive set. (In the presence of this principle, it can be shown that the scheme of Δ_0-collection and the axiom of infinity are redundant.) Instead of giving a direct computation of the Σ-ordinal of $\mathsf{KP}\omega$, Pohlers opts for giving such an analysis for Π_2-REF since, as it turns out, the Σ-ordinal of this latter theory is still the Bachmann-Howard ordinal. A simple argument can be advanced for this because it is known that the Π_2-reflection scheme is a consequence of $\mathsf{KP}\omega + \mathsf{V} = \mathsf{L}$ (the argument of theorem 11.8.1 of [19] can be adapted to get this result).

We want to point out that it is an easy exercise to show that the Π_2-reflection scheme has a set-theoretical functional S-interpretation (the interpretation as described in this section). This gives a direct way of computing the Σ-ordinal of Π_2-REF. More can be obtained since the functional interpretation is also able to compute the Σ-ordinal of $\mathsf{KP}\omega + \mathsf{V} = \mathsf{L}$.

PROPOSITION 1. *The sentence* $\mathsf{V} = \mathsf{L}$ *has a set-theoretical functional* S*-interpretation.*

PROOF. Let us start with some preliminaries. The operation $\alpha \rightsquigarrow L_\alpha$ is a Σ_1-operation in $\mathsf{KP}\omega$. This means that there is a formula $\exists w\, \psi(y,z,w)$, where ψ is a Δ_0-formula, saying that $z = L_y$ and such that

$$\mathsf{KP}\omega \vdash \forall y\,(Ord(y) \to \exists z, w\, \psi(y,z,w)).$$

Here, $Ord(y)$ is a Δ_0-formula saying that y is an ordinal. By an easy consequence of Corollary 1, there is a closed term t of type $\Omega \to \Omega$ such that:

$$\forall a \in W \forall y \in L_{a+1} \exists z \exists w \in L_{t(a)}\,(Ord(y) \to \psi(y,z,w)),$$

and, as a consequence, $\forall a \in W \exists w \in L_{t(a)}\, \psi(|a|, L_{|a|}, w)$. Let us consider now the axiom $\mathsf{V} = \mathsf{L}$ in the form $\forall x \exists y, z\,(Ord(y) \wedge z = L_y \wedge x \in z)$. Its upper S-translation can be taken to be the formula with the quantifier prefix '$\forall a \exists b, c, d$' followed by the bounded mixed formula

$$\forall x \in L_a \exists n, m, k \exists y \in L_{b\langle n\rangle} \exists z \in L_{c\langle m\rangle} \exists w \in L_{d\langle k\rangle}\,(Ord(y) \wedge \psi(y,z,w) \wedge x \in z).$$

It is clear that we can put $b = c = a + 2$ and $d = t(a) + 1$ (with $y = |a|$ and $z = L_{|a|}$). ⊣

§4. Brief digression on other functional interpretations. This is perhaps a good occasion to comment on alternative functional interpretations, namely the Diller-Nahm interpretation of [7], the bounded functional interpretation [11] of the present author and Oliva and other ones. The Diller-Nahm interpretation addresses a problem posed by the interpretation of a particular rule of logic, viz the contraction rule: from $\phi \vee \phi$ infer ϕ. Gödel's interpretation needs the presence of equality functionals in order to deal with contraction (see [25] for a perceptive discussion of this issue). The Diller-Nahm interpretation circumvents the contraction problem by relaxing the witness condition. A definite witness is not required anymore, but only that a witness is given among a finite set of possible choices. There are certain superficial similarities between the Diller-Nahm interpretation (and its generalizations, like Martin Stein's interpretations of [24]) and the bounded interpretations, where the choices of witnesses fall below a certain bound (and, along the way, the contraction problem is also circumvented). The main difference with bounded interpretations is the treatment of the interpretation of the bounded quantifiers. Take clause (4) of Definition 2. The bounded quantifier is carried across the $\forall\exists$ prefix. It is this feature that permits the interpretation of the scheme of Δ_0-collection of $\mathsf{KP}\omega$. (Of course, a suitable supply of terms is needed for the interpretation of the other axioms, specially the foundation scheme.) This is what distinguishes "bounded" functional interpretations from the Diller-Nahm variant and its generalizations (see [18] for a discussion of these variants).

