THE JOURNAL OF SYMBOLIC LOGIC
Volume 00, Number 0, XXX 0000

THE BOUNDED FUNCTIONAL INTERPRETATION OF THE DOUBLE NEGATION SHIFT

PATRÍCIA ENGRÁCIA AND FERNANDO FERREIRA

Abstract. We prove that the (non-intuitionistic) law of the double negation shift has a bounded functional interpretation with bar recursive functionals of finite type. As an application, we show that full numerical comprehension is compatible with the uniformities introduced by the characteristic principles of the bounded functional interpretation for the classical case.

§1. Introduction and background. In 1962 [14], Clifford Spector gave a remarkable characterization of the provably recursive functionals of full second-order arithmetic (a.k.a. analysis). The central result of his paper is an extension, from arithmetic to analysis, of the (then quite recent) *dialectica* interpretation of Gödel of 1958 [7]. Spector's extension relies on a form of well-founded recursion known as bar recursion. The name comes from the intuitionistic studies of L. E. J. Brouwer and his contentious bar theorem of the nineteen twenties.

Spector extends the bar notions to all finite types. There are various insights in Spector's paper, but we find that the crucial insight is that the (non-intuitionistic) law of the *double negation shift*,

$$\mathsf{DNS}: \qquad \forall n\, \neg\neg A(n) \to \neg\neg\forall n\, A(n)$$

(n is a natural number variable, A is an arbitrary formula) has a *dialectica* interpretation using bar recursive functionals of finite-type. The existence of this interpretation is enough to ensure the interpretation of the negative translation of full numerical comprehension

$$\mathsf{CA}^0: \quad \exists f^1 \forall n^0\, (f(n) =_0 0 \leftrightarrow A(n)),$$

where A is an arbitrary formula of the language of finite-order arithmetic. Here, the superscripts denote the type of the variables: type 0 is the type of natural numbers, type 1 is the type of the functions from natural numbers to natural numbers. We assume that the reader is familiar with these type-theoretic notations. [1] and the recent [12] are good sources for the *dialectica* interpretation and related issues (including bar-recursive functionals).

The bounded functional interpretation was introduced in [5]. It is an interpretation based on a new transformation of formulas $A \rightsquigarrow A^B := \tilde{\exists}\underline{a}\tilde{\forall}\underline{b} A_B(\underline{a}, \underline{b})$ and which relies essentially on majorizability notions. The *characteristic principles*

0022-4812/00/0000-0000/$00.00

of this interpretation state "uniformities" which are not set theoretically true. A conspicuous result is that the characteristic principles (for the classical case) *refute*, within a base theory, simple forms of comprehension for type 1 functions (see section 8 of [3]). In other words, the mentioned "uniformities" are not compatible with type 1 comprehension. Notwithstanding, by the soundness theorem of the interpretation, they entail (e.g.) true Π^0_2-sentences only. The reader can find in [3] some discussions and comparisons between Gödel's *dialectica* interpretation, the bounded functional interpretation and, also, the related monotone functional interpretation of Ulrich Kohlenbach (introduced in [11]). In the same article, the second author expressed the belief that the uniformities introduced by the bounded functional interpretation (for the classical case) are compatible with full *numerical* comprehension (i.e., type 0 comprehension). The results of Section 5 below confirm that this belief was correct.

The *strong majorizability* relations were introduced by Marc Bezem in [2] (after the seminal work of William Howard [10]):

$$x \leq^*_0 y := x \leq y$$
$$x \leq^*_{\rho\to\sigma} y := \forall u^\rho, v^\rho\,(u \leq^*_\rho v \to xu \leq^*_\sigma yv \wedge yu \leq^*_\sigma yv)$$

Bezem also defines the structure $\mathcal{M}^\omega$ of the strongly majorizable functionals and proved that the bar recursors are well-defined in this structure (bar recursors are not well-defined in the standard set-theoretical type structure). The bounded functional interpretation uses an *intensional* version of Bezem's majorizability relations. These relations $\trianglelefteq$ (one for each finite type) are called intensional because they are partly governed by a rule:

$$x \trianglelefteq_0 y \leftrightarrow x \leq y,$$
$$x \trianglelefteq_{\rho\to\sigma} y \to \forall u \trianglelefteq_\rho v(xu \trianglelefteq_\sigma yv \wedge yu \trianglelefteq_\sigma yv)$$
$$\frac{A_{\mathsf{bd}} \wedge u \trianglelefteq_\rho v \to su \trianglelefteq_\sigma tv \wedge tu \trianglelefteq_\sigma tv}{A_{\mathsf{bd}} \to s \trianglelefteq_{\rho\to\sigma} t}$$

where A_{bd} is an intensional bounded formula and u and v are variables which do not appear in the conclusion of the rule (named as $\mathsf{RL}_\trianglelefteq$). We assume that the reader is familiar with the intuitionistic arithmetic theory $\mathsf{HA}^\omega_\trianglelefteq$ and its bounded functional interpretation.

The main purpose of this paper is to show that DNS has a bounded functional interpretation. As discussed in [13], the *dialectica* interpretation of the intuitionistic law $\neg\neg A \wedge \neg\neg B \to \neg\neg(A \wedge B)$ can be seen as a "finite" version of the interpretation of DNS. Moreover, since this law is a theorem of $\mathsf{HA}^\omega_\trianglelefteq$, it must have a bounded functional interpretation. We work out this interpretation explicitly in the brief Section 2 as a warm up for the interpretation of DNS. The latter interpretation cannot be done solely in terms of the primitive recursive functionals in the sense of Gödel. Further terms are needed and, following the work of Spector, we effect this interpretation using terms defined by bar recursion. It turns out that the bounded functional interpretation of DNS is somewhat delicate, and we dedicate Section 4 almost entirely to it. The preceding Section 3 describes the theory in which the interpretation of DNS is verified. This theory contains the set $\Delta_{\mathcal{M}^\omega}$ of all universal sentences (with intensional bounded matrices) whose *flattenings* are true in the structure $\mathcal{M}^\omega$. This is not optimal, of course. However, we chose this route because an optimal treatment would be a distraction from the main thrust of the interpretation of DNS. Moreover, the treatment of CA^0 in Section 5 relies essentially on some facts of $\Delta_{\mathcal{M}^\omega}$.

We would like to thank Jaime Gaspar and Paulo Oliva for some discussions. We also thank an anonymous referee for pertinent observations which improved the final version of this paper. We are special grateful to the referee for pointing out an improvement in the case of arithmetical comprehension.

§2. A not so simple interpretation. Let A and B be arbitrary formulas of the language of $\mathsf{HA}^\omega_\trianglelefteq$ and suppose that A^B is $\tilde{\exists} a_1 \tilde{\forall} b_1 A_B(a_1, b_1)$ and B^B is $\tilde{\exists} a_2 \tilde{\forall} b_2 B_B(a_2, b_2)$. As a matter of fact, we should have written (possibly empty) tuples of variables in the previous quantifications. However, for ease of reading, we have omitted (and will omit) the tuple notation. In order to obtain the bounded functional interpretation of $\neg\neg A \wedge \neg\neg B \to \neg\neg(A \wedge B)$, a straightforward computation shows that we must produce monotone a_1^*, a_2^*, g_1^* and g_2^*, depending on monotone f_1, f_2, ϕ_1 and ϕ_2, such that the three clauses

(1) $\tilde{\forall} g_1 \trianglelefteq g_1^* \neg \tilde{\forall} a_1 \trianglelefteq \phi_1 g_1 \neg \tilde{\forall} b_1 \trianglelefteq g_1 a_1 \; A_B(a_1, b_1)$

(2) $\tilde{\forall} g_2 \trianglelefteq g_2^* \neg \tilde{\forall} a_2 \trianglelefteq \phi_2 g_2 \neg \tilde{\forall} b_2 \trianglelefteq g_2 a_2 \; B_B(a_2, b_2)$

(3) $\tilde{\forall} a_1 \trianglelefteq a_1^*, a_2 \trianglelefteq a_2^* \neg \tilde{\forall} b_1 \trianglelefteq f_1 a_1 a_2 \tilde{\forall} b_2 \trianglelefteq f_2 a_1 a_2 \; (A_B(a_1, b_1) \wedge B_B(a_2, b_2))$

lead to a contradiction.

