The faithfulness of $\mathbf{F_{at}}$: a proof-theoretic proof

Fernando Ferreira
Gilda Ferreira

Abstract

It is known that there is a sound and faithful translation of the full intuitionistic propositional calculus into the atomic polymorphic system $\mathbf{F_{at}}$, a predicative calculus with only two connectives: the conditional and the second-order universal quantifier. The faithfulness of the embedding was established quite recently via a model-theoretic argument based in Kripke structures. In this paper we present a purely proof-theoretic proof of faithfulness. As an application, we give a purely proof-theoretic proof of the disjunction property of the intuitionistic propositional logic in which commuting conversions are not needed.

1 Introduction

A *propositional formula* is a formula built from a stock of propositional letters (or constants) P, Q, R, etc using the propositional connectives $\perp$, $\wedge$, $\vee$ and $\to$. In [6], Prawitz defined the following translation:

$$(P)^\star :\equiv P, \text{ with } P \text{ a propositional constant}$$

$$(\perp)^\star :\equiv \forall X.X$$

$$(A \to B)^\star :\equiv A^\star \to B^\star$$

$$(A \wedge B)^\star :\equiv \forall X((A^\star \to (B^\star \to X)) \to X)$$

$$(A \vee B)^\star :\equiv \forall X((A^\star \to X) \to ((B^\star \to X) \to X)),$$

where X is a second-order propositional variable which does not occur in $A^\star$ or $B^\star$. The target language is the language of Girard's (polymorphic) system $\mathbf{F}$ (cf. [5]). It consists of the smallest class of expressions which includes the atomic formulas (propositional constants P, Q, R, ... and second-order propositional variables X, Y, Z, ...) and is closed under implication and second-order universal quantification. Note that the translation $A^\star$ of a propositional formula A is, clearly, a formula without second-order free variables. Prawitz's translation is actually an embedding of the propositional intuitionistic calculus into system $\mathbf{F}$ in the sense that if $\vdash_i A$ then $\vdash_{\mathbf{F}} A^\star$ (here $\vdash_i$ denotes provability in the intuitionistic propositional calculus and $\vdash_{\mathbf{F}}$ denotes provability in the system $\mathbf{F}$).

In 2006, the first author noticed (cf. [1]) that the above embedding still works if the target system $\mathbf{F}$ is restricted to a predicative system nowadays known as $\mathbf{F_{at}}$ (an acronym for *atomic polymorphism*). The atomic polymorphic system $\mathbf{F_{at}}$ has the same formulas as $\mathbf{F}$, but replaces the second-order universal elimination rule by a predicative variant. For definiteness, we describe the (natural deduction) rules of $\mathbf{F_{at}}$. The introduction rules are as in $\mathbf{F}$:

$$\dfrac{\begin{matrix}\langle A\rangle \\ \vdots \\ B\end{matrix}}{A \to B} \to\mathrm{I} \qquad\qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{\forall X.A} \forall\mathrm{I}$$

where the notation $\langle A\rangle$ says that the formula A is being discharged and, in the universal rule, X does not occur free in any undischarged hypothesis. The elimination rules of $\mathbf{F_{at}}$ are, however,

$$\dfrac{\begin{matrix}\vdots \\ A \to B\end{matrix} \qquad \begin{matrix}\vdots \\ A\end{matrix}}{B} \to\mathrm{E} \qquad\qquad \dfrac{\begin{matrix}\vdots \\ \forall X.A\end{matrix}}{A[C/X]} \forall\mathrm{E}$$

where C is an *atomic* formula (free for X in A), and $A[C/X]$ is the result of replacing in A all the free occurrences of X by C. Note that only atomic instantiations are permitted in the $\forall$E rule. This contrasts with the (impredicative) system $\mathbf{F}$, where C can be any formula.

