To catch one's own shadow*

Fernando Ferreira
Universidade de Lisboa

fjferreira@fc.ul.pt

If I infer 'Q' from 'P' then I am entitled to assert '$P \to Q$'. If I have '$P \to Q$' then 'Q' is *settled* by *way* of *settling* 'P': "Modus Ponendo Ponens." I have two jolly rules here, but I can't prove Peirce's law: '$((P \to Q) \to P) \to P$'.[1] It's a pretty *elementary* law, though. "If the consequence of the conditional is true then the conditional is true. If it is false, then the antecedent of the antecedent is true, because its *own* antecedent is false. Since the consequent of the antecedent is false, it follows that the antecedent of the conditional is false. Therefore, the conditional is true in this case as well."

For any P, $P \to P$. Peirce's law is not needed to justify this. Oops! Am I mixing up use and mention? Perhaps I should have said that for every *sentence* P, the sentence $\ulcorner P \to P \urcorner$ is true. Or: that the sentential *schema* '$P \to P$' is logically valid. No! I will follow the ways of G. Frege. He would have written '$\forall p(p \to p)$', with lower-case 'p' (had he used *our* notation). I will not commit this gratuitous blunder. Instead, I say that the sentence '$\forall P(P \to P)$' is a law of logic. This is a powerful *Begriffsschrift*. G. Frege was carried away by it. His insouciance was amazing. From the above law, G. Frege instantiates (e.g.) '$\forall P(P \to P) \to \forall P(P \to P)$'. Prima facie, instantiations of generalities have to be intelligible in advance of the generality itself. This is not happening here. B. Russell was more circumspect. He disliked circles, joining company with H. Poincaré. The sentence '$\forall P(P \to P)$' is a law of logic, but a law for "elementary propositions" only,[2] and the very law expresses a proposition which is not elementary (on account of the sneaky quantifier). B. Russell would be quick to add that '$\forall P(P \to P) \to \forall P(P \to P)$' is, nevertheless, an instance of a law of logic, but of *another* law viz, '$\forall P^2(P^2 \to P^2)$'. The superscript in 'P^2' indicates that the universal quantifier ranges over non-elementary propositions

*The present work is a "lite" version of the paper that was actually read at ENFA2. At the moment of this writing (December 2004), that paper is still being reworked. The interested reader will have to await for it to see a more thorough discussion of the issues raised here.

[1] This law is stated in part II of Charles S. Peirce's "On the algebra of logic: a contribution to the philosophy of notation," *The American Journal of Mathematics*, vol. 7, no. 2, pp. 180-202 (1885).

[2] 'Elementary proposition' is a technical term in Alfred North Whitehead and Bertrand Russell, *Principia Mathematica*, vol. I, second edition (Cambridge: Cambridge University Press, 1925). Cf. pp. xv-xvii.

of the *order* that we are discussing. But, of course, the sentence '$\forall P^2(P^2 \to P^2) \to \forall P^2(P^2 \to P^2)$' is no longer an instance of the law '$\forall P^2(P^2 \to P^2)$'. It is an instance of a similar law, but one of a higher-order, *etc.*, *etc.* We can keep going up the superscript numbers, ramifying, but to attempt to formulate the general law by a sentence is like attempting to catch one's own shadow. It can't be done.

Or, can it? F. Ramsey thought it could.[3] He relied on L. Wittgenstein's doctrine of "atomic propositions" in the *Tractatus*.[4] According to F. Ramsey and L. Wittgenstein, a proposition is that which expresses agreement and disagreement with complementary sets of truth-possibilities of the atomic propositions. Conversely, any set of these truth-possibilities determines a proposition, one which agrees with all the truth-possibilities in the given set, and disagrees with the remaining. On this view, there may be propositions which cannot be *presented* by sentences (for lack of linguistic resources). The sentence '$\forall P(P \to P)$' presents a proposition. It asserts the conjunction of all propositions which can be specified by conditionals whose antecedent and consequent are the very same proposition. All the conjuncts are the *same* proposition, since they all express the same truth-possibilities, viz. *whatever* (the disagreement set of truth-possibilities is void). Since a conjunction of *whatevers* is *whatever*, own '$\forall P(P \to P)$' presents the proposition that expresses *whatever*. It is *the* tautological proposition (if it were a *real* proposition, which according to F. Ramsey it isn't). This explanation is not vicious. Leaving aside the dubious doctrine of the *Tractatus* and the minutiae of propositions galore, F. Ramsey is saying that propositions are independent of their being propounded. He is rejecting the view that propositions are a mere *façon de parler* for sentences in linguistic use. He is saying that propositions have clear identity conditions and that they can be referred to (i.e., that they are *objects*). In the words of R. Carnap,[5] Ramsey is saying that propositions form an *absolute* realm. What realm is this?