Take

$$g_1^* = \lambda x. f_1 x (\phi_2(\lambda y. f_2 x y))$$
$$a_1^* = \phi_1 g_1^*$$
$$g_2^* = \lambda y. f_2 a_1^* y$$
$$a_2^* = \phi_2 g_2^*$$

Since f_1, f_2, ϕ_1 and ϕ_2 are monotone, it follows that $g_1^*, g_2^*, a_1^*, a_2^*$ are also monotone (the rule RL is heavily used in showing this). Assume that we have the clauses (1), (2) and (3) for the g_1^*, a_1^*, g_2^*, a_2^* as defined above. We must reach a contradiction.

Take a monotone a_1 with $a_1 \trianglelefteq a_1^*$, and define $g_2 := \lambda y. f_2 a_1 y$. Take, now, a monotone a_2 with $a_2 \trianglelefteq \phi_2 g_2$. Then $g_2 \trianglelefteq g_2^*$ and $a_2 \trianglelefteq a_2^*$. We get the following:

$$\tilde{\forall} b_1 \trianglelefteq g_1^* a_1 \; A_B(a_1, b_1) \wedge \tilde{\forall} b_2 \trianglelefteq g_2 a_2 \; B_B(a_2, b_2) \to$$
$$\tilde{\forall} b_1 \trianglelefteq f_1 a_1 a_2 \; A_B(a_1, b_1) \wedge \tilde{\forall} b_2 \trianglelefteq f_2 a_1 a_2 \; B_B(a_2, b_2)$$

because we have $f_1 a_1 a_2 \trianglelefteq g_1^* a_1$ by the definition of g_1^* and the fact that $a_2 \trianglelefteq \phi_2 g_2$ (note, also, that $g_2 a_2 = f_2 a_1 a_2$). By (3),

$$\tilde{\forall} b_1 \trianglelefteq f_1 a_1 a_2 \tilde{\forall} b_2 \trianglelefteq f_2 a_1 a_2 \; (A_B(a_1, b_1) \wedge B_B(a_2, b_2)) \to \bot.$$

Hence, we may conclude that $\tilde{\forall} b_1 \trianglelefteq g_1^* a_1 \; A_B(a_1, b_1) \to \neg \tilde{\forall} b_2 \trianglelefteq g_2 a_2 \; B_B(a_2, b_2)$. Due to the arbitrariness of a_2, we even get

$$\tilde{\forall} b_1 \trianglelefteq g_1^* a_1 \; A_B(a_1, b_1) \to \tilde{\forall} a_2 \trianglelefteq \phi_2 g_2 \neg \tilde{\forall} b_2 \trianglelefteq g_2 a_2 \; B_B(a_2, b_2).$$

By (2), it follows $\neg \tilde{\forall} b_1 \trianglelefteq g_1^* a_1 \; A_B(a_1, b_1)$. By the arbitrariness of a_1,

$$\tilde{\forall} a_1 \trianglelefteq \phi_1 g_1^* \neg \tilde{\forall} b_1 \trianglelefteq g_1^* a_1 \; A_B(a_1, b_1).$$

This contradicts (1) when we instantiate g_1 by g_1^*.

§3. The bounded functional interpretation extended to bar recursors. In this section, we extend the language of $\mathsf{HA}^{\omega}_{\trianglelefteq}$ with new constants $B^{\rho,\sigma}$, the *bar recursors*, and consider the following defining axioms $\mathsf{BR}_{\rho,\sigma}$:

$$\forall \psi^{(0\to\rho)\to 0}, z^{\tau_1}, u^{\tau_2}, n^0, s^{0\to\rho}\ ((\psi\overline{s,n} <_0 n \to B^{\rho,\sigma}\psi zuns =_\sigma zn\overline{s,n}) \wedge$$
$$(\psi\overline{s,n} \geq_0 n \to B^{\rho,\sigma}\psi zuns =_\sigma u(\lambda x^\rho.B^{\rho,\sigma}\psi zu(n+1)(\overline{s,n}*x))n\overline{s,n}))$$

where $\tau_1 = 0 \to ((0\to\rho)\to\sigma)$, $\tau_2 = (\rho\to\sigma)\to((0\to(0\to\rho))\to\sigma)$, and $(\overline{s,n})^{0\to\rho}$ and $(\overline{s,n}*x)^{0\to\rho}$ are defined as

$$\overline{s,n}\ k =_\rho \begin{cases} sk & \text{if } k <_0 n \\ 0 & \text{otherwise} \end{cases}$$

$$(\overline{s,n}*x)k =_\rho \begin{cases} sk & \text{if } \ k <_0 n \\ x & \text{if } \ k =_0 n \\ 0 & \text{otherwise} \end{cases}$$

Note that whereas $s^{0\to\rho}$ denotes infinite sequences of objects of type ρ, $\overline{s,n}$, although formally of type $0\to\rho$, is meant to stand for the initial subsequence of s with length n, $\langle s_0, s_1, \dots, s_{n-1}, 0, 0, \dots\rangle$, and $\overline{s,n}*x$ is the concatenation of the finite sequence $\overline{s,n}$ with x ('$\overline{s,n}*x$' is meant to be a "ternary" functional in s, n and x).

Following the treatment of Kohlenbach in [12], we officially take simultaneous bar-recursion with tuples of variables (note that the 'neutral' treatment of equality in $\mathsf{HA}^{\omega}_{\trianglelefteq}$ does not seem to allow a reduction to ordinary bar recursion without tuples). As in the previous section, we omitted (and will omit) the tuple notation. Let us write BR for the collection of all the statements of the form $\mathsf{BR}_{\rho,\sigma}$. Bar recursion is a principle of definition while bar induction is a corresponding principle of proof, in analogy with the usual recursors and induction. The scheme of bar induction BI applied to formulas P and Q is given by

$$\text{Hyp1} \wedge \text{Hyp2} \wedge \text{Hyp3} \wedge \text{Hyp4} \to \forall s \in M^{\mathbb{N}}, n \in \mathbb{N}\, Q(\overline{s,n}, n),$$

where

$$\begin{aligned} \text{Hyp1}: &\quad \forall s \in M^{\mathbb{N}} \exists n \in \mathbb{N}\ P(\overline{s,n}, n) \\ \text{Hyp2}: &\quad \forall s \in M^{\mathbb{N}}, n \in \mathbb{N}\, \forall m \leq n (P(\overline{s,m}, m) \to P(\overline{s,n}, n)) \\ \text{Hyp3}: &\quad \forall s \in M^{\mathbb{N}}, n \in \mathbb{N}\ (P(\overline{s,n}, n) \to Q(\overline{s,n}, n)) \\ \text{Hyp4}: &\quad \forall s \in M^{\mathbb{N}}, n \in \mathbb{N}\ (\forall x^\rho\ Q(\overline{s,n}*x, n+1) \to Q(\overline{s,n}, n)) \end{aligned}$$

It is well-known that we can argue by bar induction in the structure $\mathcal{M}^\omega$ (see, for instance, [12] for a closely related formulation).

Let us consider the set $\Delta_{\mathcal{M}^\omega}$ as described in the introduction: the set of all universal sentences (with intensional bounded matrices) whose flattenings happen to be true in the structure $\mathcal{M}^\omega$ of the majorizable functionals. We remind the reader that the flattening of a formula of the intensional language is obtained by replacing each sign $\trianglelefteq$ by the corresponding majorizability sign $\leq^*$ (see [4], or the end of section 6 of [5]). Even though the statements in BR are in $\Delta_{\mathcal{M}^\omega}$ (they are universal), we will write $\mathsf{HA}^{\omega}_{\trianglelefteq} + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$ instead of the shorter $\mathsf{HA}^{\omega}_{\trianglelefteq} + \Delta_{\mathcal{M}^\omega}$. The inclusion of the acronym 'BR' has the advantage of indicating that our language contains the bar recursive functionals.