The functional interpretation of the previous section does not follow the blueprint of [11], where majorizability relations play a prominent role. The reason is twofold. On the one hand, there is no need to interpret higher type quantifications (the class quantification of the next section is tame enough not to need the apparatus of majorizability). One could perhaps envisage a full-blown bounded functional interpretation of Kripke-Platek set theory, but the present objectives do not require it. On the other hand, the interpretation of Avigad and Towsner has a very nice feature: the lower S-matrices of the interpretation have only quantifications over *countably* many individuals. This opens the possibility of a reduction of $\mathsf{KP}\omega$ to a constructive theory of tree functionals, in a way similar to the reduction outlined in the second part of [2] for the theory ID_1.

Let me finally remark on the functional interpretation of $\mathsf{KP}\omega$ given by Wolfgang Burr in [5]. This interpretation is a refinement of a previous interpretation [6] whereby a choice functional is avoided by resorting to a Diller-Nahm style interpretation. Bounded formulas are interpreted by themselves, as in our interpretation. Otherwise, Burr's interpretation follows the standard functional interpretations (resorting to a Diller-Nahm maneuver), with no particular distinction between the interpretations of bounded and unbounded quantifiers. The scheme of collection is dealt with by an "infinitary" union functional. As a consequence, the term calculus underlying the interpretation has an infinitary rule and the calculus loses its finitary character. As the authors of [6] say, the notion of "normal form is no longer a mere syntactical notion."

§5. Strict-Π^1_1 reflection. Weak König's lemma plays a very important role in fragments of analysis and in reverse mathematics (see [23]). The lemma says that every infinite binary tree has an infinite path. A strict-Π^1_1 predicate $P(n)$ of natural numbers is a predicate of the form $\forall X \subseteq \mathbb{N} \exists x \in \mathbb{N}\, R(n, x, X)$, for R a recursive predicate in the oracle X. It is easy to see that we can restrict R to the predicates which are defined by bounded formulas of arithmetic (with a new second-order unary symbol X). Weak König's lemma implies (see chapter VIII of [4]) the so-called principle of strict-Π^1_1 reflection:

$$\forall X \subseteq \mathbb{N} \exists x \in \mathbb{N}\, \phi(n, x, X) \to \exists z \in \mathbb{N} \forall X \subseteq \mathbb{N} \exists x \leq z\, \phi(n, x, X),$$

for ϕ a bounded formula of arithmetic (with a new second-order unary symbol X). Given that ϕ is a bounded formula, it is easy to conclude that the formula in the consequent of the above implication is equivalent to a Σ^0_1-formula of arithmetic (do notice that the totality of exponentiation is used in this argument). This observation can also be stated by saying that the strict-Π^1_1 predicates define exactly the recursively enumerable sets.

Jon Barwise considers in [4] strict-Π^1_1 predicates and the principle of strict-Π^1_1 reflection in the framework of models of $\mathsf{KP}\omega$. A strict-Π^1_1 predicate of set theory, is a predicate $P(u)$ of the form $\forall X \exists x\, R(u, x, X)$, where R is given by a Δ_0-formula of set theory with a new second-order unary symbol X. The principle of strict-Π^1_1 reflection in set theory can be formulated as

$$\forall X \exists x\, \phi(u, x, X) \to \exists z \forall X \exists x \in z\, \phi(u, x, X),$$

for ϕ a Δ_0-formula of set theory with the second-order unary symbol X (the unary symbol X can be interpreted by *any* subset of the given admissible set). It is no longer true that strict-Π^1_1 reflection holds in every model of $\mathsf{KP}\omega$, but it is known to hold in the countable admissible sets (see [4]). Interestingly, a predicate can be strict-Π^1_1 without being equivalent to a Σ_1-predicate (see chapter VIII of Barwise's book). In hindsight, this is not really surprising for a person who is familiar with weak systems of arithmetic and analysis. There are similarites between fragments of arithmetic and admissibility (see, for instance, the preface of [20]). In this (and the next) section, we explore these similarities in the direction of formal theories.

