Zigzag and Fregean arithmetic

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Abstract

In Frege's logicism, numbers are logical objects in the sense that they are extensions of certain concepts. Frege's logical system is inconsistent, but Richard Heck showed that its restriction to predicative (second-order) quantification is consistent. This predicative fragment is, nevertheless, too weak to develop arithmetic. In this paper, I will consider an extension of Heck's system with impredicative quantifiers. In this extended system, both predicative and impredicative quantifiers co-exist but it is only permissible to take extensions of concepts formulated in the predicative fragment of the language. This system is consistent. Moreover, it *proves* the principle of reducibility applied to concepts true of only finitely many objects. With the aid of this form of reducibility, it is possible to develop arithmetic in a thoroughly Fregean way.

1 Introduction

One of the *dicta* of Frege was never to lose sight of the distinction between concept and object. When Frege wrote this maxim in p. x of [8], he was embarking on the project of reducing arithmetic to logic. The pursuit of this project eventually led him to introduce so-called value-ranges as regulated by his famous Basic Law V. In a sense, the introduction of value ranges entails that the world of concepts has a counterpart in the world of objects (by way of their extensions, which are a special kind of value-ranges). Even though the distinction between object and concept is by no mean obliterated by the introduction of extensions, the end result is that every concept has an object as a proxy and this situation proved to be fatal. For our purposes, I take Frege's system of the *Grundgesetze der Arithmetik* [9] as second-order logic (with unrestricted comprehension) together with an extension operator – attaching a first-order term $\hat{x}.A(x)$ to each formula $A(x)$ – regulated by the scheme

$$\hat{x}.A(x) = \hat{x}.B(x) \leftrightarrow \forall x(A(x) \leftrightarrow B(x)).$$

The above scheme is our version of Frege's Basic Law V. Frege was aware that this law could be disputed. In page VII of the foreword to the first volume of the *Grundgesetze*, Frege writes that

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> (...) as far as I can see, a dispute can arise only concerning my Basic Law of value-ranges (V), which perhaps has not yet been explicitly formulated by logicians although one thinks in accordance with it if, e.g., one speaks of extensions of concepts. I take it to be purely logical. At any rate, the place is hereby marked where there has to be a decision.

As it is well-known, Frege met the collapse of his system precisely at this point. Characteristically of him, Frege was forthright in his comments of Basic Law V and, after knowing of Russell's paradox, wrote – in a retrospective and melancholic mood – that he would have been glad to prescind from value-ranges had he known how to get by without them:

> Hardly anything more unwelcome can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished.
>
> This is the position into which I was put by a letter from Mr. Bertrand Russell as the printing of this volume was nearing completion. The matter concerns my Basic Law V. I have never concealed from myself that it is not as obvious as the others (...). Indeed, I pointed this very weakness in the foreword to the first volume, p. VII. I would glady have dispensed with this foundation if I had known of some substitute for it. Even now, I do not see how arithmetic can be formulated scientifically, how the numbers can be apprehended as logical objects and brought under consideration, if it is not – at least conditionally – permissible to pass from concept to its extension. May I always speak of the extension of a concept, of a class? And if not, how are the exceptions to be recognised? (Cf. afterword of volume II of *Grundgesetze der Arithmetik*.)

Indeed, how are the exceptions to be recognized? Russell's paradox refutes the simple and elegant view that there are no exceptions. The logicist view of sets as "something obtained by dividing the totality of all existing things into two categories" led to the paradoxes and today has been effectively replaced by the iterative conception of sets.[1] At present, there is no satisfactory set theory based on the logicist view. However, even it proves impossible to develop a logicist set theory (as I suspect), perhaps it is possible to develop a logicist arithmetic. By this, I mean founding arithmetic on a *strict* logicist view according to which logical objects are extensions (as in Frege), not numbers (as in neologicism) nor constructed in terms of other kinds of abstractions.

In one of his first attempts to salvage logicism from the wreck of Frege's system, Bertrand Russell toyed with so-called zigzag theories. According to him "in a zigzag theory we start from the suggestion that the propositional functions determine classes when they are fairly simple, and only fail to do so when they are complicated and recondite" ([17], pp. 145–6). If a propositional function (a concept, in Frege's terms) does not determine an extension then, given any class (set), there must be either an element of the class that does not fall under the concept or an element that falls under the concept but is not in the class. In the pictoresque terminology of Russell, propositional functions that do not have extensions must zigzag between classes. Russell never

[1] The citation is from Kurt Gödel in p. 475 of [10].

worked out a zigzag theory to his own satisfaction and eventually gave up extensions altogether and adopted a so-called "no-classes" theory.

This paper shows how to set up a zigzag theory which is sufficient to develop full second-order arithmetic (it is also sufficient to develop a nice theory of finite sets). The origins of this theory can be traced to an idea of Michael Dummett who blamed Russell's paradox not on the extension operator but on the impredicative character of Frege's system, viz. on the acceptance of the unrestricted comprehension principle (cf. pp. 218–9 of [5]). A few years later, Richard Heck proved in [12] that Dummett had a point: Frege's system is consistent provided that the comprehension principle is suitably restricted. Let us describe in some detail the theory that Heck proved consistent. This theory only differs from (our rendering of) Frege's system by restricting the comprehension scheme to formulas without second-order quantifications. So, the restricted scheme is

$$\exists F \forall x (A(x) \leftrightarrow Fx),$$

for formulas $A(x)$ in which second-order quantifications do not occur (and in which the variable F does not occur free). I denote this predicative Fregean theory by H. Heck showed that this theory is not trivial, in the sense that it is able to interpret Robinson's theory Q. However, as was suspected, H is rather weak since it cannot even interpret primitive recursive arithmetic (see [4] for a proof of this fact).

The theory H cannot be considered a zigzag theory. Even though it restricts the existence of extensions to predicative concepts, the restriction is accomplished by the drastic move of allowing only predicative concepts in the language. In fact, *all* concepts of the predicative theory have extensions. There is no zigzagging in a landscape in which every available concept has an extension.[2] Nevertheless, the toll for adopting this frugal landscape (obtained via a restriction of the comprehension scheme) is very high because H is proof-theoretically very weak. If we follow Russell's idea that concepts that have extensions must be fairly simple and if we equate recondite and complicated concepts with the impredicative ones, then the theory H falls short of these terms because it does not make room for impredicative concepts. What one needs is a theory in which impredicative concepts can be formed but in which only predicative concepts admit extensions. The business of the next section is to define such a theory, prove its consistency, and show how to develop arithmetic in it in a thoroughly Fregean way. In doing this, we isolate a certain weak form of finite reducibility. In Section 4, we *prove* the full form of finite reducibility: Every concept which is true of only finitely many elements is co-extensive with a predicative concept. This form of reducibility can be used to develop a workable theory of finite sets. The paper also includes two sections of commentary.

2 The zigzag theory

Let us consider a second-order language with a sort for first-order variables, written in lower case Latin letters $x, y, z, \ldots$, and two sorts for second-order variables: the predicative sort, given by capital Latin letters $F, G, H, \ldots$ and the impredicative sort

[2] Later on, in Secion 5, we will discuss a purported distinction between sets and extensions.

given by Gothic letters $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \ldots$. Formulas of this second-order language are defined as usual, with both kinds of second-order variables behaving syntactically like unary predicates. I also allow both kinds of second-order quantifiers $\forall F$, $\exists F$ and $\forall \mathfrak{F}$, $\exists \mathfrak{F}$. Typical of Fregean theories, an extension operator is included. This operator is, in our case, restricted: the expression $\hat{x}.A(x)$ is a well-formed (first-order) term only if impredicative variables do not occur (at all) in the formula $A(x)$. Note that the fragment of the language in which second-order impredicative variables do not occur is exactly the language of the predicative theory H. We are now ready to state the axioms of our theory PE (an acronym for predicative extensions). The theory PE is framed in classical logic and its proper axioms are those of H complemented with the following unrestricted (impredicative) comprehension scheme:

$$\exists \mathfrak{F} \forall x (A(x) \leftrightarrow \mathfrak{F}x),$$

for any formula $A(x)$ of the language (in which the variable $\mathfrak{F}$ does not occur free). Therefore, PE includes two forms of comprehension. The above impredicative form, and the (already discussed) predicative comprehension scheme that comes from the theory H:

$$\exists F \forall x (A(x) \leftrightarrow Fx),$$

for a formulas $A(x)$ in the language of H and in which second-order predicative quantifications do not occur (note that neither second-order bound impredicative variables nor second-order impredicative parameters are allowed). By impredicative comprehension, we have $\forall F \exists \mathfrak{F} \forall x (Fx \leftrightarrow \mathfrak{F}x)$, i.e., the Gothic variables have a wider range of values than the Latin variables. Basic Law V is as before, with the said restriction that terms of the form $\hat{x}.A(x)$ only make sense for formulas $A(x)$ in which second-order impredicative variables do not occur (neither free, nor bound).

