

# Comments on predicative logic

Fernando Ferreira\*  
Departamento de Matemática  
Faculdade de Ciências  
Universidade de Lisboa  
e-mail: `ferferr@cii.fc.ul.pt`

April 12, 2005

## 1 A technical note

The syntax of intuitionistic propositional second-order logic  $\text{PSOL}^i$  consists of a set of propositional constants  $P, Q, R$ , etc, a denumerable set of propositional variables  $F, G, H$ , etc, together with the primitive logical signs of the conditional and the universal propositional quantifier, and punctuation signs (parenthesis). The formulas of  $\text{PSOL}^i$  are the smallest class of expressions of the language such that:

1. Propositional constants and variables are formulas.
2. If  $A$  and  $B$  is a formula, then  $(A \rightarrow B)$  is a formula.
3. If  $A$  is a formula and  $F$  is a propositional variable, then  $\forall F(A)$  is a formula.

We sometimes write  $\forall F.A$  instead of  $\forall F(A)$ . We also follow standard usage regarding parentheses. A *predicative formula* is a formula with no second-order quantifiers. They are, in the present setting, the quantifier-free formulas. The logic of  $\text{PSOL}^i$  is *intuitionistic logic*, which we formulate in the

---

\*This work was done while visiting the Department of Philosophy of Harvard University in the Spring term of 2004. We are grateful to Richard Heck for making this visit possible, and to the Fulbright Scholar Program and FLAD (Fundação Luso-Americana para o Desenvolvimento) for the financial support. A sabbatical leave from Universidade de Lisboa freed us from teaching duties. We must also acknowledge the support of our research center CMAF (POCTI/FCT).

natural deduction calculus, with introduction and elimination rules for the conditional and the universal propositional quantifier. We need not comment on the rules for the conditional, nor on the introduction rule for the propositional quantifier. They are as usual (see [3]). If the range allowed for the  $\forall$ -elimination rule is unrestricted with respect to formulas (discounting clashing of variables), we get *impredicative* PSOL<sup>i</sup>. On the contrary, if the range is restricted to quantifier-free formulas only, we get *predicative* PSOL<sup>i</sup>. According to a 1965 celebrated work of Dag Prawitz (cf. [3]), in the impredicative case the other logical connectives can be defined in terms of the conditional and the universal propositional quantifier:

$$\begin{aligned} \neg A &=_{df} A \rightarrow \forall F.F \\ A \wedge B &=_{df} \forall F((A \rightarrow (B \rightarrow F)) \rightarrow F) \\ A \vee B &=_{df} \forall F((A \rightarrow F) \rightarrow ((B \rightarrow F) \rightarrow F)) \\ \exists G.A &=_{df} \forall F(\forall G(A \rightarrow F) \rightarrow F). \end{aligned}$$

where  $F$  is a new propositional variable. *Prima facie*, in order to prove that the above defined connectives obey the usual introduction and elimination rules of the natural deduction calculus, it is required that no restrictions are in place when instantiating generalizations. That notwithstanding, the very same definitions *do* work in the predicative case.

**Theorem 1.** *Predicative PSOL<sup>i</sup> proves the usual introduction and elimination rules of the natural deduction calculus for the connectives defined above.*

**Observation 1.** *In the  $\exists$ -introduction rule, where  $\exists G.A$  follows from  $A_P^G$  ( $P$  free for  $G$  in  $A$ ), the formula  $P$  must be quantifier-free.*

*Proof.* It is clearly sufficient to prove that, for *any* formula  $C$  of the language of PSOL<sup>i</sup>, one has:

1.  $\neg A \rightarrow (A \rightarrow C)$
2.  $A \wedge B \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow C)$
3.  $A \vee B \rightarrow ((A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow C))$
4.  $\exists G.A \rightarrow (\forall G(A \rightarrow C) \rightarrow C)$

(Of course, in 4 above,  $G$  must not occur in  $C$ .) The proof is by induction on the complexity of  $C$ . We only discuss part 3. Assume  $A \vee B$ . Suppose that  $C$  is  $C_1 \rightarrow C_2$ , and let  $A \rightarrow (C_1 \rightarrow C_2)$  and  $B \rightarrow (C_1 \rightarrow C_2)$ . Then,

