Binary models generated by their tally part

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Abstract

We introduce a class of models of the bounded arithmetic theory PV_n. These models, which are generated by their tally part, have a curious feature : they have end-extensions or satisfy $B\Sigma_n^b$ only in case they are closed under exponentiation. As an application, we show that if $I\Delta_0 + \neg exp \vdash B\Sigma_1$ then the polynomial hierarchy does not collapse.

This paper is concerned with bounded theories of arithmetic, following Buss (1986). Nonetheless, as opposed to Buss' classical setting – where the system of natural numbers forms the standard model – we work with theories that aim to describe the language $\{0,1\}^*$. Hence, we shall use the notation of Ferreira (1990a & 1990b).

To help the reader unfamiliar with the notation we briefly describe the (first-order) stringlanguage that we use. This stringlanguage consists of three constant symbols ϵ, 0 and 1, two binary function symbols $\frown$ (for *concatenation*, usually omitted) and $\times$, and a binary relation symbol $\subseteq$ (for *initial subwordness*). The interpretation of these symbols in the standard model $\{0,1\}^*$ is clear, except for the function symbol $\times$: $x \times y$ is the string x concatenated with itself length of y times. Given an element $e \in \{0,1\}^*$, we denote by $\bar{e}$ the closed term of the language obtained by

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concatenating (*via* the function symbol $\frown$) the constants 0 or 1 according to the order of the bits in e (for determinateness, we always associate $\frown$ to the left). We use the following abbreviations: $x \subseteq^* y$ (*subwordness* of x with respect to y) abbreviates $\exists z \subseteq y(z \frown x \subseteq y)$; $x \leq y$ (the length of x is less than or equal to the length of y) abbreviates $1 \times x \subseteq 1 \times y$; and $x \equiv y$ (x and y have the same length) abbreviates $x \leq y \wedge y \leq x$.

The theories studied in this paper are built upon fourteen basic open axioms, listed in Ferreira (1990a). These theories differ by the amount of induction permitted. For the convenience of the reader we provide a small lexicon that may be of some help :

$$\begin{aligned} \Sigma_n^b - \mathrm{NIA} \quad &- \quad S_2^n \\ \Sigma_n^b - \mathrm{IA} \quad &- \quad T_2^n \\ \Delta_n^b - \mathrm{NIA} \quad &- \quad \text{first-order version of } PV_n \end{aligned}$$

where $n \geq 1$. PV_n is a theory described in Krajíček et al. (1991). However, we should point out an idiosyncrasy concerning the way the theory $\Delta_n^b - \mathrm{NIA}$ is set up: its language has a function symbol for each (canonical description) of a function in $\Box_n^p$. In suitable contexts, we proceed *ad libitum* and write $f \in \Box_n^p$ instead of saying that f is a function symbol of the language of $\Delta_n^b - \mathrm{NIA}$ (in the framework of PV_n, this means that the function f of $\Box_n^p$ is given by a standard index and a standard polynomial bound).

An element u of a model M of $\Delta_1^b - \mathrm{NIA}$ is called *tally*, and we write *tally*(u), if it consists of a sequence of 1's : more formally, if $u = 1 \times u$. If M is a model of $\Delta_n^b - \mathrm{NIA}$ let,

$$\Gamma_n(M) = \{f(u) : \mathit{tally}(u) \ \& \ f \in \Box_n^p\}$$

Observe that $\Gamma_n(M)$ is a model of $\Delta_n^b - \mathrm{NIA}$, because $\Delta_n^b - \mathrm{NIA}$ is a universal theory ; moreover, it is the smallest such model having the same tally part as M. It is clear that $\Gamma_n(M) \prec_{\Delta_n^b} M$ and that $\Gamma_n(\Gamma_n(M)) = \Gamma_n(M)$.

