On the Consistency of the Δ^1_1-CA Fragment of Frege's *Grundgesetze*[*]

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Abstract

It is well known that Frege's system in the *Grundgesetze der Arithmetik* is formally inconsistent. Frege's instantiation rule for the second-order universal quantifier makes his system, except for minor differences, full (i.e., with unrestricted comprehension) second-order logic, augmented by an abstraction operator that abides to Frege's basic law V. A few years ago, Richard Heck proved the consistency of the fragment of Frege's theory obtained by restricting the comprehension schema to predicative formulae. He further conjectured that the more encompassing Δ^1_1-comprehension schema would already be inconsistent. In the present paper, we show that this is not the case.

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1 Introduction.

In the context of the recent Frege *renaissance* in the philosophy of mathematics, much attention has been paid to consistent fragments of the theory of Frege's *Grundgesetze der Arithmetik* [3]. Russell's well-known paradox arises through the interplay between second-order comprehension and Frege's value-range operator as governed by basic law V. Hence, there are essentially two options for arriving at consistent subtheories of Frege's system: restrict axiom V, or restrict the comprehension schema. We are here concerned only with the latter strategy.

The first result in this direction was obtained by Terence Parsons [7], who showed that the first-order fragment of *Grundgesetze* (that is essentially first-order logic together with a schematic version of basic law V) is free from contradiction. While Parsons' proof is model-theoretic, a constructive proof of this result has recently been given by John Burgess [2]. Warren Goldfarb [4] has shown the first-order fragment to be undecidable.

Richard Heck [5] has shown that the predicative fragment of *Grundgesetze*, i.e. the subtheory obtained by restricting the second-order comprehension schema to instances where the comprehension formula contains no second-order quantifiers, is consistent and interprets Robinson's arithmetic Q. At the same time, Heck conjectured that the more encompassing schema of Δ^1_1-comprehension would lead to inconsistency.

Wehmeier [8] defined a Fregean theory T_Δ containing the Δ^1_1-comprehension schema and proved its consistency. However, Wehmeier's technical setting is different from that of Parsons and Heck (see the last section), yielding a theory which is unable to nest some first-order abstracts and which is very weak in terms of Δ^1_1-definability. For instance, Wehmeier's model-theoretic consistency proof produces a model of T_Δ whose Δ^1_1-sets consist only of the finite and co-finite (i.e., with finite complement) sets. Even though the theory T_Δ is very limitative from this viewpoint, it did permit Wehmeier to make some interesting philosophical points concerning the existence of non-logical objects (see [8]). In the end, it remained open the question whether a contradiction would be derivable in Heck's predicative fragment augmented by the schema of Δ^1_1-comprehension.

In the present note we show that this is not the case, i.e., that the theory consisting of schema V plus Δ^1_1-comprehension is free from contradiction, thereby fully refuting Heck's conjecture. The proof goes roughly as follows. Heck proved the consistency of the predicative fragment of *Grundgesetze* by (essentially) extending Parsons' first-order model with a second-order part consisting of the first-order definable sets of that model. The extension duly results in a model of predicative comprehension. In the present paper, we carry out the same construction with the following modification: we start with a recursively saturated elementary extension of Parsons' first-order model. As a result, it follows that the second-order extended model satisfies Δ^1_1-comprehension. This happens for reasons similar to those of the following theorem of Barwise and Schlipf [1]: The class of first-order definable sets of a recursively saturated model of elementary Peano Arithmetic validates the schema of Δ^1_1-comprehension.

Finally, we note that the Δ^1_1-comprehension schema is on the verge of inconsistency. In fact, as Heck has pointed out in [5], Russell's paradox can be reproduced in the fragment of Frege's system in which the comprehension schema is restricted to Σ^1_1-formulae (equivalently, to Π^1_1-formulae).