$C_1 \rightarrow (A \rightarrow C_2)$  and  $C_1 \rightarrow (B \rightarrow C_2)$ . By induction hypothesis applied to  $C_2$ , we can conclude that  $C_1 \rightarrow C_2$ , i.e.,  $C$ . Now, suppose that  $C$  is  $\forall G.D$ , and let  $A \rightarrow \forall G.D$  and  $B \rightarrow \forall G.D$ . Then  $\forall G(A \rightarrow D)$  and  $\forall G(B \rightarrow D)$ , changing the bound variable if necessary. By induction hypothesis applied to  $D$ , we can conclude that  $\forall G.D$ , i.e.  $C$ . □

## 2 Commentary

1. The predicativist inclined philosopher takes for granted a linguistic base whose semantics is unproblematically bivalent. In our setting, this corresponds to a bivalent semantics for the propositional constants. Under these circumstances, the predicativist should accept the stability law  $\forall F(\neg\neg F \rightarrow F)$  on the grounds that the *intended* range of  $F$  is constituted by propositional sentences built up from bivalent atomic propositions using the conditional only. Let us try to articulate this idea better. Assume that the propositional constants come in *polar opposite* pairs,  $P, \bar{P}$ , where each element of the pair contradicts the other and, moreover, the components of each pair exhaust the logical space of possibilities. We are led to the following laws:

$$\text{a1. } \forall G(P \rightarrow (\bar{P} \rightarrow G))$$

$$\text{b1. } \forall G((P \rightarrow G) \rightarrow ((\bar{P} \rightarrow G) \rightarrow G))$$

If we also have *conjunction* as a primitive connective of the language (an innocent addition), we can extend the notion of a polar opposite to all quantifier-free *sentences*:  $\overline{A \rightarrow B}$  is  $A \wedge \bar{B}$  and  $\overline{A \wedge B}$  is  $A \rightarrow \bar{B}$ . It is understood that, for propositional constants  $P$ , the polar opposite of  $\bar{P}$  is  $P$  (and vice versa). It is now easy to prove by induction on the complexity of  $A$  that,

$$\text{a2. } \forall G(A \rightarrow (\bar{A} \rightarrow G))$$

$$\text{b2. } \forall G((A \rightarrow G) \rightarrow ((\bar{A} \rightarrow G) \rightarrow G))$$

for quantifier-free *sentences*  $A$ .

In predicative  $\text{PSOL}^i$ , the intended range of the quantifiers consists of the quantifier-free *sentences* (formally, of course, the universal instantiation rule must admit quantifier-free *formulas*, with their free variables envisaged as parameters for quantifier-free sentences). Therefore, in the light of the

above discussion, it is natural to work also with polar opposite pairs  $F$ ,  $\overline{F}$  of propositional *variables*, and uphold the conditions:

$$\text{a3. } \forall F \forall G (F \rightarrow (\overline{F} \rightarrow G))$$

$$\text{b3. } \forall F \forall G ((F \rightarrow G) \rightarrow ((\overline{F} \rightarrow G) \rightarrow G)),$$

with the understanding that the polar opposite of  $\overline{F}$  is  $F$  (and vice-versa). Syntactically, only propositional constants and variables can have an overstrike, and then only one. The range of a  $\forall$ -elimination rule is formed by the quantifier-free formulas and, when they come out with overstrikes, they are evaluated in the manner of the quantifier-free sentences above. The conditions (a3) and (b3) entail that  $\forall(\overline{F} \rightarrow \neg F)$  and  $\forall F(\neg F \rightarrow \overline{F})$  are derivable. The stability *law*  $\forall F(\neg\neg F \rightarrow F)$  follows easily now. This law is the base case for a proof by induction on the complexity of formulas that the conditional  $\neg\neg A \rightarrow A$  is derivable for *every* formula  $A$  of the language. This follows from well-known results in proof theory, since the connectives of our language are all “negative”: ‘ $\rightarrow$ ’, ‘ $\wedge$ ’ and ‘ $\forall$ ’. With the stability *scheme* in place we get, in effect, *classical logic*. This is the punch line.