THEOREM 3.1. $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$ *is a majorizability theory (i.e., for every closed term t there is a closed term q such that* $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} \vdash t \trianglelefteq q$*).*

PROOF. It suffices to check that the bar recursive functionals have majorants (within the theory). Let B^* be given by $B^*\psi zuns = \max_{i \leq n} B^p \psi zuis$ where $B^p\psi zuns$ is

$$\begin{cases} zn(\overline{s,n}^M) & \text{if } \psi\overline{s,n}^M < n \\ \max\{zn(\overline{s,n}^M), u(\lambda x.B^p\psi zu(n+1)(\overline{s,n} * x))n(\overline{s,n}^M)\} & \text{otherwise} \end{cases}$$

and $s^M(n)$ stands for $\max_{i \leq n} s(i)$. In Kohlenbach's recent book [12], it is shown that $\mathcal{M}^\omega \models B \leq^* B^*$. Hence, the sentence $B \trianglelefteq B^*$ is in $\Delta_{\mathcal{M}^\omega}$. ⊣

We have just seen that $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$ is a majorizability theory. Moreover, the sentences of $\mathsf{BR} + \Delta_{\mathcal{M}^\omega}$ are universal (with bounded intensional matrices) and, therefore, self-interpretable. Hence, by the main result of [5]:

THEOREM 3.2 (**Soundness**). *Let $A(\underline{z})$ be a formula of the language of* $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$ *with free variables $\underline{z}$, and assume that $A^B(\underline{z})$ is $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}A_B(\underline{z},\underline{b},\underline{c})$. If*

$$\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega[\trianglelefteq] \vdash A(\underline{z})$$

then, there are monotone closed terms $\underline{t}$ of appropriate type such that

$$\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} \vdash \tilde{\forall}\underline{a}\forall\underline{z} \trianglelefteq \underline{a}\tilde{\forall}\underline{c}\ A_B(\underline{z},\underline{ta},\underline{c}).$$

Moreover, $\mathcal{M}^\omega \models \tilde{\forall}\underline{a}\forall\underline{z} \leq^* \underline{a}\tilde{\forall}\underline{c}\ (A_B)^*(\underline{z},\underline{ta},\underline{c})$.

In the above, $\mathsf{P}^\omega[\trianglelefteq]$ consists of the characteristic principles of the bounded functional interpretation for the intuitionistic case. These principles are described in [5]. (We use the notation A^* for the flattening of the formula A.)

§4. The interpretation of the double negation shift.

This section is dedicated to the proof of the following theorem:

THEOREM 4.1. DNS *has a bounded functional interpretation in* $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$.

COROLLARY 4.2. $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega[\trianglelefteq] \vdash \mathsf{DNS}$.

PROOF. Let A be (the universal closure of) an instance of DNS. By the above theorem, $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} \vdash A^B$. The result now follows by characterization theorem (see [5]) of the bounded functional interpretation (intuitionistic case). ⊣

In order to prove Theorem 4.1, let $A(n^0)$ be an arbitrary formula of the language of $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR}$ and suppose that $A^B(n)$ is $\tilde{\exists}a\tilde{\forall}bA_B(n,a,b)$ (we simplify and omit parameters). A straightforward calculation shows that to interpret DNS, as formulated in the introductory section, we must produce monotone n^*, f^* and g^* (depending only on given monotone ϕ, ψ_1 and ψ_2) such that the statement

$$\begin{aligned} &\forall n \leq n^*\tilde{\forall}g \trianglelefteq g^*\neg\tilde{\forall}a \trianglelefteq \phi ng\neg\tilde{\forall}b \trianglelefteq ga\ A_B(n,a,b) \to \\ &\qquad \neg\tilde{\forall}f \trianglelefteq f^*\neg\forall n \leq \psi_1 f\tilde{\forall}b \trianglelefteq \psi_2 f\ A_B(n,fn,b) \end{aligned}$$

is provable in $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$ (note that, as observed in the previous section, we disregard tuples of functionals). Since the above statement is universal (in ϕ, ψ_1 and ψ_2), we need only to show that its flattening

$$\forall n \le n^* \tilde{\forall} g \le^* g^* \neg \tilde{\forall} a \le^* \phi n g \neg \tilde{\forall} b \le^* g a \; A^*_B(n,a,b) \to \\ \neg \tilde{\forall} f \le^* f^* \neg \forall n \le \psi_f \tilde{\forall} b \le^* \psi_2 f \; A^*_B(n, fn, b)$$

is true in $\mathcal{M}^\omega$ (given ϕ, ψ_1 and ψ_2 monotone in the ordinary, flattened, sense). Of course, if the concern is with *truth* in $\mathcal{M}^\omega$, then we can simplify the above formula and substitute the negative universals by appropriate existentials. That notwithstanding, we will argue intuitionistically below (in tune with the argument of Section 2). For instance, the argument given below can be adapted to show that it holds for the theory $\mathsf{HA}^\omega_{\trianglelefteq} + \mathsf{BR} + \Delta_i$, where Δ_i is the set of all universal sentences (with intensional bounded matrices) whose flattenings are provable in $\mathsf{E\text{-}HA}^\omega + \mathsf{BR} + \mathsf{BI}$. Here, the acronym E means that full extensionality is present (we are being careful at this point because our actual uses of extensionality probably do not require E). Notice that $\Delta_i \subseteq \Delta_{\mathcal{M}^\omega}$.

From here onwards and until the end of the section, we work with the ordinary majorizability sign. The statements that we prove are meant to be true in $\mathcal{M}^\omega$ (as noticed, with suitable modifications, they are even provable in $\mathsf{E\text{-}HA}^\omega + \mathsf{BR} + \mathsf{BI}$). When we use abbreviations concerning monotonicity, they are meant to be in the ordinary sense. We use $\forall x \le^* a\, A$ to abbreviate $\forall x\ (x \le^* a \to A)$, etc.

We introduce a bit of notation: if ψ is in $M_{(0\to\rho)\to\sigma}$, write ψ' for the functional of the same type defined by $\psi' s := \psi(s^M)$. In the sequel, we use some simple properties ([12] is a good reference). For instance, if $s(i)$ is monotone for $i \le n$, then $\forall i \le n(s(i) \le^* s^M(n))$. Also, $\forall i(r(i) \le^* s(i))$ entails $r^M \le^* s^M$. Finally, the following fact is handy: for monotone r, $0 \le^* r$ (here 0 denotes the zero functional of the same type as r).

Let us fix ϕ, ψ_1 and ψ_2 monotone of appropriate types. We define $B'ns$ according to the following clauses:

$$B'ns := \begin{cases} \overline{s,k} & \text{if } k \le n,\ \psi_1'\overline{s,k} < k \text{ and } \forall i < k(\psi_1'\overline{s,i} \ge i) \\ B'(n+1)(\overline{s,n} * c) & \text{if } \forall k \le n(\psi_1'\overline{s,k} \ge k) \end{cases}$$

where n is a natural number, $s \in M_\rho^{\mathbb{N}}$, $c = \phi n g_{\overline{s,n}}$ and

$$g_{\overline{s,n}} = \lambda x.\psi_2'(B'(n+1)(\overline{s,n} * x)).$$

The value $B'ns$ is in $M_\rho^{\mathbb{N}}$. In fact, we should think of this value as a finite sequence of elements of M_ρ. It is clear that B' can be defined by bar recursion.

Before we give n^*, f^* and g^*, it is convenient to study some properties of B'.