2 Terminology and Basic Notions.

The linguistic setting is as in [5]: The Frege language L_F arises from the language of pure monadic second-order logic (with first-order equality) by the addition of the value-range (VR) operator $\hat{}$, whose syntax is given by the clause: If x is an individual variable and A any formula, then $\hat{x}A$ is a term (a VR term, as we shall say). The first-order expressions of L_F are those expressions that contain no second-order variables (which we shall also call 'concept variables'). L^1_F is the first-order fragment of L_F.

Let M be a non-empty set and S a collection of subsets of M. The pair $\mathfrak{M} = (M, S)$ is a so-called generalised structure for pure monadic second-order logic: First-order variables are intended to range over M, whereas second-order variables range over S. We shall write $D_1\mathfrak{M}$ for M and $D_2\mathfrak{M}$ for S. $L^1_F(M)$ is L^1_F augmented by an individual constant c for every element $c \in M$, where the constant c is to be interpreted by the element c. $L_F(\mathfrak{M})$ is L_F augmented by an individual constant c for each $c \in D_1\mathfrak{M}$ as before and a predicate constant H for each $H \in S$, where again the predicate constant H is to be interpreted by the set H.

A structure for L_F is a pair $(\mathfrak{M}, I)$, where $\mathfrak{M}$ is a generalised second-order structure, and I is a function mapping every closed VR term $\hat{x}A$ of $L_F(\mathfrak{M})$ to an element of $D_1\mathfrak{M}$. Similarly, a structure for L^1_F is a pair (M, I) consisting of a non-empty set M, and a function I mapping every closed VR term $\hat{x}A$

of $L^1_F(M)$ to an element of M.

Given a structure $(\mathfrak{M}, I)$ for L_F, closed $L_F(\mathfrak{M})$-formulae are evaluated semantically as usual, where the denotations of the closed VR terms $\hat{x}A$ are supplied by the function I, and similarly for L^1_F.

With respect to L_F, schema V is the set of all universal closures (with respect to both first- and second-order variables) of instances of

$$\hat{x}A = \hat{y}B \leftrightarrow \forall z(A_x[z] \leftrightarrow B_y[z]),$$

for every pair A, B of L_F-formulae, where z is a fresh variable. With respect to L^1_F, schema V is the set of all such sentences for every pair A, B of L^1_F-formulae.

A predicative formula of L_F is a formula with no second-order quantifiers. A Σ^1_1-formula of L_F is a formula of the form $\exists X_1 \dots \exists X_n A$, where A is predicative. Negations of Σ^1_1-formulae are called Π^1_1-formulae. The L_F-schema of *predicative comprehension* is the set of all universal closures of all instances of

$$\exists X \forall x(Xx \leftrightarrow A),$$

where A is a predicative formula and X has no free occurrences in A. The L_F-schema of Δ^1_1-comprehension is the set of universal closures of all instances of

$$\forall x(A \leftrightarrow B) \rightarrow \exists X \forall x(Xx \leftrightarrow A),$$

where A is a Σ^1_1-formula, B is a Π^1_1-formula and X is not free in A.

Let L be a recursive first-order language. A structure M for L is recursively saturated if, for any recursive set Φ of L-formulae in at most finitely many free variables $x, \bar{y}$ and every finite tuple $\bar{n}$ of elements of M, there is a finite subset Ψ of Φ such that

$$M \models \forall x \bigvee \Phi_{\bar{y}}[\bar{n}] \rightarrow \forall x \bigvee \Psi_{\bar{y}}[\bar{n}].$$

More precisely: If for every element m of M there is a formula $A \in \Phi$ such that $M \models A_{x,\bar{y}}[m, \bar{n}]$, then there is a finite subset Ψ of Φ such that, for each $m \in M$, there is an $A \in \Psi$ with $M \models A_{x,\bar{y}}[m, \bar{n}]$ (the definition is often formulated contrapositively in the literature). The important fact about recursive saturation is the following: Given any countable structure M for a recursive language L, there exists a countable recursively saturated elementary extension N of M. For a detailed proof, see e.g. [6, pp. 148-9].