Perhaps B. Russell was right after all. Perhaps it doesn't make sense to write '$\forall P(P \to P)$' *simpliciter*. However ... R. Carnap thought he could make sense of it without falling into F. Ramsey's *absolutism*. He said that the impression of circularity arises because we are conceiving generality in the sense of "every single one." We must conceive propositional generality differently, *through logic*. From a generality we may infer any instance of it. From a justification of a "general instance," we may infer the generality. That's *all*. Carnap must *deny* that the sense of a second-order generalization is intelligible only if its in-

[3]Frank Ramsey, "The foundations of mathematics," in *Philosophical Papers*, ed. D. H. Mellor (Cambridge: Cambridge University Press, 1990). The essay was first published in 1926.

[4]Ludwig Wittgenstein, *Tractatus Logico-Philosophicus* (London:Routledge, 1961). First published in German in 1922 under the title *Logisch-Philosophische Abhandlung*. 'Atomic proposition' is another technical term. Cf. reference in note 2. Confusingly, Ludwig Wittgenstein uses the term 'elementary proposition' (Elementarsatz) instead.

[5]Rudolf Carnap, "The logicist foundations of mathematics" in *Philosophy of Mathematics*, eds. Paul Benacerraf and Hilary Putnam (Cambridge: Cambridge University Press, 1983). First published in German in 1931 under the title "Die logizistische Grundlegung der Mathematik." This is Carnap before he said that there are no morals in logic.

stantiations are intelligible in advance of the generalization itself.[6] Impressions of circularity aside, *in practice* some generalities can be justified. How are we entitled to propositional generalities? By dint of logical proof.[7] On a more somber note, with his "specific" view of generalization R. Carnap is foreclosing the notion of *truth* concerning propositional generalities. An instance of a propositional generality is not obtained by converting each occurrence of the quantified variable of its matrix into an expression that *refers*, but rather into an expression that *propounds*. The quantified variable is not apt for reference. I conclude that propositional generalities are not apt for truth (no reference, no truth: we don't want to muddle the waters). Propositional generalities are only apt for justification. R. Carnap meant *classical* justification. But how can he uphold Peirce's law if he has no principle of bivalence to rely on? The onus is on *his* side. To ignore the onus is *hocus-pocus*.

With conditional and generalization in place we have already all propositional logic (and more):[8]

$$\begin{array}{lll} \neg A & =_{df} & A \rightarrow \forall P.P \\ A \wedge B & =_{df} & \forall P((A \rightarrow (B \rightarrow P)) \rightarrow P) \\ A \vee B & =_{df} & \forall P((A \rightarrow P) \rightarrow ((B \rightarrow P) \rightarrow P)) \\ \exists Q.A & =_{df} & \forall P(\forall Q(A \rightarrow P) \rightarrow P). \end{array}$$

We get *intuitionistic* propositional logic, to be sure. (Gosh! This brings to mind L. Brouwer. Now, that's a dangerous confusion!) Having second thoughts about *negation*? If there is an alternative way of conceiving it, let it be known. Negation is not like flipping a coin anymore since certain sentences are no longer truth apt. Even though we can flip the coins of elementary sentences (Oh, boy! Did we flip the P-coin in the proof of Peirce's law.), how can we flip the *other* coins? If only we could flip all the coins...

We did catch one's own shadow, but it took a different logic.[9,10]

[6] This point was brought to my attention by note 7 of Warren Goldfarb's "Russell's reasons for ramification" in *Rereading Russell*, eds. C. Wade Savage and Anthony Anderson (Minnesota University Press, 1989).

[7] More explicitly, proofs in the natural deduction calculus as given by the introduction and elimination rules of the conditional and the propositional quantifier.

[8] The first three definitions appear informally in sections 18 and 19 of Bertrand Russell's *The Principles of Mathematics* (London: George Allen & Unwin, 1985), first published in 1903. The formal definitions are due to Dag Prawitz, *Natural Deduction: A Proof-Theoretical Study* (Stockholm: Almqvist & Wiksell, 1965).

[9] John Myhill suggests a similar venue at the end of his "The undefinability of the set of natural numbers in the ramified *Principia*," in *Bertrand Russell's Philosophy*, ed. G. Nakhnikian (New York: Barnes & Noble, 1974).

[10] 'Gratuitous blunder' appears in chapter 6, part 7, of Michael Dummett's *Frege: Philosophy of Language* (London: Duckword, 1973). 'His amazing insouciance' appears in chapter 17 of Dummett's *Frege: Philosophy of Mathematics* (Cambridge, Mass.: Harvard University Press, 1991). 'Like attempting to catch one's own shadow' appears in the p. xxxiv of *Principia Mathematica*, op. cit.