Definition. *Let M be a model of $\Delta_n^b - NIA$. We say that M is a Δ_n^b-*thin *model if $M = \Gamma_n(M)$.*

The models just defined share some curious features. In order to describe these features we introduce (remind of) some concepts. Given M and N two structures for the stringlanguage, with $M \subseteq N$, we say that N is an *end-extension* of M if whenever $a \in M$, $b \in N$ and $b \leq a$ then $b \in M$ (that is, N has no new elements having length which is smaller than or equal to the length of an element of M). The $B^t\Delta_n^b$-*collection* scheme consists of the following :

$$\forall x \leq a \; \exists u \; (tally(u) \wedge A(x,u)) \rightarrow \exists v \; (tally(v) \wedge \forall x \leq a \; \exists u \subseteq v \; A(x,u))$$

where A is a Δ_n^b-formula and v is a new variable (parameters are allowed).

Given an element a of a model M of Δ_1^b-NIA, a perspicuous way of saying that "2^a exists" is to state that there is an element $b \in M$ such that $\forall x \leq a \; (x \subseteq^* b)$. An alternative, and perhaps more ordinary way of saying that "2^a exists", involves the so-called *binary length function.* This function, denoted by $\ell h(\cdot)$, has the following recursive definition : $\ell h(\epsilon) = \epsilon$, $\ell h(x0) = \ell h(x1) = S(\ell h(x))$, where S is the *successor function* associated with the canonical linear order $<_\ell$ (see the appendix). Bearing this in mind, we can say that "2^a exists" if there is a tally element u in M such that $\ell h(u) = a$. These two definitions, albeit of a different character (observe that the first one poses an "absoluteness" question, while the second does not), coincide in models of Δ_1^b − NIA :

Lemma.

$$\Delta_1^b - NIA \vdash \exists z \; \forall x \leq a \; (x \subseteq^* z) \leftrightarrow \exists u \; (tally(u) \wedge \ell h(u) = a). \qquad \square$$

We postpone an outline of the proof of this lemma to the appendix. The axiom *exp* is the statement $\forall x$"2^x exists". It is immediate from the definitions that models of Δ_n^b − NIA + *exp* are Δ_n^b-thin.

Lemma (Main properties of Δ_n^b-thin models).

The following properties hold in the category of models of Δ_n^b − NIA:

1 – The class of Δ_n^b-thin models is elementary.

2 – In Δ_n^b-thin models not satisfying *exp*, the $B^t\Delta_n^b$-collection scheme does not hold.

3 – Δ_n^b-thin models not satisfying *exp* do not have proper end-extensions which are models of $\Delta_n^b - \mathrm{NIA}$.

4 – There are models of $\Delta_n^b - \mathrm{NIA}$ which are not Δ_n^b-thin.

Proof. 1 – The concept of an oracle Turing machine can be "smoothly" formalized in $\Delta_1^b - \mathrm{NIA}$ by a Δ_1^b-formula $T(x)$. Moreover, if $A \in \Sigma_n^b$ then the ternary relation

$\{e\}^A(x;u)\downarrow \; := T(e) \wedge tally(u) \wedge$ "the Turing machine computation of $\{e\}^A(x)$ comes to the halting state in less than length of u steps"

is defined by a Δ_{n+1}^b-formula. And the following ternary function, being in $\square_{n+1}^p$, is given by a ternary function symbol of the language of $\Delta_{n+1}^b - \mathrm{NIA}$:

$$\{e\}^A(x;u) = \begin{cases} \{e\}^A(x) & \text{if } \{e\}^A(x;u)\downarrow \\ \epsilon & \text{otherwise} \end{cases}$$

Let K_n be a natural Σ_n^b-complete set (put $K_0 = \phi$). We shall use the following fact :

(∗) Given any unary function symbol f of the language of $\Delta_n^b - \mathrm{NIA}$, there is an element $e_f \in \{0,1\}^*$ and a term t_f, called a natural time bound for f, such that the theory $\Delta_n^b - \mathrm{NIA}$ proves :

$$T(\bar{e_f}) \wedge \forall x[tally(t_f(x)) \wedge \{\bar{e_f}\}^{K_{n-1}}(x;t_f(x))\downarrow \wedge$$
$$\wedge\, f(x) = \{\bar{e_f}\}^{K_{n-1}}(x;t_f(x))]$$