3 The Model Construction.

The aim of this paper is to construct a structure $(\mathfrak{M}, I)$ for L_F satisfying every instance of schema V plus the schema of Δ^1_1-comprehension. This will be done by first constructing a recursively saturated structure for L^1_F satisfying schema V, which can easily be expanded to a structure for L_F satisfying schema V plus predicative comprehension. The recursive saturation of the first-order model will ensure that the second-order model satisfies Δ^1_1-comprehension.

We start by considering the L^1_F-structure defined in [5]. It is a structure abiding to schema V, whose domain is the set of all natural numbers, and such that there are infinitely many elements of the domain which are not denotations of VR terms.

Since L^1_F is a recursive language, there exists a countable recursively saturated elementary extension (N, I) of the above structure. By elementarity, (N, I) satisfies schema V, and still there are infinitely many elements in the domain N which are not denotaions of VR terms (these elements make room for the denotations of the impredicative VR terms yet to be considered).

Clarification. A slight complication actually arises in applying the recursive saturation theorem to L^1_F, as this is not a first-order language, owing to the presence of the VR operator. This difficulty may be circumvented by reformulating the Frege language L^1_F into the first-order language used by Burgess in [2]. Here, VR terms $\hat{x}A$ are rendered as terms $e_{x,A,\bar{y}}(\bar{y})$, where $e_{x,A,\bar{y}}$ is a function symbol and $FV(A) \setminus \{x\} = \bar{y}$. Schema V then takes the form

$$e_{r,A,\bar{v}}\bar{x} = e_{s,B,\bar{w}}\bar{y} \leftrightarrow \forall z(A_{r,\bar{v}}[z, \bar{x}] \leftrightarrow B_{s,\bar{w}}[z, \bar{y}]),$$

where z is a fresh variable, the $\bar{x}$ are free for the $\bar{v}$ in A and the $\bar{y}$ are free for the $\bar{w}$ in B. It should be noted, however, that the Burgess language is not, strictly speaking, a notational variant of L^1_F, as several translations correspond to the same VR term. Thus, $\hat{x}(x = \hat{y}(y = y))$, for instance, could be rendered as either $e_{x,x=z,z}(e_{y,y=y,\emptyset})$ or $e_{x,x=e_{y,y=y,\emptyset}\emptyset}$. Nevertheless, it is easy to see that there exists a recursive fragment of the Burgess language that is indeed a notational variant of L^1_F. We shall ignore these niceties and simply work with L^1_F, taking the applicability of the recursive saturation theorem for granted. *End of Clarification.*

This takes care of the first-order fragment. We say that a subset H of N is L^1_F-definable over (N, I) if there is an L^1_F-formula A in at most the free variables $x, \bar{y}$ and a tuple $\bar{n}$ of elements of N such that $H = \{m \in N : (N, I) \models A_{x,\bar{y}}[m, \bar{n}]\}$. Let S be the collection of all L^1_F-definable subsets of N. Let $\mathfrak{N}$ be the generalised second-order structure (N, S). In order to turn $\mathfrak{N}$ into a structure for the full Frege language L_F, we need to extend the function I to a function I^* which is also defined on (a) those closed VR terms of $L_F(\mathfrak{N})$ which have set parameters from $D_2\mathfrak{N}$, but no second-order quantifiers (predicative VR terms), and (b) those closed VR terms of $L_F(\mathfrak{N})$ in which second-order quantifiers do occur (impredicative VR terms). This can be done exactly as in [5]: for (a), replace second-order parameters by their first-order definitions, for (b), repeat the original Parsons procedure on the impredicative VR terms. As in [5], the resulting structure will satisfy both schema V and the predicative comprehension axioms. It remains to show that our structure $(\mathfrak{N}, I^*)$ is also a model of the schema of Δ^1_1-comprehension. This is the task of the next section.