The theory PE is defined in the spirit of the systems discussed by John Burgess in section 2.3d of his book [2]. The difference lies in the fact that the predicative system H over which impredicative variables of PE "float" is based on a variable-binding term-forming operator (the extension operator) and not, as in Burgess's above mentioned systems, on an extension symbol which applies to concept variables. This has the effect that – contrary to Burgess's systems – the Humean operator "number of" can be defined along Fregean lines (see the definition of card below).[3]

It is easy to see that the theory PE is consistent. Take Heck's model of the theory H. It consists of a first-order domain M (in Heck's model, this domain is actually the natural numbers), a second-order domain $S \subseteq \mathcal{P}(M)$ where the second-order variables

[3] In [1], Francesca Boccuni proposed a system with two kinds of second-order variables (plural variables and concept variables) and, like us, a variable-binding term-forming operator for getting extensions. Comprehension for plural variables is unrestricted, as with our impredicative variables. However, Boccuni's comprehension for concept variables differs from ours and cannot be understood as a predicative restriction in the traditional sense (because Boccuni's system admits comprehension for certain formulas with bound plural variables). Also, the extension operator of Boccuni is different. In particular, one cannot form the extensions of concepts given by formulas with bound concept variables, a feature which seems to prevent – as in Burgess – the definition of the operator "number of" in a Fregean manner. As a consequence, the development of arithmetic by Boccuni in [1] (and by Burgess in section 2.3d of [2]) is un-Fregean, being a mere Dedekindian development (i.e., based solely on the fact that a simply infinite system is present within a theory enjoying full impredicative comprehension).

range, and a carefully defined function that provides the interpretation of the extension operator. If we expand this structure by saying that the impredicative variables of PE range over the full power set $\mathcal{P}(M)$ of M, then it is clear that one obtains a model of PE. This simple construction proves the consistency of the theory PE.[4,5]

The blunt addition of the impredicative sort to the theory H, with apparently no interference on the predicative fragment, looks like adding a pointless idle running of language. This is not the case, however. The impredicative sort not only allows a new and important means of expression, but also increases the proof-theoretic power of the theory immensely because PE is able to interpret full second-order arithmetic. This should be compared with the theory H, which is not even able to interpret primitive recursive arithmetic. How can this be, given that (according to our conjecture in footnote 5) PE is conservative over H? The answer lies in the fact that, within the new theory, one is able to define properly (impredicatively) the concept of natural number and that, of course, arithmetic is developed for the objects falling under this concept.

Let $\mathrm{Eq}(F, G)$ abbreviate the formula which states that the (predicative) concepts F and G are equinumerous via a predicative bijection:

$$\exists R[\forall x(Fx \to \exists^1 y(Gy \wedge R\langle x, y\rangle)) \wedge \forall y(Gy \to \exists^1 x(Fx \wedge R\langle x, y\rangle))].$$

For the sake of simplicity, I formulated second-order logic with only unary concepts. We apparently need a binary predicate R in the above definition. It is nevertheless well known that R can be taken to be unary because there is a definable ordered pair operation, namely:

$$\langle x, y\rangle := \hat{z}.(z = \hat{w}.(w = x \vee w = y) \vee z = \hat{w}.(w = x)).$$

The cardinality operator is defined in the Fregean way. The number of elements falling under the concept F is the extension formed by all the extensions of concepts equinumerous with F. Formally:

$$\mathrm{card}(F) := \hat{z}.\exists H(\mathrm{Eq}(H, F) \wedge z = \hat{w}.Hw).$$

As Heck observed in [12], Hume's principle can be proved in H (and, hence, in PE), i.e.,

$$\mathsf{PE} \vdash \forall F \forall G(\mathrm{card}(F) = \mathrm{card}(G) \leftrightarrow \mathrm{Eq}(F, G)).$$

The development of arithmetic now proceeds in a thoroughly Fregean manner. The number zero is defined by $0 := \hat{z}.(z = \hat{x}.(x \neq x))$. It is clear that the theory PE proves $\forall F(\mathrm{card}(F) = 0 \leftrightarrow \neg\exists x Fx)$. The binary successor relation $S(x, y)$ is given by the following formula:

$$\exists F \exists G(x = \mathrm{card}(F) \wedge y = \mathrm{card}(G) \wedge \exists z(Gz \wedge \forall u(Fu \leftrightarrow Gu \wedge u \neq z))).$$

[4] A similar argument appears in [20] while proving the relative consistency of a theory of classes put on top of Quine's theory *New Foundations*.

[5] We conjecture the stronger statement that PE is conservative over H. A model-theoretic proof of this fact should follow well known lines. The missing ingredient is a completeness result for theories with the extension operator.

With these definitions, it is easy to prove in H (and, therefore, in PE) that 0 is not the successor of any object, that an object cannot be the successor of two different objects, and that two different objects cannot be the successor of the same object. For future reference, I list these properties:

(i) $\mathsf{PE} \vdash \forall x \neg S(x, 0)$

(ii) $\mathsf{PE} \vdash \forall x \forall y \forall z (S(x, z) \wedge S(y, z) \rightarrow x = y)$

(iii) $\mathsf{PE} \vdash \forall x \forall y \forall z (S(x, y) \wedge S(x, z) \rightarrow y = z)$

The definition of the *concept* of natural number can now be made. This concept is defined impredicatively, as it should be. First, I define the notion of *hereditarity* with respect to the successor relation:

$$\mathrm{Her}(\mathfrak{F}) := \forall x \forall y (\mathfrak{F}x \wedge S(x, y) \rightarrow \mathfrak{F}y).$$

Note that I have defined "hereditarity" for impredicative variables. An *inductive* concept is a concept which is hereditary and true of 0. A *natural number* is defined as an object which falls under every inductive concept (Frege uses the terminology "finite number"). Formally:

$$\mathbb{N}(x) := \forall \mathfrak{F} (\mathfrak{F}0 \wedge \mathrm{Her}(\mathfrak{F}) \rightarrow \mathfrak{F}x).$$

It is clear that PE proves $\mathbb{N}(0)$ and $\forall x \forall y (\mathbb{N}(x) \wedge S(x, y) \rightarrow \mathbb{N}(y))$. It is also the case that PE proves $\mathbb{N}(y) \wedge S(x, y) \rightarrow \mathbb{N}(x)$. To see this, assume that $\neg\mathbb{N}(x)$ and $S(x, y)$. By the first assumption, there is $\mathfrak{G}$ such that $\mathfrak{G}0$, $\mathrm{Her}(\mathfrak{G})$ and $\neg\mathfrak{G}x$. By (impredicative) comprehension take $\mathfrak{H}$ such that $\forall z (\mathfrak{H}z \leftrightarrow \mathfrak{G}z \wedge z \neq y)$. By (i) we have $\mathfrak{H}0$, and using (ii) it is easy to argue that $\mathrm{Her}(\mathfrak{H})$. Clearly, $\neg\mathfrak{H}(y)$ and, therefore, $\neg\mathbb{N}(y)$.

Since full comprehension for the Gothic variables is available, it is immediate to show that PE proves the full scheme of induction:

$$A(0) \wedge \forall x \forall y (\mathbb{N}(x) \wedge A(x) \wedge S(x, y) \rightarrow A(y)) \rightarrow \forall x (\mathbb{N}(x) \rightarrow A(x)),$$

for *any* formula $A(x)$ of the language of PE.

If we show that every natural number has a successor, then the concept $\mathbb{N}$ defines a simply infinite structure in the sense of Dedekind. We prove this fact again in a Fregean manner, using the Fregean trick of showing that the successor of a natural number is the number of natural numbers less than or equal to that number. The "less than or equal relation" is defined impredicatively:

$$x \leq y := \forall \mathfrak{F} (\mathfrak{F}x \wedge \mathrm{Her}(\mathfrak{F}) \rightarrow \mathfrak{F}y),$$

i.e., x is less than or equal to y if y falls under any hereditary concept which is true of x. The following are straightforward:

(iv) $\mathsf{PE} \vdash \forall x (x \leq x)$

(v) $\mathsf{PE} \vdash \forall x \forall y (S(x, y) \rightarrow x \leq y)$

(vi) $\mathsf{PE} \vdash \forall x \forall y \forall z (x \leq y \wedge y \leq z \rightarrow x \leq z)$

Given a natural number x, the concept of being less than or equal to x was defined impredicatively. However, as will shall see in the end of this section, it can be proved that this concept is co-extensive with a predicative concept. This is a form of *finite reducibility*.