As it is well-known, the second-order system of analysis WKL_0 is Π^0_2-conservative over the theory $\mathsf{I}\Sigma^0_1$ of Peano Arithmetic with induction restricted to Σ^0_1-formulas. The system WKL_0 consists of the base system RCA_0 of recursive comprehension together with weak König's lemma. These systems are described and discussed in the classic [23]. The following definitions parallel the arithmetic case.

Definition 3. *The language of second-order set theory is the extension of the language of set theory with monadic second-order quantification.*

We use the letters $X, Y, Z, \ldots$ for the monadic predicates and call them *classes*. The new atomic formulas of the language are those of the form $X(x)$, but we write instead $x \in X$. Note that the symbol $\in$ in $x \in X$ should not be confused with the primitive symbol $\in$ of the language of set theory (which infixes between first-order variables). It is a mere notational device for the official $X(x)$, and we usually read $x \in X$ by saying that x is a member of (the class) X. The notions of Δ_0-formula, Σ_1-formula, etc, are defined in the second-order language in the same old manner, but now allowing the new atomic formulas. In other words, unless otherwise stated, we allow class parameters in the definitions of these notions. In the second-order language, the scheme of Δ_0-collection is formulated with second-order parameters and the scheme of foundation is formulated for every formula of the new second-order language. However, the scheme of Δ_0-separation maintains the original formulation, i.e., *without second-order parameters.* This restriction is necessary for the results below. The theory $\mathsf{KP}\omega_2\!\restriction$ is the second-order extension of the theory $\mathsf{KP}\omega$ as described above.

DEFINITION 4. *The following schemes are defined in the language of* $\mathsf{KP}\omega_2\!\restriction$*:*

1. *The scheme of* Δ_0-CA *is* $\exists X \forall x\,(x \in X \leftrightarrow \phi(x))$*, where* $\phi(x)$ *is a* Δ_0*-formula in which* X *does not occur, possibly with first and second-order parameters.*
2. *The scheme of* Δ_1-CA *is* $\forall x\,(\phi(x) \leftrightarrow \psi(x)) \rightarrow \exists X \forall x\,(x \in X \leftrightarrow \psi(x))$*, where* $\phi(x)$ *is a* Σ_1*-formula and* $\psi(x)$ *is a* Π_1*-formula in which* X *does not occur, possibly with first and second-order parameters.*
3. *The scheme* $\mathsf{s}\Pi^1_1$-ref *of strict-*Π^1_1 *reflection is*
$$\forall X \exists x\, \phi(x, X) \rightarrow \exists z \forall X \exists x \in z\, \phi(x, X),$$
where $\phi(x, X)$ *is a* Δ_0*-formula, possibly with first and second-order parameters.*

LEMMA 3. *The theory* $\mathsf{KP}\omega_2\!\restriction + \Delta_0$-$\mathsf{CA} + \mathsf{s}\Pi^1_1$-$\mathsf{ref}$ *proves* Δ_1-CA*.*

PROOF. The argument follows closely a proof in [10] for theories of arithmetic and analysis. We reason in $\mathsf{KP}\omega_2\!\restriction + \Delta_0$-$\mathsf{CA} + \mathsf{s}\Pi^1_1$-$\mathsf{ref}$. Suppose that $\forall u(\exists y \phi(u, y) \leftrightarrow \forall z \psi(u, z))$, where ϕ and ψ are Δ_0-formulas, possibly with first and second-order parameters. We claim that

$$\forall w \exists X \forall x \in w \forall u, y, z \in x\,((\phi(u, y) \rightarrow u \in X) \wedge (u \in X \rightarrow \psi(u, z))).$$