4 The Schema of Δ^1_1-Comprehension.

We have followed Heck [5] in setting up the Frege language with concept variables of one argument-place only, although Frege himself uses also binary second-order variables in *Grundgesetze*. While the restriction to unary concept variables simplifies notation enormously, it does not, in the presence of schema V, represent any loss of generality: We may introduce ordered pairs *via* the well-known Wiener-Kuratowski definition

$$\langle u, v\rangle :\equiv \hat{x}[x = \hat{y}(y = u) \vee x = \hat{y}(y = u \vee y = v)].$$

In the presence of schema V, one easily proves

$$\forall xyuv(\langle x, y\rangle = \langle u, v\rangle \leftrightarrow (x = u \wedge y = v)).$$

With this observation, it is clear that we can speak of binary second-order variables simply by rendering them *via* unary concept variables true of the pertinent ordered pairs. Since the definition of pairs is entirely first-order, unary predicative comprehension immediately yields binary predicative comprehension: Given that A is predicative,

$$\exists X\forall x(Xx \leftrightarrow \exists uv[x = \langle u, v\rangle \wedge A])$$

is still an instance of predicative comprehension (and similarly for Δ^1_1-comprehension). The same is clearly true for ternary second-order variables, using the definition $\langle u, v, w\rangle :\equiv \langle\langle u, v\rangle, w\rangle$, *etc.*

Given a formula A of the full Frege language, R a second-order variable, and x and y first-order variables, we denote by $A_X[R_{x,y}]$ the formula obtained from A by substituting each ocurrence of Xt in A by $R\langle x, y, t\rangle$.

The main step in proving that the structure $(\mathfrak{N}, I^*)$ is a model of the schema of Δ^1_1-comprehension is the following proposition, which says in effect that $(\mathfrak{N}, I^*)$ assents to a form of Σ^1_1-choice. This is the only place in the proof where we use the recursive saturation of the structure (N, I):

Proposition. *The structure* $(\mathfrak{N}, I^*)$ *satisfies the schema*

$$\forall x \exists X A \to \exists R \forall x \exists y A_X[R_{x,y}]$$

where A *is a predicative formula, and* R *and* y *are fresh variables.*

Proof: Assume that $(\mathfrak{N}, I^*) \vDash \forall x \exists X A$. Substitute the second-order parameters of A by their first-order definitions. Without loss of generality, we may suppose that these first-order definitions only require a single first-order parameter $p \in N$: this is because we have a pairing function. (We shall use this reduction to a single parameter in the definitions whenever convenient.) Therefore, there is a formula B with no second-order parameters, and with an extra free variable w, such that $\forall x \forall X(A \leftrightarrow B_w[p])$ holds in $(\mathfrak{N}, I^*)$. Hence, by assumption, $(\mathfrak{N}, I^*) \vDash \forall x \exists X B_w[p]$. Thus, given an element $a \in N$, there is $H \in D_2\mathfrak{N}$ such that $(\mathfrak{N}, I^*) \vDash B_{w,x,X}[p, a, H]$. Since everything in $D_2\mathfrak{N}$ is L^1_F-definable, there is a first-order formula C with exactly two free variables u and y, and there is (a parameter) $n \in N$ such that $H = \{m \in N : (N, I) \vDash C_{u,y}[m, n]\}$.

Let us introduce some notation: For each first-order formula C, let $B_X[\{u : C\}]$ be the first-order formula obtained from B by substituting each occurence of the form Xt by $C_u[t]$. With this new notation, we have $(N, I) \vDash B_X[\{u : C\}]_{w,x,y}[p, a, n]$. Given that a is arbitrary, we conclude that

$$(N, I) \vDash \forall x \bigvee \Phi_w[p]$$

where Φ is the set of first-order formulae of the form $\exists y(B_X[\{u : C\}])$, with C a first-order formula in exactly the two free variables u and y. This set Φ is clearly recursive. Thus, by recursive saturation, there are first-order formulae $C^1, C^2, \dots, C^k$ such that

$$(N, I) \vDash \forall x \exists y \bigvee_{i=1}^{k} B_X[\{u : C^i\}]_w[p].$$

Now, let D be the following first-order formula:

$$(C^1 \wedge B_X[\{u : C^1\}]) \vee (C^2 \wedge B_X[\{u : C^2\}] \wedge \neg B_X[\{u : C^1\}]) \vee \dots$$

$$\ldots \vee (C^k \wedge B_X[\{u : C^k\}] \wedge \bigwedge_{i=1}^{k-1} \neg B_X[\{u : C^i\}]).$$