Lemma 1. $\mathsf{PE} \vdash \forall x(x \leq 0 \leftrightarrow x = 0)$.

Proof. The right-to-left direction is a particular case of (iv) above. For the left-to-right direction, suppose that $x \leq 0$ but $x \neq 0$. Consider the concept given by the formula $u \neq 0$. By (i) this concept is (trivially) hereditary and, by assumption, it is true of x. By hypothesis, it is therefore true of 0. This is a contradiction. □

Lemma 2. $\mathsf{PE} \vdash \forall x \forall y(S(x, y) \to \forall u(u \leq y \wedge u \neq y \to u \leq x))$.

Proof. Let us suppose that $S(x, y)$, $u \leq y$ and $u \neq y$. To see that $u \leq x$, consider $\mathfrak{F}$ a hereditary concept such that $\mathfrak{F}u$. we must show that $\mathfrak{F}x$. Assume not. By impredicative comprehension, take a concept $\mathfrak{G}$ such that $\forall v(\mathfrak{G}v \leftrightarrow \mathfrak{F}v \wedge v \neq y)$. Let v and w be such that $\mathfrak{G}v$ and $S(v, w)$. In particular, $\mathfrak{F}v$. By the hereditarity of $\mathfrak{F}$, we also have $\mathfrak{F}w$. Given that $v \neq x$ (because one has both $\mathfrak{F}v$ and, by assumption, $\neg\mathfrak{F}x$), we conclude by (ii) that $w \neq y$. Hence $\mathfrak{G}w$. We have argued that $\mathfrak{G}$ is hereditary. Since $u \leq y$ and $\mathfrak{G}u$, we get $\mathfrak{G}y$. This is absurd. □

From the above lemma and (iv), (v) and (vi), we readily have

(vii) $\mathsf{PE} \vdash \forall x \forall y(S(x, y) \to \forall u(u \leq y \leftrightarrow u \leq x \vee u = y))$

We are busy trying to show that $\mathbb{N}$, together with 0 and the successor, forms a simply infinite structure. The arguments so far do not use impredicativity in any essential way. Actually, the arguments so far do not use induction at all and the above statements are true in the full domain of objects, not only in $\mathbb{N}$. The next results of this section, however, use induction and are no longer true in the full domain of objects (they hold in $\mathbb{N}$).

Lemma 3. $\mathsf{PE} \vdash \forall x(\mathbb{N}(x) \to \neg S(x, x))$.

Proof. By induction on x. By (i), $\neg S(0, 0)$. Let us now suppose $\mathbb{N}(x)$, $\neg S(x, x)$ and $S(x, y)$. If we assume that $S(y, y)$ then, by (ii), we would get $x = y$. This is absurd. □

Lemma 4. $\mathsf{PE} \vdash \forall x \forall y(\mathbb{N}(x) \wedge S(x, y) \to \forall u(u \leq x \to u \neq y))$.

Proof. We show that $\forall y(S(x, y) \to \forall u(u \leq x \to u \neq y))$ for all natural numbers x. The proof is by induction on x. The base case $x = 0$ is a consequence of Lemma 1 and (i). To argue the induction step, assume $S(x, y)$ and $S(y, z)$ and, by induction hypothesis, $\forall u(u \leq x \to u \neq y)$. We must show that $\forall u(u \leq y \to u \neq z)$. Take u so that $u \leq y$. In order to reach a contradiction, assume $u = z$. Then $z \leq y$. By Lemma 3, $z \neq y$. Hence, by Lemma 2, $z \leq x$. By (v) and (vi), we get $y \leq x$. Using the induction hypothesis, we get the contradiction $y \neq y$. □

From the above lemma and and (v), (vi) and (vii), we get

(viii) $\mathsf{PE} \vdash \forall x \forall y(\mathbb{N}(x) \wedge S(x,y) \to \forall u(u \leq x \leftrightarrow u \leq y \wedge u \neq y))$

Proposition 1. $\mathsf{PE} \vdash \forall x(\mathbb{N}(x) \to \exists y S(x,y))$.

Proof. We prove instead the stronger and more explicit sentence

$$\forall x(\mathbb{N}(x) \to \exists F(\forall u(Fu \leftrightarrow u \leq x) \wedge S(x, \mathrm{card}(F)))).$$

(This is a roundabout way of stating that the successor of the natural number x is the number of numbers less than or equal to x. Formulating the statement this way is not without advantages because it makes clear that a weak form of finite reducibility is in operation in the argument below.) The proof is by induction on x.

The case $x = 0$ follows from Lemma 1, predicative comprehension and the definitions of 0 and S. Suppose $\mathbb{N}(x)$ and $S(x,y)$. By induction hypothesis, take a predicative concept F such that $\forall u(Fu \leftrightarrow u \leq x)$ and $S(x, \mathrm{card}(F))$. By (iii), $y = \mathrm{card}(F)$. By predicative comprehension, there is G such that $\forall u(Gu \leftrightarrow Fu \vee u = y)$. Hence, by (vii), $\forall u(Gu \leftrightarrow u \leq y)$. Also, by (viii), $\forall u(Fu \leftrightarrow Gu \wedge u \neq y)$. Given that Gy holds, we get – by the definition of the successor relation – that $S(\mathrm{card}(F), \mathrm{card}(G))$. Therefore, $S(y, \mathrm{card}(G))$, as wanted. □

As I have commented, a weak form of finite reducibility is in operation in the above argument. Let us isolate it:

Theorem 1 (Weak finite reducibility)**.** $\mathsf{PE} \vdash \forall y(\mathbb{N}(y) \to \exists F \forall x(Fx \leftrightarrow x \leq y))$.

Proof. By induction on y. For $y = 0$, use Lemma 1 and predicative comprehension. Suppose that $\mathbb{N}(y)$, $S(y,z)$ and that there is F such that $\forall x(Fx \leftrightarrow x \leq y)$. By predicative comprehension, there is a concept G such that $\forall x(Gx \leftrightarrow Fx \vee x = z)$. By (vii), we conclude that $\forall x(Gx \leftrightarrow x \leq z)$. □

3 First commentary

We have rehearsed, in our setting, Frege's development of arithmetic. We defined a concept $\mathbb{N}$ which, together with the successor operation, gives rise to a simply infinite structure (in the sense of Dedekind). By the very way $\mathbb{N}$ is defined and by the availability of unrestricted comprehension, we get induction for every formula of the language. In effect (by unrestricted comprehension again), we are able to interpret full second-order arithmetic in PE.

We should distinguish between two different issues when setting up arithmetic. One issue concerns the definition of a simply infinite structure. Another issue concerns forming a substructure of the simply infinite structure that satisfies strong forms of induction. They are mixed together in the above development but, in general, this need not be so. In the presence of unrestricted comprehension (as it is the case with the theory PE), the second issue is easily solved in the usual Frege-Dedekind way (by considering the "smallest" simply infinite structure). However, with restricted kinds of comprehension, namely forms of predicative comprehension, the definition of natural number is severely crippled and the availability of induction is very limited (way below primitive recursive induction: see [3] and [4]).