Given a set w, consider $\tilde{w}$ its transitive closure and take, by bounded comprehension, the class $X := \{u : \exists y \in \tilde{w}\, \phi(u, y)\}$. It is clear that this class X does the job. Now, by $\mathsf{s}\Pi^1_1$-ref, we get

$$\exists X \forall x \forall u, y, z \in x\,((\phi(u, y) \rightarrow u \in X) \wedge (u \in X \rightarrow \psi(u, z))),$$

and this entails the desired result. ⊣

The notion of bounded mixed formula is generalized for the second-order setting:

DEFINITION 5. *The class of* second-order bounded mixed formulas *is the smallest class of formulas of the mixed language that contains the atomic formulas and is closed under Boolean connectives, bounded quantifications and second-order quantifications.*

In a nutshell, second-order quantifications count as bounded quantifications. (The reader of like mind will imagine these second-order quantifications as characteristic functions bounded by the constant 1 function. Barwise observes in p. 316 of [4], that "the study of s-Π^1_1 predicates is one of the few places in logic where the difference between relation symbols and function symbols really matters.” In our context, this observation cashes in as the difference between bounded and unbounded quantifications.)

We can now assign to each formula ϕ of second-order set theory, a second-order bounded mixed formula $\phi_{\mathrm{S}}(a,b)$ and the corresponding formula $\phi^{\mathrm{S}} :\equiv \forall a \exists b\, \phi_{\mathrm{S}}(a,b)$. The clauses are the same as in Definition 2, together with the extra clause:

6. $(\forall X\, \phi(X))^{\mathrm{S}}$ is $\forall a \exists b\, [\forall X\, \phi_{\mathrm{S}}(a,b,X)]$.

Note that the monotonicity property of Lemma 2 is also preserved by this clause.

In the next result, second-order (class) quantifications are interpreted as ranging over *sets*:

Theorem 2. *Let ϕ be a sentence of the language of second-order set theory and suppose that $\mathsf{KP}\omega_2\!\restriction + \Delta_1\text{-}\mathsf{CA} + \mathsf{s}\Pi^1_1\text{-}\mathsf{ref} \vdash \phi$. Then there are closed terms t of $\mathcal{L}_\Omega$ such that, for appropriate types ρ, $\forall a \in S_\rho\, \phi_S(a, t(a))$.*

Proof. The logical axioms and rules extend the first-order case with new cases for substitution and universal introduction. The proof is by induction on the length of the derivation. We claim that if $\phi(x,X)$ is provable in $\mathsf{KP}\omega_2\!\restriction + \Delta_1\text{-}\mathsf{CA} + \mathsf{s}\Pi^1_1\text{-}\mathsf{ref}$, then there are closed terms t such that, for appropriate types ρ,

$$\forall c \in W \forall a \in S_\rho \forall x \in L_c \forall X\, \phi_{\mathrm{S}}(a, t(a,c), x, X).$$

In the following arguments we ignore, as usual, the parameters that do not play an important role and rely on previous notation.

Let us consider the second-order substitution axiom $\forall X\, \phi(X) \to \phi(Y)$. Its upper S-translation is

$$\forall B, a \exists \tilde{a}, b\, [\forall n^N \forall X\, \phi_{\mathrm{S}}(\tilde{a}\langle n\rangle, B(\tilde{a}\langle n\rangle), X) \to \phi_{\mathrm{S}}(a,b,Y)].$$

It is simple to check that $\tilde{a} := a+1$ and $b := B(a)$ work. Let us now consider the second-order universal introduction rule: from $\phi(X) \vee \psi$, conclude $\forall X \phi(X) \vee \psi$, where X does not occur free in ψ. The upper S-translation of the premise is

$$\forall a, d \exists b, e\, (\phi(a,b,X) \vee \psi(d,e)).$$

Hence, by induction hypothesis, there are closed terms t and q such that

$$\forall a, d \forall X\, (\phi(a, t(a,d), X) \vee \psi(d, q(a,d)))$$

holds in the intended set-theoretic interpretation. The upper S-translation of the conclusion of the rule is $\forall a, d \exists b, e\, (\forall X \phi_{\mathrm{S}}(a,b,X) \vee \psi_{\mathrm{S}}(d,e))$. Obviously, the same terms t and q witness the existential claims.