Given $a \in N$, take $n \in N$ such that

$$(N, I) \vDash \bigvee_{i=1}^{k} B_X[\{u : C^i\}]_{w,x,y}[p, a, n]$$

and, at the same time, take the least i such that $B_X[\{u : C^i\}]_{w,x,y}[p, a, n]$ holds in (N, I). The formula D was defined so that the following equality between sets holds:

$$\{m \in N : (N, I) \vDash C^i_{u,y}[m, n]\} = \{m \in N : (N, I) \vDash D_{w,x,y,u}[p, a, n, m]\}.$$

Therefore $(N, I) \vDash B_X[\{u : D\}]_{w,x,y}[p, a, n]$. We have thus argued that

$$(N, I) \vDash \forall x \exists y (B_X[\{u : D\}]_w[p]).$$

Now, the set $\{\langle a, n, m\rangle \in N : (N, I) \vDash D_{w,x,y,u}[p, a, n, m]\}$ is a predicatively defined set (ternary relation). We may conclude that

$$(\mathfrak{N}, I^*) \vDash \exists R \forall x \exists y B_X[R_{x,y}]_w[p].$$

But $B_X[R_{x,y}]_w[p]$ is $B_w[p]_X[R_{x,y}]$, and hence

$$(\mathfrak{N}, I^*) \vDash \exists R \forall x \exists y A_X[R_{x,y}].$$

□

Lemma. *The structure $(\mathfrak{N}, I^*)$ satisfies the schema*

$$\forall x \exists X \exists Y A \to \exists R \exists Q \forall x \exists y A_{X,Y}[R_{x,y}, Q_{x,y}]$$

where A is a predicative formula, and R, Q, and y are fresh variables.

Proof: This is a consequence of the previous proposition. Suppose that $\forall x \exists X \exists Y A$ holds in $(\mathfrak{N}, I^*)$. It is easy to see that $(\mathfrak{N}, I^*) \vDash \forall x \exists Z B$, where Z is a fresh variable and the formula B arises from A by substituting the occurences of the form Xt and Yt by $Z\langle 0, t\rangle$ and $Z\langle 1, t\rangle$, respectively. (Here 0 and 1 can be defined by $\hat{x}(x \neq x)$ and $\hat{x}(x = 0)$, respectively.) By the above proposition, we may conclude that

$$(\mathfrak{N}, I^*) \vDash \exists S \forall x \exists y B_Z[S_{x,y}].$$

This immediately yields our conclusion, since we may now define $R\langle x, y, u\rangle$ by $S\langle x, y, \langle 0, u\rangle\rangle$, and $Q\langle x, y, u\rangle$ by $S\langle x, y, \langle 1, u\rangle\rangle$. □

We are now ready to argue that the structure $(\mathfrak{N}, I^*)$ validates the schema of Δ^1_1-comprehension. Using the trick of the proof of the above lemma, we may collapse adjacent existential (respectively, universal) second-order quantifiers into *one* existential (respectively, universal) quantifier. Thus, without loss of generality, let A and B be predicative formulae, and suppose that

$$\forall x(\exists X A \leftrightarrow \forall Y B)$$

holds in the structure $(\mathfrak{N}, I^*)$. In particular, we have:

$$(\mathfrak{N}, I^*) \vDash \forall x \exists X \exists Y(\neg B \vee A).$$

By the above lemma,

$$(\mathfrak{N}, I^*) \vDash \exists R \exists Q \forall x \exists y(\neg B_Y[Q_{x,y}] \vee A_X[R_{x,y}]),$$

that is, there are sets $H, L \in D_2\mathfrak{N}$ such that, for each $a \in N$, we can find $n \in N$ so that

$$(\#) \qquad (\mathfrak{N}, I^*) \vDash \neg B_{x,Y}[a, H_{a,n}] \vee A_{x,X}[a, L_{a,n}],$$

where $H_{a,n} = \{m \in N : \langle a, n, m\rangle \in H\}$, $L_{a,n} = \{m \in N : \langle a, n, m\rangle \in L\}$.

Now define

$$K := \{a \in N : \exists n \in N\, (\mathfrak{N}, I^*) \vDash A_{x,X}[a, L_{a,n}]\}.$$

Note that K is predicatively defined and, thus, $K \in D_2\mathfrak{N}$. We claim that $K = \{a \in N : (\mathfrak{N}, I^*) \vDash \exists X A_x[a]\}$. Clearly, with this equality our argument will be finished.

Suppose that $a \in K$. Take $n \in N$ so that $A_{x,X}[a, L_{a,n}]$ holds in $(\mathfrak{N}, I^*)$. Therefore, $(\mathfrak{N}, I^*) \vDash \exists X A_x[a]$. Conversely, take $a \in N$ such that $\exists X A_x[a]$ holds in $(\mathfrak{N}, I^*)$. By assumption, we may suppose that $(\mathfrak{N}, I^*) \vDash \forall Y B_x[a]$. By (#), we may conclude that $(\mathfrak{N}, I^*) \vDash A_{x,X}[a, L_{a,n}]$. Hence $a \in K$.

5 Closing remarks.

The main idea of this paper is taken from the argument of Barwise and Schlipf in [1] that proves that the class of first-order definable sets of a recursively saturated model of elementary Peano Arithmetic validates the schema of Δ^1_1-comprehension. Barwise and Schlipf's proof is couched in the language of admissible set theory but, in this paper, we strove for simplicity and sidestepped this (inessential) feature. Barwise and Schlipf also showed that their model satisfies a form of Σ^1_1-choice (from which it was well known that the schema of Δ^1_1-comprehension would follow). Their Σ^1_1-choice principle is slightly different from ours. It is

$$\forall x \exists X A \rightarrow \exists R \forall x A_X[R_x],$$

where A is a predicative formula. The reason why they do not need the extra variable y (compare with our choice principle), is ultimately due to the fact that Peano Arithmetic has a canonical way (via minimization) of choosing elements from non-empty first-order definable sets. This feature is absent from our Fregean setting.

The schema of Σ^1_1-choice permits the transformation of a formula of the form $\forall x \exists X A$, with A predicative, into a Σ^1_1-formula. Thus, in the presence of Σ^1_1-choice, we may safely ignore first-order quantifications when trying to judge whether a certain given formula is (equivalent to) a Σ^1_1-formula. More precisely: Let us define the class of the *essentially* Σ^1_1-formulas as the smallest class of formulas containing all predicative formulas and closed under conjunctions, disjunctions, universal and existential first-order quantifications and existential second-order quantifications. In the presence of Σ^1_1-choice, every essentially Σ^1_1-formula is equivalent to a Σ^1_1-formula. Analogously, we define the class of the *essentially* Π^1_1-formulas, and we formulate the *essentially* Δ^1_1-comprehension schema. From the discussion above, it is clear that this schema follows from Σ^1_1-choice.