Twelve years ago, I tried to amend this kind of situation in order to obtain, within a predicative setting, the scheme of induction for all arithmetical predicates. The idea of [6] was simple enough. I worked within Heck's setting of ramified predicative arithmetic [12] (a provably consistent theory) and tried to adjoin an axiom of finite reducibility. Reducibility has a long story in logicism and was introduced by Russell and Whitehead for purely pragmatic reasons. According to Russell, "[the] axiom [of reducibility] has a purely pragmatic justification: it leads to the desired results and no others."[6] This is a justification that clearly departs from a logicist perspective and one that a logicist cannot rest contented with. In chapter XVII of [18], Russell writes that "I do not see any reason to believe that the axiom of reducibility is logically necessary, which is what would be meant by saying that it is true in all possible worlds." That notwithstanding, the restriction of the axiom of reducibility which states that a concept true of only finitely many objects is co-extensive with a predicative concept seems to be necessary in Russell's sense. In effect, if a concept is true of only finitely many objects $a_1, a_2, \ldots, a_n$ then the concept is co-extensive with the predicative concept given by the formula (with parameters) '$x = a_1 \vee x = a_2 \vee \ldots \vee x = a_n$.' I thought that a form of logicism was defensible by *postulating* the axiom of finite reducibility in Heck's ramified predicative theory and hoped that *that* would be enough to develop (in a Fregean way) first-order Peano arithmetic. This approach was tried in [6] and subjected to serious philosophical criticism in [2] (cf. p. 113). The problem lies in *stating* the axiom of finite reducibility. How does one define finiteness within a predicative theory? One would hope, perhaps, to be satisfied at first with a "deficient" definition of finiteness and then, after introducing the axiom of finite reducibility, show that the definition has, after all, the desired properties of the notion of finiteness and proves, in the end, to be right. Of course, this strategy is very delicate and unstable since the axiom of finite reducibility would be postulated concerning a *prima facie* inadequate definition, which would only be proven right after the postulation. A (hopefully) virtuous circle, as this was classified in [6]. With the benefit of hindsight, the development of [6] rests upon a presumption that goes beyond finite reducibility. It rest on the presumption that predicates of the form "there are finitely many w such that $A(x, w)$" are predicative for predicative formulas $A(x, w)$. This is a more stringent condition than that of finite reducibility and lacks an argument supporting it from the logicist viewpoint.[7] The present paper came about with the realization that if one defines finiteness impredicatively, then one can actually *prove* finite reducibility instead of "helping oneself to intuitions about finitude as axioms, not proved as theorems from logical axioms and a suitable definition of finitude" (cf. p. 113 of [2]).

Let us now turn to the first issue mentioned above, the one concerning the definition of a simply infinite structure. There is, in fact, a straightforward manner of obtaining a simply infinite structure within PE (even within H): the map $x \rightsquigarrow \hat{z}.(z = x)$ is clearly injective and not surjective (the object $\hat{z}.(z \neq z)$ is not in the image of the map). Of course, this is not the way that Frege developed arithmetic. It is an un-Fregean development. He would not have rested contented with any old simply infinite structure. He is no structuralist. For Frege, numbers are extensions formed by extensions of equinu-

[6]See p. xiv of the introduction to the second edition of volume I of *Principia Mathematica* [19].

[7]The condition was made explicit in [6]. My present view is that the arguments in [6] for accepting this condition are not founded on a logicist point of view (nor on finite reducibility).

merous concepts. For instance, the number 3 is the extension of all extensions with exactly three elements. This is an important feature of Frege's development of arithmetic. His development is not merely a technical exercise in modeling arithmetic. It obeys the constraint "that a philosophically satisfactory foundation for a mathematical theory must somehow intimately build in its possibilities of applications" (cf. p. 91 of [11]).

An interesting fact is that it is possible to obtain a simply infinite structure in H in terms of Frege's own definition of number. The development of a simply infinite structure given in Section 2 up to statement (vii) can essentially be formalized in H. For the remaining bit, it is enough to prove that every object of the form $\mathrm{card}(F)$, for some F, has a successor. Heck showed in [12] how this can be done (in an un-Fregean way) by a simple ordered pair trick.[8] An even more interesting fact, actually a rather striking one, is that this remaining bit can also be obtained in a (roughly) Fregean way within a predicative setting. This can be accomplished within Heck's ramified predicative theory described in [12]. In the ramified setting there are several rounds of second-order variables enjoying acceptable forms of comprehension for the predicativist. The zeroth round (corresponding to the second-order variables of H), the first round, the second round, etc. If in the development of arithmetic in Section 2 all the definitions are given in terms of zeroth round variables *except for* the definition of the concept of natural number (which should be defined with first round quantifications), then Lemma 3, Lemma 4, (viii), the proposition and the theorem can be proved when the predicative variables are rendered by zeroth round variables.[9] This was first shown by Heck in [14] (see, specially, Section 5) and the reader should consult his paper for details.

4 Finiteness and reducibility

Some technical choices which were made in the development of arithmetic in Section 2 may be questioned on philosophical grounds. Take, for instance, the definition of equinumerousity. By definition, $\mathrm{Eq}(F, G)$ holds whenever there is a *predicative* bijection between the objects falling under F and the objects falling under G. Is this definition faithful to the meaning of equinumerousity? Why shouldn't an impredicative bijection count as a witness of equinumerousity? The problem is compounded because our definition of equinumerousity was not really a matter of choice. The reader can easily check that the statement of equinumerousity with an impredicative bijection blocks the definition of Frege's cardinality operator card. That notwithstanding, we will see below that the issues just raised do not really arise as long as we are dealing only with concepts that are true of only finitely many elements.

In §158 of *Grundgesetze der Arithmetik*, Frege uses the locution "the cardinal number of a concept is finite" as a way of expressing that there are only finitely many el-

[8] The trick can be easily described. The predicate F can be put in bijection with the predicate H under which fall the ordered pairs of the form $\langle \hat{x}.(x \neq x), z \rangle$, with z such that Fz. Now, clearly, $\hat{x}.(x \neq x)$ does not fall under H. The successor of $\mathrm{card}(F)$ is $\mathrm{card}(G)$, where w falls under G if, and only if, $Hw \vee w = \hat{x}.(x \neq x)$.

[9] However, the form of finite reducibility stated in the theorem of Section 2 undergoes a subtle change of meaning. It becames much weaker because the "less than or equal relation" is defined with zeroth round quantifiers – one round less than the variables used in the definition of the concept of natural number.

ements falling under the given concept. This manner of speaking about finitude is not directly available in PE for impredicative concepts because the cardinality operator is only defined for predicative concepts. Fortunately, Frege also gives in the *Grundgesetze* a characterization of finitude in pure second-order (impredicative) logic – one that does not need value-ranges (extensions). This characterization provides a Fregean handle for approaching the questions of the above paragraph. Before directing our attention to this issue, we need to establish some easy facts about the finitude of predicative concepts.

Proposition 2. *The theory* PE *proves*

$$\forall F, G \forall z (\forall x (Gx \leftrightarrow Fx \vee x = z) \rightarrow (\mathbb{N}(\text{card}(F)) \leftrightarrow \mathbb{N}(\text{card}(G)))).$$

Proof. Suppose that $\forall x(Gx \leftrightarrow Fx \vee x = z)$. If Fz, then $\text{card}(G) = \text{card}(F)$ and we are done. Otherwise, by the definition of the successor relation, $S(\text{card}(F), \text{card}(G))$. The equivalence $\mathbb{N}(\text{card}(F)) \leftrightarrow \mathbb{N}(\text{card}(G))$ is now clear by the properties discussed immediately after the definition of natural number in Section 2. □

The above proposition entails that the union and the cartesian product of two finite (predicative) concepts are still finite concepts. The first statement is formally

$$\forall F, G, H \, (\mathbb{N}(\text{card}(F)) \wedge \mathbb{N}(\text{card}(G)) \wedge \forall x(Hx \leftrightarrow Fx \vee Gx) \rightarrow \mathbb{N}(\text{card}(H)))$$

and it is easily proved by induction on $\text{card}(G)$ using the above proposition. The second statment is the universal closure of

$$\mathbb{N}(\text{card}(F)) \wedge \mathbb{N}(\text{card}(G)) \wedge \forall x(Hx \leftrightarrow \exists u, v(x = \langle u, v\rangle \wedge Fu \wedge Gv)) \rightarrow \mathbb{N}(\text{card}(H))$$

and it is proved by induction on $\text{card}(G)$ using the first statement.

We are now ready to discuss finitude in the context of impredicative concepts. In the sequel, we describe (an adaptation of) Frege's second-order characterization of finitude and show that "finite" concepts – according to this characterization – are co-extensive with predicative concepts of finite cardinality (finite reducibility). We also argue that impredicative bijections between two "finite" concepts are co-extensive with predicative ones. In fact, a robust theory of finite sets can be developed in PE.

Frege's characterization of finitude in terms of pure second-order logic is discussed between §158 and §179 of the volume I of *Grundgesetze der Arithmetik*. We will not follow Frege's treatment in a strict manner (see Heck's paper [13] for a discussion of why Frege sometimes did not chose the simplest way), but opt for a streamlined analysis adapted to our present purposes. The two theorems of this section can be seen as versions of Theorems 327 and 348 of Frege's *Grundgesetze* adapted to the setting of PE.