Now we can show that the class of Δ_n^b-thin models is elementary. Consider the following predicate and function symbols of the language of $\Delta_n^b - \mathrm{NIA}$:

$$Q_n(x) := tally(x) \wedge \{\ell h((x)_1)\}^{K_{n-1}}((x)_2;(x)_3)\downarrow$$

$$thin_n(x) := \begin{cases} \{\ell h((x)_1)\}^{K_{n-1}}((x)_2;(x)_3) & \text{if } Q_n(x) \\ \epsilon & \text{otherwise} \end{cases}$$

Here $\lambda x.(x)_1$, $\lambda x.(x)_2$ and $\lambda x.(x)_3$ are the decoding functions of a tally word viewed as a triple of tally words (observe that the set of tally words of a model of $\Delta_1^b - \mathrm{NIA}$ is naturally a model of $I\Delta_0$; hence, we can use twice our favorite Δ_0-pairing function to code a triple of tally words u_1, u_2, u_3 by a tally word $\langle u_1, u_2, u_3 \rangle$). Note that $thin_n$ defines, indeed, a function in $\square_n^p$.

We claim that a model M of $\Delta_n^b - \mathrm{NIA}$ is Δ_n^b-thin if, and only if,

$$M \models \forall x \, \exists u (Q_n(u) \wedge thin_n(u) = x).$$

The "sufficient" direction is an immediate consequence of the definition of a Δ_n^b-thin model. Conversely, suppose that M is Δ_n^b-thin and let a be an arbitrary element of M. By the thinness of M there is a function symbol f of the language of $\Delta_n^b - \mathrm{NIA}$ and a tally element u such that $M \models a = f(u)$. Using $(*)$, pick $e_f \in \{0,1\}^*$ the Gdel number of the (natural) Turing machine associated with the function symbol f and let t_f be its natural time bound. Then,

$$M \models f(u) = \{\bar{e_f}\}(u; t_f(u)).$$

Now, take $o \in \{1\}^*$ with $\ell h(o) = e_f$. Clearly,

$$M \models Q_n(\langle \bar{o}, u, t_f(u) \rangle) \wedge thin_n(\langle \bar{o}, u, t_f(u) \rangle) = a.$$

2 – Let M be a Δ_n^b-thin model in which the $B^t\Delta_n^b$-collection scheme holds. We claim that M satisfies exp. Let a be an arbitrary element of M. By thinness and the proof of 1,

$$M \models \forall x \leq a \, \exists u (Q_n(u) \wedge thin_n(u) = x).$$

Applying $B^t\Delta_n^b$-collection we get

$$M \models \forall x \leq a \, \exists u \subseteq v (Q_n(u) \wedge thin_n(u) = x)$$

for a certain tally element v of M. Let b be the concatenation of all the elements $thin_n(u)$, with $u \subseteq v$: the notation is $b = \sum_{u \subseteq v} thin_n(u)$ – note that this makes sense, because $\lambda v. \sum_{u \subseteq v} thin_n(u)$

is in $\square_n^p$. Clearly,

$$M \models \forall x \leq a(x \subseteq^* b).$$

Hence "2^a exists".

3 – The proof of this property uses an underspill argument in the manner of Paris & Kirby (1978). However, a direct underspill argument purporting to show the implication

$$M \subset_e N \models \Delta_n^b - \mathrm{NIA} \implies M \models B^t\Delta_n^b$$

fails. The reason stems from the fact that we (do not seem to have) induction on notation for Π_n^b-formulae in models of Δ_n^b – NIA. Hence, a preliminary maneuver is needed. Consider M a Δ_n^b-thin model with a proper end-extension N satisfying Δ_n^b – NIA. Let a be an arbitrary element of M. The following is true,

$$b \in M \Rightarrow N \models b \subseteq^* \sum_{u \subseteq v} thin_n(u)$$

for any tally element $M < v \in N$.