The reason why, despite the restriction of the $\forall$E-rule, the system $\mathbf{F_{at}}$ is still able to embed full intuitionistic propositional calculus lies in the availability of *instantiation overflow*, i.e., for the three types of universal formulas occurring in Prawitz's translation, it is possible to derive in $\mathbf{F_{at}}$ the formulas resulting from instantiations of the second-order variable X by *any* formula, not only the atomic ones. For a complete description of instantiation overflow and of the embedding see [1, 2]. In the former reference, it is also shown that $\mathbf{F_{at}}$ has both the subformula property (for normal derivations) and an appropriate form of the disjunction property. (The notion of subformula only needs explanation for universal formulas. The proper subformulas of a formula of the form $\forall X.A[X]$ are the subformulas of the formulas of the form $A[C/X]$, for C an *atomic* formula free for X in A.) The latter reference is a study on the translation of the commuting conversions of the intuitionistic propositional calculus into $\mathbf{F_{at}}$. Note that, since the connectives $\perp$, $\vee$ and $\exists$ are absent from $\mathbf{F_{at}}$, this system has no commuting conversions. For more on $\mathbf{F_{at}}$, including a proof that the system is strongly normalizable for $\beta\eta$-conversions, see [3].

As we have discussed, Prawitz's translation $(\cdot)^\star$ gives a sound embedding of the intuitionistic propositional calculus into $\mathbf{F_{at}}$, that is: If $\vdash_i A$ then $\vdash_{\mathbf{F_{at}}} A^\star$. The translation is also faithful. I.e.:

If $\vdash_{\mathbf{F_{at}}} A^\star$ then $\vdash_i A$.

This latter fact was recently proved using a model-theoretic argument in [4]. In the present paper, we give a pure proof-theoretic proof of the faithfulness of $\mathbf{F_{at}}$. We believe that this approach is interesting in its own right. Furthermore, it shows how to obtain a proof-theoretic proof of the disjunction property for the intuitionistic propositional calculus via natural deduction *without* the need of commuting conversions. As we have suggested in previous papers (cf. [2, 3]), the need for the *ad hoc* commuting conversions is a reflection of the fact that we are not considering intuitionistic propositional logic in its proper setting, viz the wider setting of $\mathbf{F_{at}}$.

The paper is organized in three sections. After this introduction, Section 2 presents the new proof-theoretic proof of the faithfulness of $\mathbf{F_{at}}$. The alternative proof of the disjunction property of the intuitionistic propositional calculus is presented in Section 3.

2 A proof-theoretic proof of faithfulness

A second-order universal formula which is a subformula of a formula of the form $A^\star$ (A a propositional formula) must take one of three forms: $\forall X.X$, $\forall X((C^\star \to (D^\star \to X)) \to X)$ or $\forall X((C^\star \to X) \to ((D^\star \to X) \to X))$, with C and D propositional formulas. Hence, the following definition is in good standing:

Definition 2.1. Let A be a propositional formula. For B any subformula of $A^\star$, we define a formula $\tilde{B}$ in the language of propositional calculus ($\bot$, $\wedge$, $\vee$, $\to$) *extended with* second-order variables (but without second-order quantifications) in the following way:

If B is atomic, then $\tilde{B} :\equiv B$.

If $B :\equiv C \to D$, then $\tilde{B} :\equiv \tilde{C} \to \tilde{D}$.

If $B :\equiv \forall X.X$, then $\tilde{B} :\equiv \bot$.

If $B :\equiv \forall X((C^\star \to (D^\star \to X)) \to X)$, then $\tilde{B} :\equiv C \wedge D$.

If $B :\equiv \forall X((C^\star \to X) \to ((D^\star \to X) \to X))$, then $\tilde{B} :\equiv C \vee D$.

Note that B and $\tilde{B}$ have the same free variables. Also, when C is a propositional formula, $\widetilde{C^\star}$ is just C.