In Section 2, we defined the notion of hereditarity with respect to the successor relation. We need that notion with respect to an arbitrary relation $\mathfrak{R}$:

$$\text{Her}_{\mathfrak{R}}(\mathfrak{F}) := \forall x \forall y (\mathfrak{F}x \wedge \mathfrak{R}\langle x, y\rangle \rightarrow \mathfrak{F}y),$$

as well as the associated *ancestral* relation $\mathfrak{R}^*$ of $\mathfrak{R}$:

$$\mathfrak{R}^*\langle x, y\rangle := \forall \mathfrak{F} \, (\mathfrak{F}x \wedge \text{Her}_{\mathfrak{R}}(\mathfrak{F}) \rightarrow \mathfrak{F}y).$$

It is clear that $\mathfrak{R}\langle x, y\rangle \to \mathfrak{R}^*\langle x, y\rangle$ and that the relation given by $\mathfrak{R}^*$ is reflexive and transitive. We will need to rely on some facts about ancestral relations. These are known facts of pure second-order (impredicative) logic. As a matter of fact, ancestral relations made their appearance in Frege's first book [7], where they were studied.

We lay the needed facts in the form of three lemmas and, for the sake of completeness, we provide their proofs in the appendix of this paper.

Lemma 5. $\mathsf{PE} \vdash \mathfrak{R}^*\langle x, y\rangle \wedge \forall z(\mathfrak{R}\langle x, z\rangle \to z = x) \to y = x.$

Lemma 6. *The theory* PE *proves the universal closure of the conditional formula whose antecedent is*

$$\forall u \forall v\, (\mathfrak{R}\langle u, v\rangle \wedge \mathfrak{R}^*\langle x, u\rangle \wedge \mathfrak{R}^*\langle u, y\rangle \wedge \mathfrak{R}^*\langle x, v\rangle \wedge \mathfrak{R}^*\langle v, y\rangle \to \mathfrak{Q}\langle u, v\rangle),$$

and whose consequent is $\mathfrak{R}^*\langle x, y\rangle \to \mathfrak{Q}^*\langle x, y\rangle$.

In particular, it proves $\forall u \forall v(\mathfrak{R}\langle u, v\rangle \to \mathfrak{Q}\langle u, v\rangle) \wedge \mathfrak{R}^*\langle x, y\rangle \to \mathfrak{Q}^*\langle x, y\rangle$.

We say that $\mathfrak{R}$ is *functional* if $\forall x \forall y \forall z\, (\mathfrak{R}\langle x, y\rangle \wedge \mathfrak{R}\langle x, z\rangle \to y = z)$, and we write $\mathrm{Func}(\mathfrak{R})$.

Lemma 7. *The following formulae are provable in* PE*:*

1. $\mathrm{Func}(\mathfrak{R}) \wedge x \neq y \wedge \mathfrak{R}^*\langle x, y\rangle \wedge \mathfrak{R}\langle x, z\rangle \to \mathfrak{R}^*\langle z, y\rangle$.
2. $\mathrm{Func}(\mathfrak{R}) \wedge \mathfrak{R}^*\langle z, x\rangle \wedge \mathfrak{R}^*\langle z, y\rangle \to \mathfrak{R}^*\langle y, x\rangle \vee \mathfrak{R}^*\langle x, y\rangle$.
3. $\mathrm{Func}(\mathfrak{R}) \wedge x \neq y \wedge \mathfrak{R}^*\langle x, y\rangle \wedge \mathfrak{R}^*\langle y, x\rangle \wedge \mathfrak{R}^*\langle x, z\rangle \to \mathfrak{R}^*\langle z, x\rangle$.

Frege's characterization of finiteness is based on the following definition:

$$\mathrm{Btw}(\mathfrak{R}, a, b, x) := \mathrm{Func}(\mathfrak{R}) \wedge \mathfrak{R}^*\langle a, x\rangle \wedge \mathfrak{R}^*\langle x, b\rangle \wedge \neg\exists z(\mathfrak{R}\langle b, z\rangle \wedge \mathfrak{R}^*\langle z, b\rangle).$$

In the words of Frege, according to the translation [9], "x belongs to the $\mathfrak{R}$-series starting with a and ending with b" (see §158 of the *Grundgesetze*). We also say that x lies between a and b in the $\mathfrak{R}$-series.

Definition. *We say that* $\mathfrak{F}$ *is* Fregean finite*, and write* $\mathrm{Fin}(\mathfrak{F})$*, just in case*

$$\exists \mathfrak{R} \exists a \exists b \forall x(\mathfrak{F}x \leftrightarrow \mathrm{Btw}(\mathfrak{R}, a, b, x))$$

or, else, $\mathfrak{F}$ *is an empty concept.*

We also use the notation $\mathrm{Fin}(F)$ for predicative variables F. We could have permitted this case in the above definition. Equivalently, we can see it as abbreviating $\exists \mathfrak{F}(\forall x(\mathfrak{F}x \leftrightarrow Fx) \wedge \mathrm{Fin}(\mathfrak{F}))$.

Lemma 8. $\mathsf{PE} \vdash \mathrm{Btw}(\mathfrak{R}, a, b, x) \to \neg\exists z(\mathfrak{R}\langle x, z\rangle \wedge \mathfrak{R}^*\langle z, x\rangle)$.

Proof. Suppose that $\mathrm{Btw}(\mathfrak{R}, a, b, x)$ and assume that there is z such that $\mathfrak{R}\langle x, z\rangle$ and $\mathfrak{R}^*\langle z, x\rangle$. By definition, $x \neq b$. Since $\mathfrak{R}^*\langle x, b\rangle$, by (3) of Lemma 7, either $x = z$ or $\mathfrak{R}^*\langle b, x\rangle$. If $x = z$, by the functionality of $\mathfrak{R}$ and Lemma 5, we get $x = b$ which is impossible. If $\mathfrak{R}^*\langle b, x\rangle$, let y be such that $\mathfrak{R}\langle b, y\rangle$: this y exists by Lemma 5. By (1) of Lemma 7 and the fact that $x \neq b$, we get $\mathfrak{R}^*\langle y, b\rangle$. This contradicts the last clause of the definition of $\mathrm{Btw}(\mathfrak{R}, a, b, x)$. □

Lemma 9. *The theory* PE *proves the universal closure of the conditional formula whose antecedent is* $\mathrm{Btw}(\mathfrak{R}, a, b, x) \wedge \mathrm{Btw}(\mathfrak{R}, a, b, y) \wedge \mathfrak{R}\langle x, y\rangle$ *and whose consequent is*

$$\forall w(\mathfrak{R}^*\langle a, w\rangle \wedge \mathfrak{R}^*\langle w, y\rangle \to \mathfrak{R}^*\langle w, x\rangle \vee w = y).$$

Proof. Suppose that $\mathrm{Btw}(\mathfrak{R}, a, b, x)$, $\mathrm{Btw}(\mathfrak{R}, a, b, y)$, $\mathfrak{R}\langle x, y\rangle$, $\mathfrak{R}^*\langle a, w\rangle$ and $\mathfrak{R}^*\langle w, y\rangle$. By (2) of Lemma 7, either $\mathfrak{R}^*\langle w, x\rangle$ or $\mathfrak{R}^*\langle x, w\rangle$. In the first case, we are done. Assume $\mathfrak{R}^*\langle x, w\rangle$. By (1) of Lemma 7, either $x = w$ or $\mathfrak{R}^*\langle y, w\rangle$. In the former case, $\mathfrak{R}^*\langle w, x\rangle$ and we are done. The following situation remains to be studied: $\mathfrak{R}^*\langle x, w\rangle$ and $\mathfrak{R}^*\langle y, w\rangle$. If $y = w$, we are done. Otherwise $y \neq w$. In this case, by Lemma 5, there is z with $\mathfrak{R}\langle y, z\rangle$. Now, by (1) of Lemma 7, $\mathfrak{R}^*\langle z, w\rangle$ and, as a consequence, $\mathfrak{R}^*\langle z, y\rangle$. This is impossible by the previous lemma. □

Our version of Theorem 327 of the *Grundgesetze der Arithmetik* is the result below. Its formulation necessarily incorporates the principle of finite reducibility:

Theorem 2. $\mathsf{PE} \vdash \forall \mathfrak{F}\,(\mathrm{Fin}(\mathfrak{F}) \to \exists F(\forall x(Fx \leftrightarrow \mathfrak{F}x) \wedge \mathbb{N}(\mathrm{card}(F))))$.