LEMMA 4.3. *Take $n \in \mathbb{N}$ and $s \in M_\rho^{\mathbb{N}}$, then*

$$\forall i \le n\ (\psi_1'\overline{s,i} \ge i) \to \forall i < n\ (\overline{s,n}\ i = B'n(\overline{s,n})i).$$

PROOF. We argue by bar induction. Take

$$P(s,n) = \exists i \le n\ (\psi_1'\overline{s,i} < i)$$
$$Q(s,n) = \forall i \le n\ (\psi_1'\overline{s,i} \ge i) \to \forall i < n\ (\overline{s,n}\ i = B'n(\overline{s,n})i).$$

Let us see that we have Hyp1-Hyp4 of bar induction. As we know, Hyp1 holds in the structure of majorizable functionals. Hyp2 and Hyp3 are clear. Let us focus on Hyp4. Take arbitrary s and n and assume that, for every $x \in M_\rho$,

$$\forall i \le n{+}1\ (\psi_1'(\overline{\overline{s,n} * x, i}) \ge i) \to \forall i < n{+}1\ ((\overline{s,n}{*}x)\ i = B'(n{+}1)(\overline{s,n}{*}x)i).$$

We must show $Q(\overline{s,n}, n)$. Suppose that $\forall i \le n\ (\psi_1'\overline{s,i} \ge i)$. By definition of B', $B'n(\overline{s,n}) = B'(n{+}1)(\overline{s,n}{*}c)$ with c given by $c = \phi n(\lambda x.\psi_2'(B'(n{+}1)(\overline{s,n}{*}x)))$.

Either $\psi'_1(\overline{s,n} * c) < n+1$ or $\psi'_1(\overline{s,n} * c) \geq n+1$. If the first case occurs, then $B'(n+1)(\overline{s,n} * c) = \overline{s,n} * c$ and also $B'n\overline{s,n} = \overline{s,n} * c$. From this it follows that $\forall i < n\ (\overline{s,n}\, i = B'n(\overline{s,n})i)$. On the other hand, if $\psi'_1(\overline{s,n}*c) \geq n+1$, then, by the initial assumption with $x = c$, we get $\forall i < n+1((\overline{s,n} * c)i = B'(n+1)(\overline{s,n} * c)i)$. It clearly follows that $\forall i < n\ (\overline{s,n}i = B'n(\overline{s,n})i)$, as desired. ⊣

The following lemma (and respective proof) is similar to the corresponding result concerning the majorability proof of section 11.5 of [12].

Lemma 4.4. *If* $n \in \mathbb{N}$ *and* $s, r \in M_\rho^{\mathbb{N}}$, *then*

$$\forall i < n\ (si \leq^* ri) \rightarrow \forall j\ (B'nsj \leq^* B'nrj).$$

Proof. We argue by bar induction. Take

$$P(r,n) := \exists k \leq n\ (\psi'_1\overline{r,k} < k)$$
$$Q(r,n) := \forall s\ (\forall i < n(si \leq^* ri) \rightarrow \forall j(B'n\overline{s,n}j \leq^* B'nrj)).$$

As in the lemma above, Hyp1 and Hyp2 hold. Let us check that Hyp3 obtains. Suppose that $P(\overline{r,n}, n)$. Take s such that $\forall i < n(si \leq^* ri)$. Let k_0 be the least natural number such that $\psi'_1\overline{r,k_0} < k_0$. Note that $k_0 \leq n$. By the monotonicity of ψ_1 and the observation that $\overline{s,k_0}^M \leq^* \overline{r,k_0}^M$, $\psi'_1\overline{s,k_0} \leq \psi'_1\overline{r,k_0} < k_0$. Take k_1 least such that $\psi'_1\overline{s,k_1} < k_1$. Note that $k_1 \leq k_0$. Therefore, we have $B'n\overline{s,n} = B'ns = \overline{s,k_1}$ and $B'n\overline{r,n} = B'nr = \overline{r,k_0}$. Hence, $\forall j(B'n\overline{s,n}j \leq^* B'n\overline{r,n}j)$. So, $Q(\overline{r,n}, n)$.

It remains to see Hyp4, i.e., $\forall x Q(\overline{r,n} * x, n+1) \rightarrow Q(\overline{r,n}, n)$. So, assume that $\forall x Q(\overline{r,n} * x, n+1)$. If $\exists k \leq n(\psi'_1\overline{r,k} < k)$, then by what was shown in Hyp3 we get $Q(\overline{r,n}, n)$. We are restricted to the case $\forall k \leq n(\psi'_1\overline{r,k} \geq k)$. Let s be given such that $\forall i < n(si \leq^* ri)$. By definition of B', $B'n\overline{r,n} = B'(n+1)(\overline{r,n} * c)$, where $c = \phi n g_{\overline{r,n}}$ and $g_{\overline{r,n}} = \lambda x.\psi'_2(B'(n+1)(\overline{r,n} * x))$.

We claim that $g_{\overline{r,n}}$ is monotone. We must show that

$$x \leq^* z \rightarrow \psi'_2(B'(n+1)(\overline{r,n} * x)) \leq^* \psi'_2(B'(n+1)(\overline{r,n} * z)).$$

Given that $x \leq^* z$, it is clear that $\forall i < n+1((\overline{r,n} * x)i \leq^* (\overline{r,n} * z)i)$. Since we have $Q(\overline{r,n} * z, n+1)$ we may conclude that

$$\forall j(B'(n+1)(\overline{r,n} * x)j \leq^* B'(n+1)(\overline{r,n} * z)j),$$

and, therefore, by the monotonicity of ψ_2, it follows that $\psi'_2(B'(n+1)(\overline{r,n}*x)) \leq^* \psi'_2(B'(n+1)(\overline{r,n} * z))$.

We also claim that c is monotone. However, this is an immediate consequence of the definition of c and the previous claim, given that ϕ is monotone.

With these two claims proved, we show that $\forall j(B'n\overline{s,n}j \leq^* B'n\overline{r,n}j)$. We discuss two cases.

The first case is when $\forall k \leq n(\psi'_1\overline{s,k} \geq k)$. In this case, we have $B'n\overline{s,n} = B'(n+1)(\overline{s,n} * d)$, where $d = \phi n g_{\overline{s,n}}$ and $g_{\overline{s,n}} = \lambda x.\psi'_2(B'(n+1)(\overline{s,n} * x))$.

We prove that $g_{\overline{s,n}} \leq^* g_{\overline{r,n}}$. It is sufficient to show that

$$x \leq^* z \rightarrow \psi'_2(B'(n+1)(\overline{s,n} * x)) \leq^* \psi'_2(B'(n+1)(\overline{r,n} * z)).$$

Well, if $x \leq^* z$ then $\forall i < n+1((\overline{s,n} * x)i \leq^* (\overline{r,n} * z)i)$. By $Q(\overline{r,n} * z, n+1)$ and the monotonicity of ψ_2, the claim follows.

It is now clear that $d \leq^* c$. Therefore, $\forall i < n+1((\overline{s,n} * d)i \leq^* (\overline{r,n} * c)i)$. By $Q(\overline{r,n} * c, n+1)$ we may infer $\forall j(B'(n+1)(\overline{s,n} * d)j \leq^* B'(n+1)(\overline{r,n} * c)j)$. At this point we only have to observe that $B'(n+1)(\overline{s,n}*d) = B'n\overline{s,n}$ and that $B'(n+1)(\overline{r,n}*c) = B'n\overline{r,n}$.

Finally, the second case is when $\exists k \leq n(\psi'_1\overline{s,k} < k)$. Take k_0 least such that $\psi'_1\overline{s,k_0} < k_0$. Note that $k_0 \leq n$. By definition of B', $B'n\overline{s,n} = \overline{s,k_0}$. By the previous lemma, we have $\forall i < n(\overline{r,n}i = B'n\overline{r,n}i)$. It readily follows that $\forall j < k_0(B'n\overline{s,n}j \leq^* B'n\overline{r,n}j)$. The claim also extends for $j \geq k_0$ provided that all the entries of the sequence $B'n\overline{r,n}$ are monotone (and, therefore, majorize 0). This is easily seen to be the case. Observe that $Q(\overline{r,n}*c, n+1)$ implies that, for all j, $B'(n+1)(\overline{r,n}*c)j$ is monotone. But, as we know, $B'(n+1)(\overline{r,n}*c) = B'n\overline{r,n}$. ⊣

The following is an immediate consequence of the above lemma:

Corollary 4.5. *Let $n \in \mathbb{N}$. Consider $s, r \in M_\rho^{\mathbb{N}}$ and suppose that $si \leq^* ri$, for all $i < n$. Then*

$$\lambda x.\psi'_2(B'(n+1)(\overline{s,n}*x)) \leq^* \lambda x.\psi'_2(B'(n+1)(\overline{r,n}*x)).$$

In particular, given $r \in M_\rho^{\mathbb{N}}$ such that, for each $i < n$, ri is monotone, then $\lambda x.\psi'_2(B'(n+1)(\overline{r,n}*x))$ is monotone.