The treatment of the mathematical axioms of $\mathsf{KP}\omega_2\!\restriction$ repeats the case of $\mathsf{KP}\omega$. Do notice, however, that the argument for Δ_0-separation does not extend to formulas with second-order parameters. In fact, it is known that if we admit parameters in the Δ_0-separation axiom, then the power set axiom is provable (I thank Andreas Cantini for having brought this fact to my attention; see p. 42 of [21]).

By the previous lemma, we can check $\Delta_0\text{-}\mathsf{CA}$ instead of $\Delta_1\text{-}\mathsf{CA}$. An instance of the scheme $\Delta_0\text{-}\mathsf{CA}$ has the form $\forall W, w \exists X \forall x\, (x \in X \leftrightarrow \phi(x, w, W))$, where

ϕ is a Δ_0-formula in which X does not occur. The upper S-translation of this instance is:

$$\forall d^\Omega, a^\Omega \forall W \forall w \in L_d \exists X \forall n^N \forall x \in L_{a\langle n\rangle}(x \in X \leftrightarrow \phi(x, w, W)).$$

This statement holds with the *set* $X := \{x \in L_\alpha : \phi(x, w, W)\}$, where $\alpha = |a|$.

It remains to check the principle of strict-Π^1_1 reflection. The upper S-translation of this principle is:

$$\forall a^\Omega \exists c^\Omega [\forall X \exists n^N \exists x \in L_{a\langle n\rangle}\, \phi(x, X) \rightarrow \exists z \in L_c \forall X \exists x \in z\, \phi(x, X)].$$

Since $L_{a\langle n\rangle} \subseteq L_a$, for all natural numbers n, and $L_a \in L_{a+1}$, it is clear that $c := a + 1$ (with $z = L_a$) witnesses the above statement. $\dashv$

As before, the following are immediate consequences of the above interpretation theorem:

COROLLARY 3. *If* $\mathsf{KP}\omega_2\!\upharpoonright + \Delta_1\text{-}\mathsf{CA} + \mathsf{s}\Pi^1_1\text{-}\mathsf{ref} \vdash \forall x \exists y \phi(x, y)$, *where* $\phi(x, y) \in \Delta_0$ *(x and y are the only free variables), then there is a closed term t of type* $\Omega \rightarrow \Omega$ *such that*

$$\forall a \in W \forall x \in L_a \exists y \in L_{t(a)}\, \phi(x, y).$$

COROLLARY 4. *If* $\mathsf{KP}\omega_2\!\upharpoonright + \Delta_1\text{-}\mathsf{CA} + \mathsf{s}\Pi^1_1\text{-}\mathsf{ref} \vdash \exists x \phi(x)$, *where* $\phi(x) \in \Delta_0$ *(x is the only free variable), then there is an ordinal* α *smaller than the Bachmann-Howard ordinal such that* $L_\alpha \models \exists x \phi(x)$.

§6. Internalizing the interpretation. In Section 2, we described the set-theoretic interpretation of the term language $\mathcal{L}_\Omega$. There are other interpretations, of course. In this section, we describe (following section 9.4 of [1], but without the μ-operator) an interpretation H analogous to the interpretation of Gödel's T by the hereditarily recursive operations. The interpretation H_ρ, for each type ρ, is given as follows:

1. $H_N = \mathbb{N}$
2. H_Ω is the smallest set $X \subseteq \mathbb{N}$ which contains 0 and whenever $f \in \mathbb{N}$ is such that $\forall n \in \mathbb{N}\,(\{f\}(n)\downarrow \wedge \{f\}(n) \in X)$ then $\langle 1, f\rangle \in X$. (Here we are using Kleene's bracket notation for the partial recursive functions, and $\langle\, ,\, \rangle$ is a pairing function from $\mathbb{N}^2$ to $\mathbb{N} \setminus \{0\}$.)
3. $H_{\rho\rightarrow\tau} = \{f \in \mathbb{N} : \forall n \in \mathbb{N}\,(n \in H_\rho \rightarrow \{f\}(n)\downarrow \wedge \{f\}(n) \in H_\tau)\}$

The interpretations of the combinators and the arithmetical constants are done as in the arithmetical case of the hereditarily recursive operations (see [26] for details). We let $(0_\Omega)^H$ be 0 and take $Sup^H \in \mathbb{N}$ such that, for all $f \in \mathbb{N}$, $\{Sup^H\}(f) = \langle 1, f\rangle$. In order to interpret the tree recursors R^Ω_ρ, we use the recursion theorem to produce natural numbers r_ρ satisfying

$$\{r_\rho\}(0, a, e) = a \text{ and } \{r_\rho\}(\langle 1, f\rangle, a, e) \simeq \{\{e\}(f)\}(\lambda x^N.\{r_\rho\}(\{f\}(x), a, e)).$$

We must argue that, for $c \in H_\Omega$, $a \in H_\rho$ and $e \in H_{\rho\rightarrow(N\rightarrow\rho)\rightarrow\rho}$, $\{r_\rho\}(c, a, e)$ is defined. As in Section 2, we can associate to each element c of H_Ω a countable ordinal $|c|$ so that, $|0| = 0$ and, for each $f \in H_{N\rightarrow\Omega}$, $|\langle 1, f\rangle| = \sup\{|\{f\}(n)| + 1 : n \in \mathbb{N}\}$. It is now easy to show, by transfinite induction on $|c|$, that $\{r_\rho\}(c, a, e)$ is defined. The interpretation of R^ρ_Ω is the natural number $(R^\rho_\Omega)^H$ such that $\{\{\{(R^\rho_\Omega)^H\}(c)\}(a)\}(e) \simeq r_\rho(c, a, e)$, for all c, a and e.

Finally, we define the application operation between an element f of $H_{\rho\to\tau}$ and an element e of H_ρ as $\{f\}(e)$. This finishes the description of the interpretation H of $\mathcal{L}_\Omega$. We claim that this interpretation can be done inside the theory $\mathsf{KP}\omega$. Within $\mathsf{KP}\omega$, the H_ρ's are not sets. They are given by formulas. This is because H_Ω is given by an arithmetical monotone inductive definition. Therefore, it is given by a Σ_1-formula of the language of set theory and $\mathsf{KP}\omega$ does not have comprehension for these formulas. The higher type H_ρ's build on H_Ω and, in fact, have definitions of unbounded formula complexity.

Let us look more closely at H_Ω. The positive inductive operator associated with it is

$$\Gamma(X) := \{c \in \omega : c = 0 \vee \exists f \in \omega\, (c = \langle 1, f\rangle \wedge \forall n \in \omega\, (\{f\}(n)\downarrow \wedge \{f\}(n) \in X))\}.$$

As it is well-known, we can define within $\mathsf{KP}\omega$ the Σ_1-operation $\alpha \rightsquigarrow I^\alpha_\Gamma$, from ordinals to subsets of ω, such that $I^\alpha_\Gamma = \Gamma(\bigcup_{\beta<\alpha} I^\beta_\Gamma)$, for all ordinals α. Of course, for all $c \in \omega$, c lies in H_Ω if, and only if, $\exists \alpha\, (c \in I^\alpha_\Gamma)$. Within $\mathsf{KP}\omega$, we can now associate to each element c of H_Ω the unique ordinal α such that $c \in I^\alpha_\Omega \wedge \forall \beta < \alpha\, (c \notin I^\beta_\Omega)$. By abuse of language, we abbreviate this formula by writing $|c| = \alpha$ and say that the height of c is α. The "height function" enjoys the right properties:

LEMMA 4 ($\mathsf{KP}\omega$). $|0| = 0$ *and, given* $f \in H_{N\to\Omega}$, $|\{Sup^H\}(f)|$ *is the least upper bound of all ordinals of the form* $|\{f\}(n)| + 1$, *with* $n \in \omega$.