Proof. Let $\mathfrak{F}$ be such that $\mathrm{Fin}(\mathfrak{F})$. If $\mathfrak{F}$ is an empty concept, the conclusion is clear. Otherwise, take $\mathfrak{R}$, a and b such that $\forall x(\mathfrak{F}x \leftrightarrow \mathrm{Btw}(\mathfrak{R}, a, b, x))$. Using Lemma 6, we may suppose without loss of generality that

$$\forall u \forall v(\mathfrak{R}\langle u, v\rangle \to \mathfrak{R}^*\langle a, u\rangle \wedge \mathfrak{R}^*\langle u, b\rangle \wedge \mathfrak{R}^*\langle a, v\rangle \wedge \mathfrak{R}^*\langle v, b\rangle).$$

By impredicative comprehension, let $\mathfrak{G}$ be such that for all w, $\mathfrak{G}w$ if, and only if,

$$\exists G(\mathbb{N}(\mathrm{card}(G)) \wedge \forall x(Gx \leftrightarrow \mathfrak{R}^*\langle a, x\rangle \wedge \mathfrak{R}^*\langle x, w\rangle))).$$

We first claim that $\mathrm{Her}_{\mathfrak{R}}(\mathfrak{G})$. To see this, suppose that $\mathfrak{G}u$ and $\mathfrak{R}\langle u, v\rangle$. Hence, there is G such that $\mathbb{N}(\mathrm{card}(G))$ and $\forall x(Gx \leftrightarrow \mathfrak{R}^*\langle a, x\rangle \wedge \mathfrak{R}^*\langle x, u\rangle)$. By the previous lemma, $\forall x(\mathfrak{R}^*\langle a, x\rangle \wedge \mathfrak{R}^*\langle x, v\rangle \to \mathfrak{R}^*\langle x, u\rangle \vee x = v)$. Therefore,

$$\forall x(\mathfrak{R}^*\langle a, x\rangle \wedge \mathfrak{R}^*\langle x, v\rangle \leftrightarrow Gx \vee x = v).$$

By predicative comprehension (and the fact that the sucessor of a natural number is a natural number), it follows that $\mathfrak{G}v$.

We also claim that $\mathfrak{G}a$ holds. This easily follows from

$$\forall x(\mathfrak{R}^*\langle a, x\rangle \wedge \mathfrak{R}^*\langle x, a\rangle \to x = a).$$

To see this, assume that $\mathfrak{R}^*\langle a, x\rangle$, $\mathfrak{R}^*\langle x, a\rangle$ but $x \neq a$. By Lemma 5, take z such that $\mathfrak{R}\langle a, z\rangle$. By (1) of Lemma 7, we get $\mathfrak{R}^*\langle z, x\rangle$ and, so, $\mathfrak{R}^*\langle z, a\rangle$. This contradicts Lemma 8.

Since $\mathfrak{R}^*\langle a, b\rangle$, we conclude $\mathfrak{G}b$. This is what we want. □

Theorem 348 of Frege's *Grundgesetze* says that "if the cardinal number of a concept is finite, then the objects that fall under it can be ordered into a simple series running from a specific object to a specific object" (cf. §172 of [9]). In our setting, this is stated as $\forall F(\mathbb{N}(\mathrm{card}(F)) \to \mathrm{Fin}(F))$. For convenience, we prove a stronger property.

Theorem 3. $\mathsf{PE} \vdash \forall F \forall \mathfrak{H}(\mathbb{N}(\text{card}(F)) \wedge \forall x(\mathfrak{H}x \to Fx) \to \text{Fin}(\mathfrak{H}))$.

Proof. By induction on $\text{card}(F)$. If $\text{card}(F) = 0$, $\mathfrak{H}$ is empty and there is nothing to prove. Suppose that $\mathbb{N}(n)$, $S(n,m)$, $\text{card}(F) = m$ and $\forall x(\mathfrak{H}x \to Fx)$. Take c such that $\mathfrak{H}c$ (otherwise, $\mathfrak{H}$ is empty and there is nothing to prove). If G is such that $\forall x(Gx \leftrightarrow Fx \wedge x \neq c)$, clearly $\text{card}(G) = n$,. By induction hypothesis, $\text{Fin}(\mathfrak{L})$, where $\mathfrak{L}$ is such that $\forall x(\mathfrak{L}x \leftrightarrow \mathfrak{H}x \wedge x \neq c)$. If $\mathfrak{L}$ is empty then $\mathfrak{H}$ is true of only the element c and the Fregean finiteness of $\mathfrak{H}$ easily follows. Otherwise, take $\mathfrak{R}$, a and b so that $\forall x(\mathfrak{L}x \leftrightarrow \text{Btw}(\mathfrak{R}, a, b, x))$. The idea is clear now: We want to tack c onto the end of the series given by $\mathfrak{R}$, a and b.

As a preliminary step, by Lemma 6, we may suppose that

$$\forall u \forall v(\mathfrak{R}\langle u, v\rangle \to \mathfrak{R}^*\langle a, u\rangle \wedge \mathfrak{R}^*\langle u, b\rangle \wedge \mathfrak{R}^*\langle a, v\rangle \wedge \mathfrak{R}^*\langle v, b\rangle).$$

By impredicative comprehension, let $\mathfrak{Q}(x)$ be defined as

$$\exists u \exists v[x = \langle u, v\rangle \wedge (\mathfrak{R}\langle u, v\rangle \vee (u = b \wedge v = c))].$$

We have $\text{Func}(\mathfrak{Q})$ and, since $\neg\exists z \mathfrak{Q}\langle c, z\rangle$, *a fortiori* $\neg\exists z(\mathfrak{Q}\langle c, z\rangle \wedge \mathfrak{Q}^*\langle z, c\rangle)$. We argue that $\forall x(\mathfrak{H}x \leftrightarrow \text{Btw}(\mathfrak{Q}, a, c, x))$, and this shows that $\mathfrak{H}$ is Fregean finite. So, we must argue

$$\forall x\,((\text{Btw}(\mathfrak{R}, a, b, x) \vee x = c) \leftrightarrow \text{Btw}(\mathfrak{Q}, a, c, x)).$$

If $\text{Btw}(\mathfrak{R}, a, b, x)$ then, by Lemma 6 (its particular case), $\text{Btw}(\mathfrak{Q}, a, b, x)$. It readily follows that $\text{Btw}(\mathfrak{Q}, a, c, x)$. Of course, $\mathfrak{Q}^*\langle a, b\rangle$. Therefore $\mathfrak{Q}^*\langle a, c\rangle$, and we get $\text{Btw}(\mathfrak{Q}, a, c, c)$. Conversely, assume that $\text{Btw}(\mathfrak{Q}, a, c, x)$. We may suppose that $x \neq c$. We have $\mathfrak{Q}^*\langle a, x\rangle$. By Lemma 6, $\mathfrak{R}^*\langle a, x\rangle$. On the other hand, by (2) of Lemma 7, $\mathfrak{Q}^*\langle x, b\rangle$ or $\mathfrak{Q}^*\langle b, x\rangle$. In the first case, by Lemma 6, we get $\mathfrak{R}^*\langle x, b\rangle$, and we are done. In the second case, we are in a situation where both $\mathfrak{Q}^*\langle b, x\rangle$ and $\mathfrak{Q}\langle b, c\rangle$ hold. By (1) of Lemma 7, either $b = x$ or $\mathfrak{Q}^*\langle c, x\rangle$. In the first case, $\mathfrak{R}^*\langle x, b\rangle$ and we are done. The second case is impossible by Lemma 5. □

Theorems 2 and 3 permit the development in PE of a very robust theory of finiteness. As an illustration, we show that an impredicate bijection between two predicative concepts of finite cardinality is co-extensive with a predicative concept. So, let us consider $\mathfrak{R}$ a bijection between concepts F and G of finite cardinality. Of course, $\mathfrak{R}$ is a sub-relation of the cartesian product between F and G. The latter has finite cardinality, as we have observed at the beginning of the present section. Hence, by Theorem 3, $\text{Fin}(\mathfrak{R})$. The conclusion follows from an application of Theorem 2.