In order to ease readability, we write $\langle s0, s1, \ldots, s(n-1), 0, 0, \ldots\rangle$ to denote $s \in M_\rho^{\mathbb{N}}$ such that $si = 0$ for $i \geq n$.

Let us define recursively

$$\begin{aligned} g_0^* &= \lambda x.\psi'_2(B'1\langle x, 0, 0, \ldots\rangle) \\ a_0^* &= \phi 0 g_0^* \\ g_{i+1}^* &= \lambda x.\psi'_2(B'(i+2)\langle a_0^*, a_1^*, \ldots, a_i^*, x, 0, 0, \ldots\rangle) \\ a_{i+1}^* &= \phi(i+1)g_{i+1}^* \end{aligned}$$

Using the above corollary, it is clear by induction that the a_i^*'s and the g_i^*'s are monotone. Define:

$$\begin{aligned} f^* &= \langle a_0^*, a_1^*, a_2^*, \ldots\rangle^M \\ n^* &= \psi_1 f^* \\ g^* &= \max_{i \leq n^*} g_i^*. \end{aligned}$$

Observe that f^* and g^* are monotone.

The remainder of the section is dedicated to proving that the monotone functionals n^*, f^* and g^* defined above (which depend only on the given monotone ϕ, ψ_1 and ψ_2) lend themselves to interpret DNS. More precisely, we show that the two statements

$$\forall n \leq n^* \tilde{\forall} g \leq^* g^* \neg\tilde{\forall} a \leq^* \phi n g \neg\tilde{\forall} b \leq^* ga\ A_B(n, a, b) \tag{4}$$

$$\tilde{\forall} f \leq^* f^* \neg\forall n \leq \psi_1 f \tilde{\forall} b \leq^* \psi_2 f\ A_B(n, fn, b). \tag{5}$$

entail a contradiction.

Definition 4.6. A sequence of monotone elements $a_0, \ldots, a_n$ of M_ρ is *nice* if, for each $0 \leq i \leq n$, $a_i \leq^* \phi i g_i$, where

$$g_i = \lambda x.\psi'_2(B'(i+1)\langle a_0, \ldots, a_{i-1}, x, 0, 0, \ldots\rangle).$$

Note that each g_i above depends only on $a_0, \dots, a_{i-1}$ for its definition. We prove some facts about nice sequences.

LEMMA 4.7. *Consider $a_0, \dots, a_n$ a nice sequence, with associated functions $g_0 \dots, g_n, g_{n+1}$. For all $i \leq n+1$, g_i is monotone, $g_i \leq^* g_i^*$ and, for $i \leq n$, $a_i \leq^* a_i^*$. Moreover, if $i \leq n^*$ then $g_i \leq^* g^*$.*

PROOF. The result is easily proved by complete induction on $i \leq n$ using Corollary 4.5. ⊣

At this point, we can already prove the following:

PROPOSITION 4.8. *Under the hypothesis (4) we have, for all $n \leq n^*$,*

$$\neg\tilde{\forall} a_0, \dots, a_n \neg\forall i \leq n \ (a_i \leq^* \phi i g_i \wedge \tilde{\forall} b \leq^* g_i a_i \ A_B(i, a_i, b)).$$

PROOF. The proof is made by induction on n. For $n = 0$, the conclusion comes from (4):

$$\neg\tilde{\forall} a \leq^* \phi 0 g_0 \neg\tilde{\forall} b \leq^* g_0 a \ A_B(0, a, b).$$

To prove the induction step, take the induction hypothesis:

$$\neg\tilde{\forall} a_0, \dots, a_n \neg\forall i \leq n \ (a_i \leq^* \phi i g_i \wedge \tilde{\forall} b \leq^* g_i a_i \ A_B(i, a_i, b))$$

with $n < n^*$ and assume

$$\tilde{\forall} a_0, \dots, a_{n+1} \neg\forall i \leq n+1 \ (a_i \leq^* \phi i g_i \wedge \tilde{\forall} b \leq^* g_i a_i \ A_B(i, a_i, b)),$$

which is equivalent to

$$\begin{cases} \tilde{\forall} a_0, \dots, a_n \tilde{\forall} a_{n+1} \neg(\forall i \leq n \ (a_i \leq^* \phi i g_i \wedge \tilde{\forall} b \leq^* g_i a_i \ A_B(i, a_i, b)) \wedge \\ \qquad a_{n+1} \leq^* \phi(n+1) g_{n+1} \wedge \tilde{\forall} b \leq^* g_{n+1} a_{n+1} \ A_B(n+1, a_{n+1}, b)). \end{cases}$$

By (4), if $a_0, \dots, a_n$ is a nice sequence and g_{n+1} is the $(n+1)$th associated function, then $\neg\tilde{\forall} a \leq^* \phi(n+1) g_{n+1} \neg\tilde{\forall} b \leq^* g_{n+1} a \ A_B(n+1, a, b)$. That is:

$$\begin{aligned} &\tilde{\forall} a_0, \dots, a_n \ (\forall i \leq n \ (a_i \leq^* \phi i g_i) \to \\ &\quad \neg\tilde{\forall} a_{n+1} \neg(a_{n+1} \leq^* \phi(n+1) g_{n+1} \wedge \tilde{\forall} b \leq^* g_{n+1} a_{n+1} A_B(n+1, a_{n+1}, b))). \end{aligned}$$

Applying the intuitionist rule

$$\frac{\forall x \forall z \ \neg(H(x) \wedge A(x) \wedge B(x, z)) \quad \forall x \ (H(x) \to \neg\forall z \ \neg B(x, z))}{\forall x \ \neg(H(x) \wedge A(x))}$$

we get

$$\tilde{\forall} a_0, \dots, a_n \neg\forall i \leq n \ (a_i \leq^* \phi i g_i \wedge \tilde{\forall} b \leq^* g_i a_i \ A_B(i, a_i, b)).$$

The contradiction follows from the induction hypothesis. ⊣

In particular, under hypothesis (4), we have:

(6) $$\neg\tilde{\forall} a_0, \dots, a_{n^*} \neg\forall i \leq n^* \ (a_i \leq^* \phi i g_i \wedge \tilde{\forall} b \leq^* g_i a_i \ A_B(i, a_i, b)).$$

We will show that the above leads to a contradiction under the hypothesis (5). Firstly, we need to prove some further facts about nice sequences:

LEMMA 4.9. *Let $a_0, \dots, a_{n^*}$ be a nice sequence and $g_0, \dots, g_{n^*}$ (and g_{n^*+1}) its associated functions. Then we have $\forall n < n^* (g_{n+1} a_{n+1} \leq^* g_n a_n)$.*

PROOF. Let $n < n^*$. By definition, we have

$$g_n a_n = \psi_2'(B'(n+1)\langle a_0 \dots , a_n, 0, 0, \dots \rangle)$$
$$g_{n+1} a_{n+1} = \psi_2'(B'(n+2)\langle a_0 \dots , a_n, a_{n+1}, 0, 0, \dots \rangle).$$

We consider two cases. Suppose that there is $k \leq n$ such that

$\psi_1' \langle a_0, \dots , a_k, 0, 0, \dots \rangle < k + 1.$

Let k_0 be the least such k. Then, by definition of B'

$$B'(n+1)\langle a_0, \dots , a_n, 0, 0, \dots \rangle = \langle a_0, \dots , a_{k_0}, 0, 0, \dots \rangle$$
$$B'(n+2)\langle a_0, \dots , a_n, a_{n+1}, 0, 0, \dots \rangle = \langle a_0, \dots , a_{k_0}, 0, 0, \dots \rangle$$

Therefore, $g_{n+1} a_{n+1} = g_n a_n$. Note that $g_n a_n$ is monotone, since $a_0, \dots , a_{k_0}$ are monotone.