PROOF. Take f in $H_{N\to\Omega}$ and let $\alpha = |\{Sup^H\}(f)|$. Hence, $\langle 1, f\rangle \in I^\alpha_\Gamma$ and, for each $n \in \omega$, $\{f\}(n) \in \bigcup_{\beta<\alpha} I^\beta_\Omega$. Therefore, $\forall n \in \omega \exists \beta < \alpha\, (|\{f\}(n)| \le \beta)$. It follows that α is an upper bound for the ordinals of the form $|\{f\}(n)| + 1$, with $n \in \omega$.

Let $\beta < \alpha$. By definition of α, $\langle 1, f\rangle \notin I^\beta_\Omega$. Therefore, it is not the case that, for every $n \in \omega$, $\{f\}(n) \in \bigcup_{\gamma<\beta} I^\gamma_\Gamma$. Take $n_0 \in \omega$ with $\forall \gamma < \beta\, (\{f\}(n_0) \notin I^\gamma_\Gamma)$. We conclude that $\beta \le |\{f\}(n_0)| < |\{f\}(n_0)| + 1$. Therefore, β is not an upper bound for the ordinals of the form $|\{f\}(n)| + 1$, with $n \in \omega$. ⊣

We have seen how the H_ρ's are given by predicates within $\mathsf{KP}\omega$. (In the sequel, as it is usually done, we use the membership sign for saying that an element falls under a predicate H_ρ, instead of subsuming the element under the predicate.) We have also seen that each element in H_Ω has an ordinal height and that this "height function" has the properties stated in the above lemma. There is no difficulty in interpreting the combinators and the arithmetical constants within $\mathsf{KP}\omega$. The interpretation of the tree recursors is as before, via the recursion theorem. The totality of these recursors for appropriate inputs is provable by ordinal induction on the height of c (notation as before). Note that this ordinal induction is available in $\mathsf{KP}\omega$ because of the (unrestricted) scheme of foundation. The scheme of foundation is indeed heavily used in justifying the totality of the tree recursors.

Given a term $t(a)$ of type ρ, with a free variable a^τ, it is now clear that we can give a sense to $t^H(x)$, for x in H_τ. Formally, it is enough to define $(t(x) = y)^H$ for x in H_τ and y in H_ρ. This is done by a straightforward induction on the build-up of the term t. Note that, under the previous conditions, $(t(x) = y)^H$

is a Δ^0_1 (i.e., recursive) relation. Let us now give a nice application of the H-interpretation and show that the Bachmann-Howard ordinal is a lower bound for $||\mathsf{KP}\omega||_\Sigma$. In view of Corollary 2, $||\mathsf{KP}\omega||_\Sigma$ is exactly the Bachmann-Howard ordinal.

PROPOSITION 2. *For each ordinal β smaller than the Bachmann-Howard ordinal, there is a Σ_1-sentence provable in $\mathsf{KP}\omega$ which fails in L_β.*