5 Second commentary

Frege's system of the *Grundgesetze* is a theory of extensions (in fact, a theory of value-ranges). Extensions are, of course, extensions *of* concepts and they satisfy the form of extensionality given by Frege's Basic Law V. Sets, for Frege, are extensions (of concepts). They are not autonomous from concepts, nor is their basic relation: membership. In Frege's system of the *Grundgesetze*, membership is a defined notion given by

$$x \in y :\equiv \exists F(y = \hat{z}.Fz \wedge Fx).$$

This defined notion is fully operational in Frege's *Grundgesetze* in the sense that the *law of concretion* is derivable:

$$\forall x(x \in \hat{z}.A(z) \leftrightarrow A(x)),$$

for every formula of the language A.[10] The problem, of course, is that this law leads to Russell's paradox. This situation was analyzed in detail by Heck in [12] who observed that, in the predicative setting H, the law of concretion does not hold. The derivation from right to left is blocked by lack of concept-comprehension.

It is very instructive to discuss Heck's theory H because it brings to light some interesting phenomena. As with Frege's (inconsistent) theory, in (the consistent) H the extension operator applies to every formula of the language. Also, extensions satisfy the form of extensionality given by Frege's Basic Law V.

Lemma 10. *The theory* H *proves the following form of the law of concretion:*

$$\forall F \forall x(x \in \hat{z}.Fz \leftrightarrow Fx).$$

Proof. Let F be given. Suppose that $x \in \hat{z}.Fz$. By the definition of membership, there is G such that $\hat{z}.Fz = \hat{z}.Gz \wedge Gx$. By Basic Law V, we conclude that Fx. Conversely, given Fx it is clear (by the definition of membership) that $x \in \hat{z}.Fz$. □

The following fact is illuminating:

Proposition 3. *The theory* H *proves* $\exists y \forall x(x \in y \leftrightarrow A(x)) \leftrightarrow \exists F \forall x(Fx \leftrightarrow A(x))$, *for every formula* A *of the language (in which the variables* y *and* F *do not occur free).*

Proof. Suppose that there is y such that $\forall x(x \in y \leftrightarrow A(x))$. If there is no element x such that $A(x)$, then clearly $\exists F \forall x(Fx \leftrightarrow A(x))$: just take the predicative concept associated with the formula '$x \neq x$'. Suppose that there is w such that $A(w)$. Hence $w \in y$ and, in particular, there is F such that $y = \hat{z}.Fz$. We claim that, for this F, one has $\forall x(Fx \leftrightarrow A(x))$. By the above Lemma, $\forall x(x \in y \leftrightarrow Fx)$. The claim follows.

Let us now assume that there is F such that $\forall x(Fx \leftrightarrow A(x))$. Take y as $\hat{z}.Fz$. Applying the above Lemma again, we immediately get $\forall x(x \in y \leftrightarrow A(x))$. □

The above proposition says that set-comprehension and concept-comprehension go hand in hand. The right notion of set in H, one abiding by the law of concretion, is to say that y is a *set* if $\exists F(y = \hat{z}.Fz)$. In Heck's predicative setting, one must distinguish concepts which are in the range of second-order variables from (more generally) concepts expressed by a formula of the language. On purely philosophical grounds,

[10]Frege works with value-ranges, of which extensions are a particular case. The analogue of membership for value-ranges is Frege's application operator $\frown$ as defined in §34 of [9]. The analogue of the law of concretion for value-ranges is discussed by Frege in the same Section and formally proved in §55 of the *Grundgesetze*. The designation 'law of concretion' comes from Quine (see p. 16 of [15]).

the predicativist can accept every concept expressed by the language of H. However, he is at technical odds to make comprehension available for the latter concepts (but not to obtain their extensions, since every concept expressed by the language of H has an extension in H). One must not comprehend these concepts with the same round of second-order variables (i.e., within the language of H), on pain of falling into impredicative comprehension and, indeed, on pain of contradiction. But, of course, these concepts can be comprehended in an enlargement of the language with a *new* round of concept variables. In short, when the predicativist brand of logicism is formalized in a language, there are concepts which have extensions and which are fit for being sets (by enlarging the language) but which are not sets according to the language. We may dub these extensions which are not sets as 'protosets.' Interestingly, the existence of these protosets allows the theory H (and the theory PE) to explore some features of the (technically) missing corresponding sets. For instance, protosets are instrumental in defining Frege's cardinality operator card (in H) and in developing numbers in a Fregean manner (in PE).

As discussed in the last paragraph, protosets in H may be gathered as sets (via concept-comprehension) with a new round of second-order variables. Of course, with this move, new protosets will emerge in the extended language and a further round of second-order variables is necessary to gather them as sets. Etcetera. Etcetera. One is led into theories of ramification and into envisaging ramified systems in which the extension operator applies to every formula of the language and in which Basic Law V is unrestricted (as Heck does in [12]). If there is a last round of second-order variables, a most general membership relation can be defined, but the law of concretion fails. If the ramified theory has no last round of variables, membership relations necessarily ramify. Even if one takes the position that no formal theory is able to express all the predicative concepts (presumably because of the inexistence of a natural ordinal bound for ramification), in this informal theory there is no overall membership relation, but only local ones for fixed rounds. Be that as it may, we do not get a decent theory of sets for all extensions (because either there is no global membership relation or the law of concretion fails for extensions).

How does the theory PE fare with respect to the above issues? Well, PE is essentially a round of impredicative variables on top of H. The membership operation $x \in y$ has to be defined with predicative variables because the alternative '$\exists \mathfrak{F}(y = \hat{z}.\mathfrak{F}z \wedge \mathfrak{F}x)$' is not a well-formed formula (we remind the reader that the extension operator does not apply to formulas in which impredicative variables occur). As with H, the theory PE has extensions which are not sets. However, the situation is quite different for *finite* extensions. Let us say that an extension $\hat{x}.A(x)$ is finite if $\exists \mathfrak{F}(\mathrm{Fin}(\mathfrak{F}) \wedge \forall x(\mathfrak{F}x \leftrightarrow A(x)))$. This is a well-defined notion for extensions (thanks to Basic Law V). Now, the axiom of finite reducibility entails that finite extensions are sets. In other words, as long as we are only dealing with finite extensions, no new round of variables is necessary to gather finite extensions as sets. There are no finite protosets in PE, and this permits the development of a robust theory of finite sets.

The main goal of this paper was to convince the reader that arithmetic can be developed in a strict logicist manner by working on a (consistent) subsystem of Frege's original (inconsistent) theory. The restriction is simple and memorable: one is only allowed to take extensions of predicative concepts. Indeed, on this view of extensions, we believe that we have shown that Frege's own logicist program, when restricted to arithmetic, is successful.

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7 Appendix

In order to prove Lemma 5, let x and y be given such that $\mathfrak{R}^*\langle x, y\rangle$. Suppose that $\forall z(\mathfrak{R}\langle x, z\rangle \to z = x)$. Take $\mathfrak{F}$ such that $\forall w(\mathfrak{F}w \leftrightarrow w = x)$. Clearly, $\mathrm{Her}_{\mathfrak{R}}(\mathfrak{F})$ and $\mathfrak{F}x$. By $\mathfrak{R}^*\langle x, y\rangle$, we get $\mathfrak{F}y$, i.e., $y = x$.

To see Lemma 6, assume the antecedent condition and $\mathfrak{R}^*\langle x, y\rangle$. By impredicative comprehension, let $\mathfrak{F}$ be such that $\forall w(\mathfrak{F}w \leftrightarrow \mathfrak{R}^*\langle x, w\rangle \wedge (\mathfrak{R}^*\langle w, y\rangle \to \mathfrak{Q}^*\langle x, w\rangle))$. $\mathfrak{F}x$ is immediate. We claim that $\mathrm{Her}_{\mathfrak{R}}(\mathfrak{F})$. Suppose $\mathfrak{F}u$ and $\mathfrak{R}\langle u, v\rangle$. We get $\mathfrak{R}^*\langle x, u\rangle$ and, therefore, $\mathfrak{R}^*\langle x, v\rangle$. Now, suppose that $\mathfrak{R}^*\langle v, y\rangle$. We infer $\mathfrak{R}^*\langle u, y\rangle$ and, by $\mathfrak{F}u$, we also get $\mathfrak{Q}^*\langle x, u\rangle$. By the antecedent condition, it is clear that we have $\mathfrak{Q}\langle u, v\rangle$. Hence, $\mathfrak{Q}^*\langle x, v\rangle$. We have shown $\mathfrak{F}v$ and, therefore, proved $\mathrm{Her}_{\mathfrak{R}}(\mathfrak{F})$. Given that we have $\mathfrak{R}^*\langle x, y\rangle$, we may conclude $\mathfrak{F}y$ and, hence, $\mathfrak{Q}^*\langle x, y\rangle$.