Now, using the fact that N is an end-extension of M we get,

$$N \models \forall x \subseteq^* \sum_{u \subseteq v} thin_n(u)\,[x \leq a \rightarrow (x0 \subseteq^* \sum_{u \subseteq v} thin_n(u) \wedge \\ \wedge\ x1 \subseteq^* \sum_{u \subseteq v} thin_n(u))]$$

The formula above, following the symbol "$\models$", is Δ_n^b. Hence, by an underspill argument, there is a tally element u_0 in M such that,

$$N \models \forall x \subseteq^* \sum_{u \subseteq u_0} thin(u)\,[x \leq a \rightarrow (x0 \subseteq^* \sum_{u \subseteq u_0} thin_n(u) \wedge \\ \wedge\ x1 \subseteq^* \sum_{u \subseteq u_0} thin_n(u))]$$

By absoluteness, this also holds in M. It is easy to see, by induction on notation on x, that $M \models x \leq a \rightarrow x \subseteq^* \sum_{u \subseteq u_0} thin_n(u)$. Therefore, "$2^a$ exists".

4 – By the previous result, any model of $\Delta_n^b - \mathrm{NIA} + \neg exp$ with an end-extension to a model of $\Delta_n^b - \mathrm{NIA}$ will do. □

The general form of the collection scheme is as follows:

$$\forall x \leq a\ \exists y\ A(x,y) \rightarrow \exists z\ \forall x \leq a\ \exists y \leq z\ A(x,y)$$

where z is a new variable (parameters are allowed). If we only permit bounded formulae A, we have the *bounded collection* scheme, a.k.a. the $B\Sigma_1$-scheme. If the formulae A are further restricted to the n$^{\text{th}}$-level of the hierarchy of bounded formulae, we have the $B\Sigma_n^b$-scheme. Note that $B\Sigma_n^b \Rightarrow B^t\Delta_n^b$.

Corollary. $PV_n + \neg exp \nvdash B\Sigma_n^b$. □

Corollary. *If S_2 collapses then $S_2 + \neg exp \nvdash B\Sigma_1$. In particular, if $I\Delta_0$ is finitely axiomatizable then $I\Delta_0 + \neg exp \nvdash B\Sigma_1$.* □

There is a proof (but not a statement) of the latter result in Paris, Wilkie & Woods (1988). As a matter of fact, we have more.

Lemma. *Let $M \subseteq N$ be models of Δ_1^b-NIA and suppose that $\{x \in N : tally(x)\} \subseteq M$. If M satisfies* exp *then so does N.*

Proof. Assume that M satisfies exp and take a an arbitrary element of N. By hypothesis, $1 \times a \in M$. Hence, there is $u \in M \subseteq N$ such that $N \models tally(u) \wedge lh(u) = a$. We use the following property:

($) $$\Delta_1^b - \mathrm{NIA} \vdash x \leq_\ell \ell h(u) \rightarrow \exists v \subseteq u\ (x = \ell h(v)).$$

Due to the fact that $a \leq_\ell 1 \times a$, this entails that $N \models \exists v \subseteq u(lh(v) = a)$. Hence, $N \models$ "2^a exists". The property ($), and other properties that we will use in the appendix, are easy to prove within Δ_1^b-NIA: the details can be found in Ferreira (1988). □

Theorem. *If $\Sigma_n^p = \Delta_n^p$ then $S_2 + \neg exp \nvdash B\Sigma_n^b$.*

Proof. Let us work under the assumption that $\Sigma_n^p = \Delta_n^p$. Take N a non-standard model of the true theory of $\{0,1\}^*$ and let M be an initial segment of N closed under multiplication of lengths, but in which exp fails. Clearly M is a model of S_2 and, hence, a model of $\Delta_n^b - \mathrm{NIA}$. We claim that bounded formulae are absolute between $\Gamma_n(M)$ and M. This is due to the collapse of the polynomial hierarchy at the n$^{\mathrm{th}}$-level : in effect, this collapse entails that every bounded existential assumption can be witnessed via a $\Box_n^p$-function on the parameters of the formula. Hence, $\Gamma_n(M)$ is a model of S_2 (and, by the previous lemma, of $\neg exp$). By main property 2 of the Δ_n^b-thin models, the $B^t\Delta_n^b$-collection scheme does not hold in $\Gamma_n(M)$. In particular, $\Gamma_n(M) \not\models B\Sigma_n^b$. $\Box$