With the terminology introduced in the last paragraph, we may finally compare the result of this paper with Wehmeier's mathematical result of [8]. Wehmeier's setting is (monadic) second-order logic with equality augmented by a unary function symbol ϵ that, when attached to a second-order variable X, yields a first-order term ϵX. Wehmeier's theory T_Δ is axiomatic second-order logic with the axiom of comprehension restricted to Δ^1_1-formulas (in these formulas one allows the occurence of terms of the form ϵX) together with the *single* Fregean axiom V:

$$\forall X \forall Y (\epsilon X = \epsilon Y \leftrightarrow \forall x (Xx \leftrightarrow Yx)).$$

The rendering of the abstractor as a function symbol proper, instead of a term-building operator as in the Parsons-Heck tradition, has the effect of restricting severely the uses of nested (first-order) abstraction. As Wehmeier remarks in his paper [8], it is impossible to Δ^1_1-define in T_Δ the set of singletons, i.e., the set of elements x such that $\exists z(x = \hat{y}(y = z))$. More precisely, the theory T_Δ does not prove the sentence

$$\exists X \forall x (Xx \leftrightarrow \exists z \exists Z (\forall y (Zy \leftrightarrow y = z) \wedge x = \epsilon Z)).$$

Consequently, Wehmeier's T_Δ does not have a counterpart for the nested (first-order) abstract $\hat{x}(\exists z(x = \hat{y}(y = z)))$. Note, however, that the property of being a singleton does indeed have an *essentially* Δ^1_1-definition in Wehmeier's setting. In effect, axiom V readily implies the equivalence:

$$\forall x(\exists z \exists Z(\forall y(Zy \leftrightarrow y = z) \wedge x = \epsilon Z) \leftrightarrow \exists z \forall Z(\forall y(Zy \leftrightarrow y = z) \rightarrow x = \epsilon Z)).$$

Hence, T_Δ does not validate the comprehension schema for essentially Δ^1_1-formulae and, *a fortiori*, does not validate Σ^1_1-choice. It is this combination of an abstraction function together with the failure of the essentially Δ^1_1-comprehension schema that makes Wehmeier's theory T_Δ a rather weak one from the definability viewpoint, and therefore unable to nest some first-order abstracts.

As our theory extends T_Δ, however, Wehmeier's philosophical points continue to hold here: Our theory proves the non-existence of the value-range concept (it proves $\neg\exists X \forall x(Xx \leftrightarrow \exists Y x = \hat{y}Yy)$), as well as the existence of arbitrarily finitely many non-value ranges (for each natural number n, our theory proves $\exists x_1, \ldots, x_n(\bigwedge_{i \neq j} x_i \neq x_j \wedge \forall X \bigwedge_i x_i \neq \hat{x}Xx)$). Thus, regardless of how much of mathematics can be carried out in this theory (conjecture: not much more than in Heck's predicative fragment), for reasons discussed in [8], it does not seem to be an attractive option for logicists.

References

[1] Barwise, Jon & John Schlipf 1975: On Recursively Saturated Models of Arithmetic. In *Model Theory and Algebra*, ed. D. H. Saracino and V. B. Weispfenning, Lecture Notes in Mathematics **498**, Springer-Verlag, 42-55.

[2] Burgess, John 1998: On a Consistent Subsystem of Frege's *Grundgesetze. Notre Dame Journal of Formal Logic* **39**, 274-278.

[3] Frege, Gottlob 1893: *Grundgesetze der Arithmetik, Band I.* Hermann Pohle, Jena.

[4] Goldfarb, Warren 2001: First-Order Frege Theory is Undecidable. *Journal of Philosophical Logic* **30**, 613-616.

[5] Heck, Richard 1996: The Consistency of Predicative Fragments of Frege's *Grundgesetze der Arithmetik. History and Philosophy of Logic* **17**, 209-220.

[6] Kaye, Richard 1991: *Models of Peano Arithmetic*. Clarendon Press, Oxford.

[7] Parsons, Terence 1987: On the Consistency of the First-Order Portion of Frege's Logical System. *Notre Dame Journal of Formal Logic* **28**, 161-168.

[8] Wehmeier, Kai F. 1999: Consistent Fragments of *Grundgesetze* and the Existence of Non-Logical Objects. *Synthese* **121**, 309-328.