PROOF. Let β be an ordinal smaller than the Bachmann-Howard ordinal. Take q a closed term of $\mathcal{L}_\Omega$ of type Ω such that $\beta \leq |q^H|$ (obviously, the ordinal height of q is the same in the set-theoretic interpretation and in the interpretation H). By the discussions above, $\mathsf{KP}\omega \vdash \exists\alpha\,(q^H \in I^\alpha_\Gamma)$. Note that the formula $q^H \in I^\alpha_\Gamma$ is, formally,

$$\exists y\,(y \in I^\alpha_\Gamma \wedge (q = y)^H)$$

where, as we have discussed, the predicate $(q = y)^H$ is Δ^0_1. Therefore, the formula $q^H \in I^\alpha_\Gamma$ is Σ_1 and, hence, so is $\exists\alpha\,(q^H \in I^\alpha_\Gamma)$. If $L_\beta \models \exists\alpha\,(q^H \in I^\alpha_\Gamma)$, then we would have $q^H \in I^\alpha_\Gamma$ for some $\alpha < \beta$. This is absurd. ⊣

It is clear how to interpret the second-order bounded mixed formulas of Definition 5 via the H-interpretation within $\mathsf{KP}\omega$. Given a formula

$$\phi(x_1, \dots, x_n, X_1, \dots, X_r)$$

of second-order set theory, it makes sense to write

$$(\phi_{\mathrm{S}})^H(a_1, \dots, a_k, b_1, \dots, b_m, x_1, \dots, x_n, X_1, \dots, X_r),$$

with the a's and the b's in appropriate levels of the H_ρ's. The important thing to note, is that the second-order variables are interpreted as ranging over *sets* (we could have used lower case letters, but it would be artificial to use different letters in ϕ and in $(\phi_{\mathrm{S}})^H$).

The arguments of Theorems 1 and 2 can be internalized in $\mathsf{KP}\omega$ and we get:

THEOREM 3. *Let ϕ be a sentence of the language of second-order set theory and suppose that $\mathsf{KP}\omega_2\!\upharpoonright + \Delta_1\text{-}\mathsf{CA} + \mathsf{s}\Pi^1_1\text{-}\mathsf{ref} \vdash \phi$. Then there is a closed term t of $\mathcal{L}_\Omega$ such that, for appropriate type ρ, $\mathsf{KP}\omega \vdash \forall a \in H_\rho\,(\phi_{\mathrm{S}})^H(a, t^H(a))$.*

The following is an improvement of Corollary 4:

COROLLARY 5. *The theory $\mathsf{KP}\omega_2\!\upharpoonright + \Delta_1\text{-}\mathsf{CA} + \mathsf{s}\Pi^1_1\text{-}\mathsf{ref}$ is Σ_1-conservative over $\mathsf{KP}\omega$.*

PROOF. Suppose that $\mathsf{KP}\omega_2\!\upharpoonright + \Delta_1\text{-}\mathsf{CA} + \mathsf{s}\Pi^1_1\text{-}\mathsf{ref} \vdash \exists x\phi(x)$, where $\phi(x) \in \Delta_0$ (x is the only free variable). By the above soundness theorem, there is a closed term q of type Ω such that $\mathsf{KP}\omega \vdash \exists\alpha\,(|q^H| = \alpha \wedge \exists x \in L_\alpha\,\phi(x))$. This entails $\mathsf{KP}\omega \vdash \exists x\phi(x)$. ⊣

We conjecture that $\mathsf{KP}\omega_2\!\upharpoonright + \Delta_1\text{-}\mathsf{CA} + \mathsf{s}\Pi^1_1\text{-}\mathsf{ref}$ is Π_2-conservative over the theory $\mathsf{KP}\omega$. This conjecture is in line with the analogous result in arithmetic and analysis, whereby WKL_0 is Π^0_2-conservative over $\mathsf{I}\Sigma^0_1$. There is an (unpublished) result of Leo Harrington that shows that the second-order theory WKL_0 is even fully conservative over the first-order theory $\mathsf{I}\Sigma^0_1$. Harrington's proof uses a forcing argument and has been reported in [23] (there is also a pure proof-theoretic proof of Harrington's conservation result, given in [10]). We can, of course, also pose the question whether the theory $\mathsf{KP}\omega_2\restriction + \Delta_1\text{-}\mathsf{CA} + \mathsf{s}\Pi^1_1\text{-}\mathsf{ref}$ is fully conservative over $\mathsf{KP}\omega$.

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