Now, for the second case: $\forall k \leq n \; \psi_1' \langle a_0, \dots , a_k, 0, 0, \dots \rangle \geq k+1$. In this case

$B'(n+1)\langle a_0, \dots , a_n, 0, 0, \dots \rangle = B'(n+2)\langle a_0, \dots , a_n, c, 0, 0, \dots \rangle,$

where $c = \phi(n+1)g_{n+1}$. Since, $a_{n+1} \leq^* \phi(n+1)g_{n+1} = c$, then

$$\psi_2'(B'(n+2)\langle a_0, \dots , a_n, a_{n+1}, 0, 0, \dots \rangle) \leq^* \psi_2'(B'(n+2)\langle a_0, \dots , a_n, c, 0, 0, \dots \rangle),$$

as desired. ⊣

Given $\overline{a} = a_0, \dots , a_{n^*}$ a nice sequence, $\psi_1 \langle a_0, \dots , a_{n^*}, 0, 0, \dots \rangle^M \leq \psi_1 f^* = n^* < n^* + 1$. Let k_0 be the least natural number $\leq n^*$ such that

$\psi_1 \langle a_0, \dots , a_{k_0}, 0, 0, \dots \rangle^M < k_0 + 1.$

Define $f_{\overline{a}}$ as $\langle a_0, \dots , a_{k_0}, 0, 0, \dots \rangle^M$. Observe that $f_{\overline{a}} \leq^* f^*$ and $\psi_1 f_{\overline{a}} \leq n^*$.

LEMMA 4.10. *Let $\overline{a} = a_0, \dots , a_{n^*}$ be a nice sequence, with associated functions $g_0, \dots , g_{n^*}$ (and g_{n^*+1}). Take $f_{\overline{a}}$ as defined above. Then, $\psi_2 f_{\overline{a}} \leq^* g_n a_n$, for all $n \leq n^*$.*

PROOF. We show that $\psi_2 f_{\overline{a}} = g_{n^*} a_{n^*}$. With the help of the previous lemma, this entails our result. By definition, $f_{\overline{a}} = \langle a_0, \dots , a_{k_0}, 0, 0, \dots \rangle^M$, where k_0 is least satisfying $\psi_1 \langle a_0, \dots , a_{k_0}, 0, 0, \dots \rangle^M < k_0 + 1$. According to the definition of B',

$B'(n^*+1)\langle a_0, \dots , a_{n^*}, 0, 0, \dots \rangle = \langle a_0, \dots , a_{k_0}, 0, 0, \dots \rangle.$

Therefore: $\psi_2 f_{\overline{a}} = \psi_2'(B'(n^*+1)\langle a_0, \dots , a_{n^*}, 0, 0, \dots \rangle) = g_{n^*} a_{n^*}$. ⊣

LEMMA 4.11. *Assume that statement (5) holds, and let $\overline{a} = a_0, \dots , a_{n^*}$ be a nice sequence, with associated functions $g_0, \dots , g_{n^*}$ (and g_{n^*+1}). In this situation,*

$\neg \forall n \leq \psi_1 f_{\overline{a}} \tilde{\forall} b \leq^* g_n a_n \; A_B(n, a_n, b),$

for $f_{\overline{a}}$ defined as above.

PROOF. Assume $\forall n \leq \psi_1 f_{\overline{a}} \tilde{\forall} b \leq^* g_n a_n A_B(n, a_n, b)$. By the above lemma,

$\forall n \leq \psi_1 f_{\overline{a}} \tilde{\forall} b \leq^* \psi_2 f_{\overline{a}} \; A_B(n, a_n, b).$

Let $f_{\overline{a}} = \langle a_0, \dots , a_{k_0}, 0, 0, \dots \rangle^M$. By definition, $\psi_1 f_{\overline{a}} \leq k_0$. Now, if $n \leq \psi_1 f_{\overline{a}}$, we clearly have $a_n \leq^* f_{\overline{a}} n$. Using the monotonicity of A_B in the entry of a_n, we get $\forall n \leq \psi_1 f_{\overline{a}} \tilde{\forall} b \leq^* \psi_2 f_{\overline{a}} \; A_B(n, f_{\overline{a}} n, b)$. This contradicts (5). ⊣

Let us take stock. We have showed in the previous lemma that, under the hypothesis (5),

$$\tilde{\forall} a_0, \dots, a_{n^*} (\forall n \leq n^* (a_n \leq^* \phi n g_n) \to \neg \forall n \leq \psi_1 f_{\overline{a}} \tilde{\forall} b \leq g_n a_n \; A_B(n, a_n, b)).$$

Equivalently,

$$\tilde{\forall} a_0, \dots, a_{n^*} \neg (\forall n \leq n^* (a_n \leq^* \phi n g_n) \wedge \forall n \leq \psi_1 f_{\overline{a}} \tilde{\forall} b \leq g_n a_n \; A_B(n, a_n, b)).$$

Now, since $\psi_1 f_{\overline{a}} \leq n^*$, this entails

$$\tilde{\forall} a_0, \dots, a_{n^*} \neg \forall n \leq n^* (a_n \leq^* \phi n g_n \wedge \tilde{\forall} b \leq g_n a_n \; A_B(n, a_n, b)).$$

We have reached a contradiction with (6).

Theorem 4.1 is now proved.

§5. The interpretation of full numerical comprehension. As mentioned in the introduction, Spector introduced bar recursive functionals in order to effect a *dialectica* interpretation of full numerical comprehension. The interpretation is done within the classical setting via a negative (Gödel-Gentzen like) translation $A \rightsquigarrow A^g$ of formulas. The soundness theorem of the bounded functional interpretation within the classical setting reads as follows:

THEOREM 5.1 (**Soundness**). *Let $A(\underline{z})$ be a formula of the language of $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$ with free variables $\underline{z}$, and assume that $(A^g)^B(\underline{z})$ is $\tilde{\exists} \underline{b} \tilde{\forall} \underline{c} \; (A^g)_B(\underline{z}, \underline{b}, \underline{c})$. If*

$$\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq] \vdash A(\underline{z})$$

then there are monotone closed terms $\underline{t}$ of appropriate type such that

$$\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} \vdash \tilde{\forall} \underline{a} \forall \underline{z} \trianglelefteq \underline{a} \tilde{\forall} \underline{c} \; (A^g)_B(\underline{z}, \underline{ta}, \underline{c}).$$

Moreover, $\mathcal{M}^\omega \models \tilde{\forall} \underline{a} \forall \underline{z} \leq^* \underline{a} \tilde{\forall} \underline{c} \; ((A^g)_B)^*(\underline{z}, \underline{ta}, \underline{c})$.

In the above, $\mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq]$ is constituted by the characteristic principles of the bounded functional interpretation for the classical case. These principles are described in [5] and, in a more perspicuous form, in [4]. Our aim is to show that

$$\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq] \vdash \mathsf{CA}^0.$$

As discussed in [3] and [4], the principles in $\mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq]$ embody uniformities which are absent from the set-theoretic world. Already very simple instances of comprehension for type 1 functionals are incompatible with these uniformities. It was nevertheless suggested in [3] that full numerical comprehension is compatible with such uniformities. The above soundness theorem, together with the fact that CA^0 is a consequence of $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq]$, shows that this is indeed the case. We argue this fact in a rather indirect way, relying on the work of the previous section. We will show that $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq] \vdash \mathsf{AC}^{0,\omega}$, where $\mathsf{AC}^{0,\omega}$ is the form of choice $\forall n^0 \exists x A(n, x) \to \exists f \forall n A(n, fn)$, for *arbitrary* formulas A (x can be of any type). It is well known how to derive CA^0 from $\mathsf{AC}^{0,\omega}$ in a classical setting. In effect, let $A(n)$ be an arbitrary formula. By classical logic, $\forall n \exists k \, ((k = 0 \wedge A(n)) \vee (k = 1 \wedge \neg A(n)))$. By $\mathsf{AC}^{0,\omega}$ (only $\mathsf{AC}^{0,0}$ is needed), there is $f^{0 \to 0}$ which witnesses such k. Of course, we get $\forall n (fn = 0 \leftrightarrow A(n))$, as desired.