Lemma 2.2. Let Γ be a tuple of formulas in $\mathbf{F_{at}}$ and A be a formula in $\mathbf{F_{at}}$ with their free variables among the variables in $\bar{X}$. If there is a proof (say $\mathcal{D}$) in $\mathbf{F_{at}}$ of $A[\bar{X}]$ from $\Gamma[\bar{X}]$ in which all formulas (occurring in $\mathcal{D}$ and $\Gamma[\bar{X}]$) are subformulas of formulas of the form $D^\star$ (D a propositional formula), then

$$\tilde{\Gamma}[\bar{F}/\bar{X}] \vdash_i \tilde{A}[\bar{F}/\bar{X}]$$

for any tuple of propositional formulas $\bar{F}$. For $\Gamma[\bar{X}] :\equiv A_1[\bar{X}], \ldots, A_n[\bar{X}]$, $\tilde{\Gamma}[\bar{F}/\bar{X}]$ denotes the tuple of propositional formulas $\tilde{A}_1[\bar{F}/\bar{X}]$, $\ldots$, $\tilde{A}_n[\bar{F}/\bar{X}]$. (Of course, the reading of $\tilde{A}[\bar{F}/\bar{X}]$ is to first consider the transformed formula $\tilde{A}$ and, afterwards, effect the substitution $[\bar{F}/\bar{X}]$ in it. The alternative reading does not make sense in general.)

Proof. By induction on the length of the derivation $\mathcal{D}$.

If $\mathcal{D}$ is a one node proof-tree, then $A[\bar{X}]$ is in $\Gamma[\bar{X}]$. The result is trivial since for any tuple $\bar{F}$ of propositional formulas we have $\tilde{A}[\bar{F}/\bar{X}] \vdash_i \tilde{A}[\bar{F}/\bar{X}]$.

• Case where the last rule is a $\to$I:

$$\dfrac{\begin{array}{c}\langle A[\bar{X}]\rangle \qquad \Gamma[\bar{X}] \\ \vdots \\ B[\bar{X}]\end{array}}{A[\bar{X}] \to B[\bar{X}]}$$

Fix $\bar{F}$ a tuple of propositional formulas. The aim is to prove that $\tilde{\Gamma}[\bar{F}/\bar{X}] \vdash_i \tilde{A}[\bar{F}/\bar{X}] \to \tilde{B}[\bar{F}/\bar{X}]$. According to the induction hypothesis, we have $\tilde{A}[\bar{F}/\bar{X}], \tilde{\Gamma}[\bar{F}/\bar{X}] \vdash_i \tilde{B}[\bar{F}/\bar{X}]$. Thus, adding an introduction rule for implication which discharges $\tilde{A}[\bar{F}/\bar{X}]$, we get the desired result.

• Case where the last rule is a $\to$E:

$$\dfrac{\begin{array}{c}\Gamma[\bar{X}] \\ \vdots \\ A[\bar{X}]\end{array} \qquad \begin{array}{c}\Gamma[\bar{X}] \\ \vdots \\ A[\bar{X}] \to B[\bar{X}]\end{array}}{B[\bar{X}]}$$

Fix $\bar{F}$ a tuple of propositional formulas. By induction hypothesis, we have both $\tilde{\Gamma}[\bar{F}/\bar{X}] \vdash_i \tilde{A}[\bar{F}/\bar{X}]$ and $\tilde{\Gamma}[\bar{F}/\bar{X}] \vdash_i \tilde{A}[\bar{F}/\bar{X}] \to \tilde{B}[\bar{F}/\bar{X}]$. Applying the elimination rule for implication, we get $\tilde{\Gamma}[\bar{F}/\bar{X}] \vdash_i \tilde{B}[\bar{F}/\bar{X}]$.

• Case where the last rule is a $\forall$I:

$$\dfrac{\begin{array}{c}\Gamma[\bar{Y}] \\ \vdots \\ A[\bar{Y}, X]\end{array}}{\forall X. A[\bar{Y}, X]}$$

Since $\forall X. A[\bar{Y}, X]$ is a subformula of a translated formula $D^\star$, with D a propositional formula, we know that only three cases may occur: (i) A is X; (ii) A has the form $(C^\star \to (E^\star \to X)) \to X$ or (iii) A has the form $(C^\star \to X) \to ((E^\star \to X) \to X)$ with C and E propositional formulas. In any of the cases, the only free variable in A is X. So, in the scheme above, $A[\bar{Y}, X]$ and $\forall X. A[\bar{Y}, X]$ may be replaced by $A[X]$ and $\forall X. A[X]$ respectively.