2. The philosophical cast of predicativism is nonrealistic, in the sense that the second-order entities do not subsist independently of us but are, in a way, legitimized by our being able to present (define) them. Warren Goldfard voices in [2] the concern that this non-realism poses a difficulty for the legitimacy of the use of classical logic. The previous comment can be seen as a response to such an objection. Of course, in this paper we restricted ourselves to *propositional* second-order logic, whereas the real action of predicativism is in settings with an underlying *first-order* domain, like in the traditional predicativist position in the foundation of mathematics associated with the names of Henri Poincaré, Hermann Weyl, Solomon Feferman *et al.* In order to encompass these cases, the discussion above would have to be extended to include first-order formulas, and that brings problems of its own. We think that these problems can be met, but we leave them at this juncture.

3. In §10.1 of [1], Jean-Yves Girard *et al.* say that the elimination rules for *falsum*, disjunction and existential quantification are *very bad*. They add that “what is catastrophic about them is the presence of a parasitic formula  $C$  which has no structural link with the formula which is being eliminated.” In fact, (e.g.) the rule of disjunction elimination permits the inference of  $C$  from  $A \vee B$  if one has deductions of  $C$  from  $A$  and of  $C$  from  $B$ . They say that “the adoption of these rules (and let us say that there is currently

no alternative) contradicts the idea that natural deductions are the “real objects” behind the proofs.” Indeed, certain “commuting” conversions, quite apart from the natural conversions associated with the connectives (removal of *local peaks*), must be made available in order to have a decent theory of normalization of derivations.

Girard *et al.* point out that there is currently no alternative to the *bad* rules, although they hasten to say that the fragment of the calculus with *falsum*, disjunction and existential quantification “is [not] etched on tablets of stone.” Here is a preliminary alternative: embed the intuitionistic propositional calculus into *predicative* PSOL<sup>i</sup>, where there are no *bad* rules. This is a nice alternative, but we discuss another one. Namely, restrict the range of the  $\forall$ -elimination rule to *atomic* formulas. For lack of a better name, let us call this restricted calculus *atomic* PSOL<sup>i</sup>. Observe that Theorem 1 still goes through with *atomic* PSOL<sup>i</sup> (instead of *predicative* PSOL<sup>i</sup>). The notion of a subformula is defined as usual, with the extra clause that says that the subformulas of a quantification  $\forall F.A$  are the quantification itself together with the subformulas of formulas of the form  $A_P^F$ , where  $P$  is either a propositional variable (free for  $F$  in  $A$ ) or a propositional *constant*.

A normalization theorem for *atomic* PSOL<sup>i</sup> can be easily proved (in the manner of Theorem 2 of Section V of [3]). We also make the usual conventions about Eigenvariables of  $\forall$ -introduction rules, namely that an Eigenvariable in a derivation is the Eigenvariable of at most one inference, and that the Eigenvariable occurs only above the conclusion of the inference. From this we have the following result (after [1]):

**Theorem 2.** *Let  $\Delta$  be a normal derivation in atomic PSOL<sup>i</sup>. Then*

- i) every formula in  $\Delta$  is a subformula of the conclusion or of a (undischarged) hypothesis of  $\Delta$  (Subformula Property);*
- ii) if  $\Delta$  ends in an elimination rule, it has a principal branch, i.e. a sequence of formulas  $A_0, A_1, \dots, A_n$  such that*
  - $A_0$  is an (undischarged) hypothesis;*
  - $A_n$  is the conclusion;*
  - for  $i = 0, \dots, n - 1$ ,  $A_i$  is the principal premise of an elimination rule whose conclusion is  $A_{i+1}$ .*

*In particular,  $A_n$  is a subformula of  $A_0$ .*

*Proof.* The proof is by induction on the number of inferences in  $\Delta$ . The base for induction is when  $\Delta$  consists solely of a hypothesis, in which case there is nothing to do. If  $\Delta$  ends in an introduction rule, say in

$$\frac{\vdots}{\forall G.A} \forall i$$

just apply the induction hypothesis to the derivation that ends in  $A$ . If  $\Delta$  ends in an elimination rule, say in

$$\frac{\vdots}{A_P^G} \forall e$$

it is not possible that the penultimate inference rule is an introduction, since otherwise the derivation wouldn't be normal. Therefore, it is an elimination rule and we now apply the induction hypothesis again.  $\square$