Corollary. *If the polynomial hierarchy collapses then $I\Delta_0 + \neg exp \nvdash B\Sigma_1$.* $\Box$

Wilkie & Paris (1989) asked whether $I\Delta_0 + \neg exp \vdash B\Sigma_1$. The previous corollary shows that a positive answer to this question is at least as difficult as proving that the polynomial hierarchy (a.k.a. the Meyer-Stockmeyer hierarchy) does not collapse. We do not know if the hypothesis of the previous corollary can be replaced by the seemingly more natural "the linear hierarchy collapses". A referee of this paper wrote that our results "are (for me) somewhat surprising". In fact, the previous theorem and corollary *seem* to run against a dictum in bounded arithmetic, namely that *more proofs* $\Rightarrow$ *more algorithms.* (The beautiful and seminal result of Krajíček et al. (1991), saying that if the theory S_2 collapses (more proofs) then so does the polynomial hierarchy (more algorithms), is the most important manifestation of this dictum.) We consider the above corollary as evidence that the theory $I\Delta_0 + \neg exp$ does not prove the scheme of collection for bounded formulae.

Let us finish with the following independence result for relativized theories :

Proposition. $S_2(\alpha) + \neg exp \nvdash B\Sigma_1(\alpha)$.

Proof. Take N a non-standard model of the true theory of $\{0,1\}^*$, with an extra unary predicate symbol α interpreted by a (fixed) PSPACE complete set. As in the proof of the previous theorem, let M be an initial segment of N closed under multiplication of lengths, but in which exp fails. Clearly M is a model of $S_2(\alpha)$. Consider,

$$\Gamma_1^\alpha(M) = \{f(u) : tally(u) \ \& \ f \in \square_1^p(\alpha)\}.$$

It is easy to convince ourselves that $\Gamma_1^\alpha(M) \models S_2(\alpha)$. Moreover, by the same argument that proves the main property 2 of thinness, $\Gamma_1^\alpha(M) \not\models B\Sigma_1^b(\alpha)$. $\square$

Appendix

The successor function S is defined by $S(\epsilon) = 0$, $S(x0) = x1$ and $S(x1) = S(x)0$. The $<_\ell$ canonical linear order of $\{0,1\}^*$ is as follows :

$$x <_\ell y \Leftrightarrow (x \leq y \wedge x \not\equiv y) \vee (x \equiv y \wedge \exists z \subseteq x(z0 \subseteq x \wedge z1 \subseteq y)).$$

One of the implications of the equivalence between the two notions of "2^a exists" results from the fact that if $\ell h(u) = a$, then $\forall x \leq a \ \left(x \subseteq^* \sum_{v \subseteq u}(\ell h(v) \frown 0 \frown \ell h(v) \frown 1)\right)$. Let us explain this bound. Take $x \leq a$, with $x \neq \epsilon$, and denote by x^- the string x without its last bit. It is clear that $x^- <_\ell a$, and hence, by property ($) in the main text, there is $v \subseteq u$ with $x^- = \ell h(v)$. We conclude that either $x = \ell h(v) \frown 0$ or $x = \ell h(v) \frown 1$. This justifies the bound.

The converse implication is more involved and mainly consists of some *programming*. Suppose that $\forall x \leq a \ (x \subseteq^* b)$. We consider four tasks :

<u>1st task</u>. Fix a tally element $\epsilon \neq \ell \leq a$. Let $c = \sum_{x \subseteq b} \sum_{y \subseteq x} d_\ell(y,x)$, where $y \frown d_\ell(y,x) = x$ if there is such $d_\ell(y,x)$ with the same length as ℓ (otherwise, $d_\ell(y,x) = \epsilon$).

<u>Comments</u>. c is a concatenation of all subwords of b of length equal to that of ℓ : hence $c \equiv \ell \times t$,

for a certain tally t. Moreover, it is clear that $t \leq b \times b$. Notice that all this is true in models of $\Delta_1^b - \mathrm{NIA}$.