We first prove the weaker statement:

$$\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq] \vdash \mathsf{bAC}^{0,\omega},$$

where $\mathsf{bAC}^{0,\omega}$ is the principle $\forall n^0 \exists x A(n,x) \to \tilde{\exists} f \forall n \exists x \trianglelefteq fn A(n,x)$, for *arbitrary* formulas A (x can be of any type). Observe that if we follow the argument above, this weaker statement is found wanting for deriving CA^0 because $\mathsf{bAC}^{0,\omega}$ only provides a bound for the k (1 is a trivial bound), not an exact k.

In order to prove the weaker statement, we show that

$$\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega[\trianglelefteq] \vdash (\mathsf{bAC}^{0,\omega})^g.$$

Let us see why this does the job. On the one hand, by Theorem 3.2, we get $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} \vdash ((\mathsf{bAC}^{0,\omega})^g)^B$ (we must see each instance of $\mathsf{bAC}^{0,\omega}$ as given by its corresponding universal closure). On the other hand, by the characterization theorem of the bounded functional interpretation for the classical case (see [4]), we have $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq] \vdash \mathsf{bAC}^{0,\omega} \leftrightarrow ((\mathsf{bAC}^{0,\omega})^g)^B$. The result follows.

We should point out that the characterization theorem of [4] was formulated for a *direct* bounded functional interpretation of the *classical* theory $\mathsf{PA}^\omega_\trianglelefteq$, whereas here we are applying it to the indirect interpretation $A \rightsquigarrow (A^g)^B$, via a negative translation. That notwithstanding, the characterization theorem still holds in this indirect case. For instance, we can rely on Jaime Gaspar's factorization [6] of the direct interpretation in terms of a negative translation and the (intuitionistic) bounded functional interpretation. Even though Gaspar's factorization concerns the so-called Krivine negative translation, it is not difficult to see that it also applies to the Gödel-Gentzen translation using the fact that both translations are intuitionistically equivalent.

The presence of the bar recursors and of the corresponding axioms BR are paramount for proving the next result.

Proposition 5.2. $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega[\trianglelefteq] \vdash (\mathsf{bAC}^{0,\omega})^g$.

Proof. This relies on the adaptation of a well-known argument. The negative translation of $\mathsf{bAC}^{0,\omega}$ may be taken to be

$$\forall n \neg\neg \exists x A^g(n,x) \to \neg\neg \tilde{\exists} f \forall n \neg\neg \exists x \trianglelefteq fn A^g(n,x).$$

Assume $\forall n \neg\neg \exists x\ A^g(n,x)$. At this juncture, we rely on the work of the previous section, namely on Corollary 4.2. Therefore, we get $\neg\neg \forall n \exists x A^g(n,x)$. Since $\forall n \exists x A^g(n,x) \to \tilde{\exists} f \forall n \exists x \trianglelefteq fn A^g(n,x)$ is included in $\mathsf{P}^\omega[\trianglelefteq]$, we get (by intuitionistic logic)

$$\neg\neg \forall n \exists x A^g(n,x) \to \neg\neg \tilde{\exists} f \forall n \exists x \trianglelefteq fn A^g(n,x)$$

and, therefore, by Modus Ponens, $\neg\neg \tilde{\exists} f \forall n \exists x \trianglelefteq fn A^g(n,x)$. This entails our result. ⊣

We now prove $\mathsf{AC}^{0,\omega}$ from $\mathsf{bAC}^{0,\omega}$ within $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq]$, and this fact proves our aim. It is convenient to introduce the following form of (ineffective) choice, which we call tameAC:

$$\tilde{\forall} f \exists h \trianglelefteq f \forall x\ (\exists z \trianglelefteq fx A_{\mathsf{bd}}(x,z) \to A_{\mathsf{bd}}(x,hx)),$$

for (intensional) bounded formulas A_{bd}.

Lemma 5.3. *The flattening of the instances of* tameAC *are true in* $\mathcal{M}^\omega$.

PROOF. We present the following (ineffective) proof. Given f monotone in $M_{\sigma\to\rho}$, define h such that

$$x^\sigma \rightsquigarrow \begin{cases} z & \text{for some } z \in M_\rho \text{ such that } z \leq^*_\rho fx \wedge A^*_{\mathsf{bd}}(x,z) \\ 0^\rho & \text{otherwise} \end{cases}$$

Such h exists by the axiom of choice, and it is clear that $h \leq^* f$. Therefore, h is in $M_{\sigma\to\rho}$ and obviously witnesses the truth of the flattening of the given instance of tameAC. ⊣

The principle tameAC is not a universal statement. However, the following weakening, dubbed w-tameAC, is universal (with bounded intensional matrix):

$$\tilde{\forall} f \tilde{\forall} b \exists h \trianglelefteq f \forall x \trianglelefteq b\ (\exists z \trianglelefteq fx A_{\mathsf{bd}}(x,z) \to A_{\mathsf{bd}}(x,hx)).$$

By the above lemma, the flattenings of the instances of w-tameAC are true in $\mathcal{M}^\omega$. Since each such instance has the right syntactic form, they are in $\Delta_{\mathcal{M}^\omega}$.

LEMMA 5.4. $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq] \vdash$ tameAC.

PROOF. The principle tameAC is an immediate consequence of w-tameAC in the presence of the bounded (contra) collection principle of $\mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq]$. ⊣

LEMMA 5.5. *The theory* $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq]$ *proves*

$$\tilde{\forall} f(\tilde{\forall} a \exists b \trianglelefteq fa A(a,b) \to \exists h \trianglelefteq f \tilde{\forall} a A(a,ha)),$$

where A is an arbitrary universal formula (with bounded intensional matrix).

PROOF. Let f be monotone and assume that $\tilde{\forall} a \exists b \trianglelefteq fa \forall z\ A_{\mathsf{bd}}(a,b,z)$, where A_{bd} is a bounded formula. *A fortiori*, $\tilde{\forall} d \tilde{\forall} a \exists b \trianglelefteq fa \forall z \trianglelefteq d\ A_{\mathsf{bd}}(a,b,z)$ and, by tameAC, it follows that

$$\tilde{\forall} d \exists h \trianglelefteq f \tilde{\forall} a \forall z \trianglelefteq d\ A_{\mathsf{bd}}(a,ha,z)$$

and, therefore, $\tilde{\forall} c,d \exists h \trianglelefteq f \tilde{\forall} a \trianglelefteq c \forall z \trianglelefteq d\ A_{\mathsf{bd}}(a,ha,z)$. By bounded (contra) collection (which is included in $\mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq]$), we get $\exists h \trianglelefteq f \tilde{\forall} a \forall z\ A_{\mathsf{bd}}(a,ha,z)$. ⊣

We can finally prove the following:

PROPOSITION 5.6. $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} + \mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq] + \mathsf{bAC}^{0,\omega} \vdash \mathsf{AC}^{0,\omega}$.

PROOF. Suppose $\forall n \exists x A(n,x)$, with arbitrary A. It is a consequence of the characterization theorem of the bounded functional interpretation (classical case) that, within $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR} + \mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq]$, the formula $A(x,n)$ is equivalent to a formula of the form $\tilde{\exists} a \tilde{\forall} b B_{\mathsf{bd}}(a,b,n,x)$, with B_{bd} bounded. Hence, $\forall n \exists x \tilde{\exists} a \tilde{\forall} b B_{\mathsf{bd}}(a,b,n,x)$. By $\mathsf{bAC}^{0,\omega}$, we get $\tilde{\exists} f,g \forall n \exists x \trianglelefteq fn \tilde{\exists} a \trianglelefteq gn \tilde{\forall} b\, B_{\mathsf{bd}}(a,b,n,x)$.