As an application of the above discussion, we prove the following disjunction property. Let  $A$  and  $B$  be the translations into the language of  $\text{PSOL}^i$  of given formulas of the intuitionistic propositional calculus (according to the embedding discussed above). If  $A \vee B$  is derivable in *atomic*  $\text{PSOL}^i$  then so is either  $A$  or  $B$  (or both). Take a normal derivation of  $A \vee B$ , i.e., of

$$\forall G((A \rightarrow G) \rightarrow ((B \rightarrow G) \rightarrow G)).$$

By the last remark of part (ii) of Theorem 2, the last rule of this derivation must be an introduction rule. Therefore, the normal derivation has the form:

$$\frac{\vdots}{(A \rightarrow G) \rightarrow ((B \rightarrow G) \rightarrow G)} \forall i$$

By (ii) of Theorem 2 again, the penultimate inference rule must also be an introduction. We thus have:

$$\frac{[A \rightarrow G]_*}{\frac{\vdots}{(B \rightarrow G) \rightarrow G} \rightarrow i^*} \forall i$$

where  $A \rightarrow G$  may be absent as a discharged assumption. We now argue that the last rule of the normal derivation

$$\begin{array}{c} A \rightarrow G \\ \vdots \\ (B \rightarrow G) \rightarrow G \end{array}$$

can't be an elimination rule. If it were, take its principal branch. Its top formula is an undischarged assumption, viz.  $A \rightarrow G$ . This assumption must be the principal premise of an elimination rule. Therefore, this rule must be of the form:

$$\frac{A \rightarrow G \quad \begin{array}{c} \vdots \\ A \end{array}}{G} \rightarrow e$$

where ' $A \rightarrow G$ ' on the left hand side is the top formula of the principal branch. This is impossible because the derivation would have to terminate at that point (i.e., at  $G$ ).

We argued that the last rule of a (normal) derivation of  $(B \rightarrow G) \rightarrow G$  from  $A \rightarrow G$  is an introduction rule. Hence, we have a normal derivation of the form:

$$\begin{array}{ccc} A \rightarrow G & & B \rightarrow G \\ \ddots & & \ddots \\ & G & \end{array}$$

The last line of this derivation is *per force* an elimination rule. Consider the principal branch of this derivation. Its top is either the formula  $A \rightarrow G$  or the formula  $B \rightarrow G$ . Suppose it is  $A \rightarrow G$  (the other case is similar). Then, it must be the principal premise of an elimination rule. Therefore, the above derivation must have the form:

$$\frac{A \rightarrow G \quad \begin{array}{c} \ddots \\ A \end{array}}{G} \rightarrow e$$

where the ' $A \rightarrow G$ ' on the left side is the top formula of the principal branch, which consists just of  $A \rightarrow G$  followed by  $G$ .

We argue now that substituting  $G$  for any formula  $C$  on the top right side derivation above yields a derivation of  $A$  from the premises  $A \rightarrow C$  and  $B \rightarrow C$  (one or both of which may be absent). If both  $A \rightarrow G$  and  $B \rightarrow G$

are absent, then  $A$  is derivable without premises (as wanted). Otherwise, the only thing to check is that all rules in the derivation involving formulas where  $G$  appears can be mimicked by rules where  $C$  appears instead. This is *obviously* true for the implication rules, as well as for the quantification rules *provided* that in a  $\forall$ -introduction  $G$  is not the Eigenvariable and that the conclusion of a  $\forall$ -elimination is not obtained by substituting the bound variable of the universal quantifier by the atomic variable  $G$ . Well, a  $\forall$ -introduction rule with Eigenvariable  $G$  never occurs because  $G$  is in a undischarged hypothesis above it. One must now study the  $\forall$ -elimination rule. By the subformula property, all universal quantifications in the derivation are either *falsum* (i.e., of the form  $\forall F.F$ ), a conjunction or a disjunction (see the definitions *à la* Prawitz). And for *these* universal quantifications, we may suppose that their range consists of *all* formulas of the language (cf. Theorem 1) and, in particular, the formula  $C$ .

To finish the proof of the disjunction property, just take  $C$  to be  $G \rightarrow G$ .

## References

- [1] J.-Y. Girard, Y. Lafont and P. Taylor. *Proofs and Types*. Cambridge Tracts in Computer Science. Cambridge University Press, Cambridge, 1989.
- [2] W. Goldfarb. Russell's reasons for ramification. In C. Savage and C. Anderson, editors, *Rereading Russell*, pages 25–39. University of Minnesota Press, Minneapolis, 1989.
- [3] D. Prawitz. *Natural Deduction*. Almqvist & Wiksell, Stockholm, 1965.