Let us now prove Lemma 7.

For (1), assume $\mathrm{Func}(\mathfrak{R})$, $x \neq y$, $\mathfrak{R}^*\langle x, y\rangle$ and $\mathfrak{R}\langle x, z\rangle$. By impredicative comprehension, take $\mathfrak{F}$ such that $\forall w(\mathfrak{F}w \leftrightarrow (x = w \vee \mathfrak{R}^*\langle z, w\rangle))$. We claim that $\mathrm{Her}_{\mathfrak{R}}(\mathfrak{F})$. To see this, take u and v such that $\mathfrak{F}u$ and $\mathfrak{R}\langle u, v\rangle$. Then either $x = u$ or $\mathfrak{R}^*\langle z, u\rangle$. The latter case obviously entails $\mathfrak{R}^*\langle z, v\rangle$ and, therefore, $\mathfrak{F}v$. If $x = u$, then we have both $\mathfrak{R}\langle x, z\rangle$ and $\mathfrak{R}\langle x, v\rangle$. By the functionality of $\mathfrak{R}$, $v = z$. Hence $\mathfrak{R}^*\langle z, v\rangle$, as wished. Now that we have established $\mathrm{Her}_{\mathfrak{R}}(\mathfrak{F})$, note that $\mathfrak{F}x$ and (by assumption) $\mathfrak{R}^*\langle x, y\rangle$. Therefore, $\mathfrak{F}y$ and we are done.

To see (2), assume $\mathrm{Func}(\mathfrak{R})$, $\mathfrak{R}^*\langle z, x\rangle$ and $\mathfrak{R}^*\langle z, y\rangle$. By impredicative comprehension, take $\mathfrak{F}$ such that $\forall w(\mathfrak{F}w \leftrightarrow \mathfrak{R}^*\langle y, w\rangle \vee \mathfrak{R}^*\langle w, y\rangle)$. We claim that $\mathrm{Her}_{\mathfrak{R}}(\mathfrak{F})$. To see this, take u and v such that $\mathfrak{F}u$ and $\mathfrak{R}\langle u, v\rangle$. Either $\mathfrak{R}^*\langle y, u\rangle$ or $\mathfrak{R}^*\langle u, y\rangle$. The former case entails $\mathfrak{R}^*\langle y, v\rangle$. In the latter case, by (1) above, either $u = y$ or $\mathfrak{R}^*\langle v, y\rangle$. Note that $u = y$ together with $\mathfrak{R}\langle u, v\rangle$ entails $\mathfrak{R}^*\langle y, v\rangle$. In both cases, $\mathfrak{R}^*\langle y, v\rangle \vee \mathfrak{R}^*\langle v, y\rangle$. We have showed $\mathfrak{F}v$ and, hence, proved $\mathrm{Her}_{\mathfrak{R}}(\mathfrak{F})$. Clearly $\mathfrak{F}z$ and, by assumption, $\mathfrak{R}^*\langle z, x\rangle$. Hence $\mathfrak{F}x$, and we are done.

Finally, assume $\mathrm{Func}(\mathfrak{R})$, $x \neq y$, $\mathfrak{R}^*\langle x, y\rangle$, $\mathfrak{R}^*\langle y, x\rangle$ and $\mathfrak{R}^*\langle x, z\rangle$. By (2), either $\mathfrak{R}^*\langle z, y\rangle$ or $\mathfrak{R}^*\langle y, z\rangle$. In the former case, we get our conclusion $\mathfrak{R}^*\langle z, x\rangle$. We study the case $\mathfrak{R}^*\langle y, z\rangle$. By comprehension, take $\mathfrak{F}$ such that $\forall w(\mathfrak{F}w \leftrightarrow \mathfrak{R}^*\langle w, x\rangle)$. We claim that $\mathrm{Her}_{\mathfrak{R}}(\mathfrak{F})$. To see this, take u and v such that $\mathfrak{F}u$ and $\mathfrak{R}\langle u, v\rangle$. By (1), either $u = x$ or $\mathfrak{R}^*\langle v, x\rangle$ (i.e., $\mathfrak{F}v$). We only need to study the case $u = x$. We have $\mathfrak{R}\langle x, v\rangle$ and, since $\mathfrak{R}^*\langle x, y\rangle$ and $x \neq y$, we get $\mathfrak{R}^*\langle v, y\rangle$ by another application of (1). Given that $\mathfrak{R}^*\langle y, x\rangle$, by transitivity, we conclude $\mathfrak{R}^*\langle v, x\rangle$ (i.e., $\mathfrak{F}v$). We have just proved $\mathrm{Her}_{\mathfrak{R}}(\mathfrak{F})$. Since $\mathfrak{R}^*\langle x, z\rangle$ and $\mathfrak{F}x$, we conclude $\mathfrak{F}z$, i.e., $\mathfrak{R}^*\langle z, x\rangle$.

References

[1] F. Boccuni. Plural *Grundgezetze*. *Studia Logica*, 96(2):315–330, 2010.

[2] J. Burgess. *Fixing Frege*. Princeton University Press, 2005.

[3] J. Burgess and A. Hazen. Predicative logic and formal arithmetic. *Notre Dame Journal of Formal Logic*, 39:1–17, 1998.

[4] L. Cruz-Filipe and F. Ferreira. The finitistic consistency of Heck's predicative Fregean system. *Notre Dame Journal of Formal Logic*, 56(1):61–79, 2015.

[5] M. Dummett. *Frege. Philosophy of Mathematics.* Harvard University Press, 1991.

[6] F. Ferreira. Amending Frege's *Grundgesetze der Arithmetik*. *Synthese*, 147:3–19, 2005.

[7] G. Frege. *Begriffsschrift, a Formula Language, Modeled upon that of Arithmetic, for Pure Thought*. In J. van Heijenoort, editor, *From Frege to Gödel*, pages 5 – 82. Harvard University Press, 1967. A translation of Frege's *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, which appeared in German in 1879. Translated by J. van Heijenoort.

[8] G. Frege. *The Foundations of Arithmetic*. Northwestern University Press, 1980. A translation of Frege's *Die Grundlagen der Arithmetik*, which appeared in German in 1884. Translated by J. L. Austin.

[9] G. Frege. *Basic Laws of Arithmetic*. Oxford University Press, 2013. A translation of Frege's two volumes of the *Grundgesetze der Arithmetik*, which appeared in German in 1893 and 1903. Translated and edited by P. A. Ebert and M. Rossberg with a foreword of Crispin Wright.

[10] K. Gödel. What is Cantor's continuum problem. In P. Benacerraf and H. Putnam, editors, *Philosophy of Mathematics (selected readings)*, pages 470 – 485. Cambridge University Press, 2005. This is a revised and expanded version of a paper first published in 1947.

[11] B. Hale and C. Wright. Logicism in the twenty-first century. In S. Shapiro, editor, *The Oxford Handbook of Philosophy of Mathematics and Logic*, pages 166 – 202. Oxford University Press, 2005.

[12] R. Heck. The consistency of predicative fragments of Frege's *Grundgesetze der Arithmetik*. *History and Philosophy of Logic*, 17:209–220, 1996.

[13] R. Heck. The finite and the infinite in Frege's *Grundgesetze der Arithmetik*. In M. Schirn, editor, *The Philosophy of Mathematics Today*, pages 429–466. Clarendon Press Oxford, 1998.

[14] R. Heck. Ramified Frege arithmetic. *Journal of Philosophical Logic*, 40(6):715–735, 2011.

[15] W. O. Quine. *Set Theory and its Logic*. Harvard University Press, 1963.

[16] C. Parsons. Developing arithmetic in set theory without infinity: some historical remarks. *History and Philosophy of Logic*, 8:201–213, 1987.

[17] B. Russell. On some difficulties in the theory of transfinite numbers and order types. In D. Lackey, editor, *Essays in Analysis*, pages 135–164. George Allen and Unwin Ltd., 1973. This paper was first published in 1906.

[18] B. Russell. *Introduction to Mathematical Philosophy*. Dover Publications, 1993. First published in 1919.

[19] B. Russell and A. N. Whitehead. *Principia Mathematica*. Cambridge University Press (2nd edition), 1927.

[20] H. Wang. A formal system of logic. *The Journal of Symbolic Logic*, 15:25–32, 1950.