By Lemma 5.5, there are h and s so that $\forall n \tilde{\forall} b(sn \trianglelefteq sn \wedge B_{\mathsf{bd}}(sn,b,n,hn))$. In particular, we have that $\forall n \tilde{\exists} a \tilde{\forall} b B_{\mathsf{bd}}(a,b,n,hn)$, i.e., $\forall n A(n,hn)$. As desired. ⊣

Notice that this argument relies on the fact that the flattenings of the instances of w-tameAC are among the set of sentences $\Delta_{\mathcal{M}^\omega}$. Therefore, it is no longer possible (as it was in Section 4) to replace this set of sentences by universal sentences provable using BI. The use of the ineffective principle stemming from w-tameAC is essential in the *above proof.* However, it is not clear if the result *per se* requires the use of such ineffective principle. In point of fact, the anonymous referee pointed out that the principle can be avoided in the important special case of arithmetical comprehension. Let us see why.

On the one hand, it is well known that arithmetical comprehension follows from Π^0_1-comprehension. On the other hand, this latter case of comprehension obviously follows from Π^0_1-$\mathsf{AC}^{0,0}$ (i.e., $\mathsf{AC}^{0,0}$ restricted to Π^0_1-matrices). We are left to prove this form of choice. Let us assume $\forall n \exists k \forall r A_{\mathsf{qf}}(n,k,r)$, where n, k and r are numerical variables and $A_{\mathsf{qf}}(n,k,r)$ is a quantifier-free formula (possibly with parameters). By $\mathsf{bAC}^{0,\omega}$, there is a monotone f such that $\forall n \exists k \leq fn \forall r A_{\mathsf{qf}}(n,k,r)$. In particular, $\forall l \forall n \exists k \leq fn \forall r \leq l A_{\mathsf{qf}}(n,k,r)$. By bounded search, we get $\forall l \exists h \leq_1 f \forall n \forall r \leq l A_{\mathsf{qf}}(n,hn,r)$. We may suppose

$$\forall l \exists h \trianglelefteq_1 f \forall n \forall r \leq l A_{\mathsf{qf}}(n,hn,r),$$

either by appealing to the construction of h or, alternatively, by replacing h by $\min_1(h,f)$ and noticing that h appears only in the (extensional) context "hn" in the matrix (observe that, for monotone f, $\min_1(h,f) \trianglelefteq_1 f$). Hence

$$\forall s \forall l \exists h \trianglelefteq_1 f \forall n \leq s \forall r \leq l A_{\mathsf{qf}}(n,hn,r),$$

and, by bounded (contra) collection, $\exists h \trianglelefteq_1 f \forall n \forall r A_{\mathsf{qf}}(n,hn,r)$. We conclude $\exists h \forall n \forall r A_{\mathsf{qf}}(n,hn,r)$, as wanted.

Our last result subsumes the fact that the theory $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{AC}^{0,\omega} + \mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq]$ has a bounded functional interpretation by bar recursive functionals (verifiable in $\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega}$):

Corollary 5.7. *Let $A(\underline{z})$ be a formula of the language of $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{BR}$ with free variables $\underline{z}$, and assume that $(A^g)^B(\underline{z})$ is $\tilde{\exists}\underline{b}\tilde{\forall}\underline{c}\ (A^g)_B(\underline{z},\underline{b},\underline{c})$. If*

$$\mathsf{PA}^\omega_\trianglelefteq + \mathsf{AC}^{0,\omega} + \mathsf{P}^\omega_{\mathsf{bd}}[\trianglelefteq] \vdash A(\underline{z})$$

then there are monotone closed terms $\underline{t}$ of appropriate type such that

$$\mathsf{HA}^\omega_\trianglelefteq + \mathsf{BR} + \Delta_{\mathcal{M}^\omega} \vdash \tilde{\forall}\underline{a}\forall \underline{z} \trianglelefteq \underline{a}\tilde{\forall}\underline{c}\ (A^g)_B(\underline{z},\underline{ta},\underline{c}).$$

Moreover, $\mathcal{M}^\omega \models \tilde{\forall}\underline{a}\forall\underline{z} \leq^* \underline{a}\tilde{\forall}\underline{c}\ ((A^g)_B)^*(\underline{z},\underline{ta},\underline{c})$.

Spector's result of 1962 was subsequently improved by Howard [9], where it is shown that (the negative translation of) the principle of dependent choices has a *dialectica* interpretation using bar recursive functionals. We conjecture that "dependent choices" has also a bounded functional interpretation.

REFERENCES

[1] J. Avigad and S. Feferman, *Gödel's functional ("Dialectica") interpretation*, ***Handbook of proof theory*** (S. R. Buss, editor), Studies in Logic and the Foundations of Mathematics, vol. 137, North Holland, Amsterdam, 1998, pp. 337–405.

[2] M. Bezem, *Strongly majorizable functionals of finite type: a model for bar recursion containing discontinuous functionals*, this Journal, vol. 50 (1985), pp. 652–660.

[3] F. Ferreira, *A most artistic package of a jumble of ideas*, ***dialectica***, vol. 62 (2008), pp. 205–222, Special Issue: Gödel's *dialectica* Interpretation. Guest editor: Thomas Strahm.

[4] ———, *Injecting uniformities into Peano arithmetic*, ***Annals of Pure and Applied Logic***, vol. 157 (2009), pp. 122–129, Special Issue: Kurt Gödel Centenary Research Prize Fellowships. Editors: Sergei Artemov, Matthias Baaz and Harvey Friedman.

[5] F. Ferreira and P. Oliva, *Bounded functional interpretation*, ***Annals of Pure and Applied Logic***, vol. 135 (2005), pp. 73–112.

[6] J. Gaspar, *Factorization of the Shoenfield-like bounded functional interpretation*, ***Notre Dame Journal of Formal Logic***, vol. 50 (2009), pp. 53–60.

[7] K. Gödel, *Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes*, ***dialectica***, vol. 12 (1958), pp. 280–287, Reprinted with an English translation in [8].

[8] ———, ***Collected works, vol. II***, S. Feferman *et al.*, eds. Oxford University Press, Oxford, 1990.

[9] W. A. Howard, *Functional interpretation of bar induction by bar recursion*, ***Compositio Mathematica***, vol. 20 (1968), pp. 107–124.

[10] ———, *Hereditarily majorizable functionals of finite type*, ***Metamathematical investigation of intuitionistic Arithmetic and Analysis*** (A. S. Troelstra, editor), Lecture Notes in Mathematics, vol. 344, Springer, Berlin, 1973, pp. 454–461.

[11] U. Kohlenbach, *Analysing proofs in analysis*, ***Logic: from foundations to applications*** (W. Hodges, M. Hyland, C. Steinhorn, and J. Truss, editors), European Logic Colloquium (Keele, 1993), Oxford University Press, 1996, pp. 225–260.

[12] ———, ***Applied proof theory: Proof interpretations and their use in mathematics***, Springer Monographs in Mathematics, Springer, Berlin, 2008.

[13] P. Oliva, *Understanding and using Spector's bar recursive interpretation of classical analysis*, ***Logical approaches to computational barriers*** (B. Löwe A. Beckmann, U. Berger and J. Tucker, editors), Lecture Notes in Computer Science, vol. 3988, Springer, New York, 2006, pp. 423–434.

[14] C. Spector, *Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles in current intuitionistic mathematics*, ***Recursive function theory*** (F. D. E. Dekker, editor), vol. 5, American Mathematical Society, Providence, Rhode Island, 1962, pp. 1–27.

DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDADE DE LISBOA
LISBOA, PORTUGAL
E-mail: p.engracia@gmail.com

DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDADE DE LISBOA
LISBOA, PORTUGAL
E-mail: ferferr@cii.fc.ul.pt