2nd task. The theory $\Delta_1^b - \mathrm{NIA}$ proves the following

$$\forall u\, \forall x\, \forall y\, (tally(u) \wedge x \equiv \ell \times u \rightarrow \exists v \subseteq u (\forall w \subset v\, d_\ell(x|_{\ell \times w}, x|_{\ell \times w1}) <_\ell y \wedge$$

$$\wedge\, (v \neq u \rightarrow y \leq_\ell d_\ell(x|_{\ell \times v}, x|_{\ell \times v1}))))$$

where $w \subset v$ abbreviates $(w \subseteq v \wedge w \neq v)$ and $x|_y$ is the truncation of x at the length of y. By Herbrand analysis there is a function symbol $q(u,x,y)$ of the language of $\Delta_1^b - \mathrm{NIA}$ witnessing v in the above formula.

Comments. In case the length of x is a multiple of the length of ℓ (that is, $x \equiv \ell \times u$), x can be considered a sequence of words, each of length ℓ ; $q(u,x,y)$ picks the first place of this sequence in which the corresponding word is lexicographically greater than or equal to y; if there is no such place, it picks u.

3rd task. Consider the $\square_1^p$-function,

$$p(u,x,y) = \begin{cases} x & \text{if} \quad y \subseteq d(x|_{\ell \times q(u,x,y)}, x) \\ x|_{\ell \times q(u,x,y)} \frown y \frown d(x|_{\ell \times q(u,x,y)}, x) & \text{otherwise} \end{cases}$$

where $d(y,x)$ is such that $y \frown d(y,x) = x$, if there is such $d(y,x)$; otherwise $d(y,x) = \epsilon$.

Comments. $p(u,x,y)$ is the result of inserting the word y at the $q(u,x,y)^{\text{th}}$-place of the sequence given by x, provided y is not there.

4th task. If the length of x is a multiple of the length of ℓ, the quotient of these lengths, given by the length of u, is determined by x (and by the fixed ℓ). Hence, we may consider p and q as functions of only two arguments x and y. Define, by bounded recursion on notation,

$$r(\epsilon) = \epsilon$$

$$r(w0) = r(w1) = \begin{cases} p\left(r(w), d\left(c|_{\ell\times w}, c|_{\ell\times w1}\right)\right) & \text{if} \quad w1 \leq t \\ r(w) & \text{otherwise} \end{cases}$$

This is a legal definition by bounded recursion, because $\ell \times t$ is a bound.

<u>Comments</u>. $r(t)$ is the result of rearranging the sequence of words in c according to the lexicographic order $<_\ell$. Repetitions are omitted. It is easy to show that $\Delta_1^b - \text{NIA}$ proves this to be the case. Notice that, in particular, the last element of the sequence $r(t)$ is the word ℓ.

We have shown that for each tally ℓ, with $\ell \leq a$, there is an element e_ℓ, with $e_\ell \leq \ell \times b \times b$, which consists of the sequence of all elements of length ℓ according to the lexicographic order.

We claim that $\forall \ell \subseteq 1 \times a \;\; \exists z \subseteq \ell \times b \times b \; (\ell h(z) = \ell)$. In particular, there is z with $\ell h(z) = 1 \times a$. Due to ($), this solves the problem. The proof is by induction on ℓ. The base case $\ell = \epsilon$ is immediate. Suppose $\ell 1 \subseteq 1 \times a$ and assume, by induction hypothesis, that there is $z \subseteq \ell \times b \times b$ such that $\ell h(z) = \ell$. Take $e_{\ell 1}$ as above. By construction $e_{\ell 1} \equiv \ell 1 \times h$, for some tally $h \subseteq b \times b$. It is easy to show, by induction on g, that

$$\ell h(z \frown g \frown 1) = d\left(e_{\ell 1}|_{\ell 1 \times g}, e_{\ell 1}|_{\ell 1 \times g1}\right),$$

for all $g \subset h$. In particular, $\ell h(z \frown h) = \ell 1$. Finally, observe that

$$z \frown h \subseteq (\ell \times b \times b) \frown h \subseteq (\ell \times b \times b) \frown (1 \times b \times b) = \ell 1 \times b \times b.$$

We are done.

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