In case (i), fix $\bar{F}$ a tuple of propositional formulas and let us prove that $\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i \bot$. By induction hypothesis we know that $\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i X[G/X]$ for every propositional formula G. Just take G as being $\bot$.

In case (ii), we need to prove that $\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i C \wedge E$, for every tuple $\bar{F}$ of propositional formulas. Fix $\bar{F}$. By induction hypothesis, we know that $\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i \tilde{A}[G/X]$ for any propositional formula G. In particular, for $G :\equiv C \wedge E$, we have

$$\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i (C \to (E \to C \wedge E)) \to C \wedge E.$$

Thus, in the natural deduction calculus for the intuitionistic propositional calculus, we have the following proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\langle C\rangle \qquad \langle E\rangle}{C \wedge E}}{E \to C \wedge E}}{C \to (E \to C \wedge E)} \qquad \begin{array}{c}\tilde{\Gamma}[\bar{F}/\bar{Y}]\\ \vdots \\ (C \to (E \to C \wedge E)) \to C \wedge E\end{array}}{C \wedge E}$$

Therefore, $\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i C \wedge E$.

In case (iii), we need to prove that $\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i C \vee E$, for every tuple $\bar{F}$ of propositional formulas. Fix $\bar{F}$. By induction hypothesis, we know that $\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i \tilde{A}[G/X]$, for any propositional formula G. In particular, for $G :\equiv C \vee E$, we have

$$\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i (C \to C \vee E) \to ((E \to C \vee E) \to C \vee E).$$

Thus, in the intuitionistic propositional calculus, we have the following proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\langle C\rangle}{C \vee E}}{C \to C \vee E} \qquad \begin{array}{c}\tilde{\Gamma}[\bar{F}/\bar{Y}]\\ \vdots \\ (C \to C \vee E) \to ((E \to C \vee E) \to C \vee E)\end{array}}{(E \to C \vee E) \to C \vee E} \qquad \dfrac{\dfrac{\langle E\rangle}{C \vee E}}{E \to C \vee E}}{C \vee E}$$

Therefore, $\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i C \vee E$.

- Case where the last rule is a $\forall$E:

$$\dfrac{\begin{array}{c}\Gamma[\bar{Y}]\\ \vdots \\ \forall X.A[X, \bar{Y}]\end{array}}{A[C/X, \bar{Y}]}$$

with C an atomic formula in $\mathbf{F_{at}}$, i.e., C is a propositional constant or a second-order variable. We assume w.l.o.g that if C is a second-order variable then C is among the variables $\bar{Y}$, say Y_i.

By hypothesis, since $\forall X.A[X, \bar{Y}]$ is a subformula of a translated formula, we know that this formula falls into one of the following three cases: (i) it is the translation of $\bot$; (ii) it is the translation of a conjunction; or (iii) it is the translation of a disjunction. Moreover, $\forall X.A[X, \bar{Y}]$ has no free variables and so, in the scheme above we can replace $\forall X.A[X, \bar{Y}]$ and $A[C/X, \bar{Y}]$ by $\forall X.A[X]$ and $A[C/X]$, respectively.

In case (i), we have the following proof in $\mathbf{F_{at}}$

$$\dfrac{\begin{array}{c}\Gamma[\bar{Y}]\\ \vdots \\ \forall X.X\end{array}}{C}$$

and we want to prove that $\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i C[F_i/Y_i]$, for any tuple $\bar{F}$ of propositional formulas. By F_i we denote the formula of the tuple $\bar{F}$ which instantiates Y_i in $\tilde{\Gamma}[\bar{F}/\bar{Y}]$.

Fix $\bar{F}$. By induction hypothesis we know that $\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i \bot$. As a consequence, in the intuitionistic propositional calculus we have the following proof

$$\dfrac{\begin{array}{c}\tilde{\Gamma}[\bar{F}/\bar{Y}]\\ \vdots \\ \bot\end{array}}{C[F_i/Y_i]}$$

Hence, $\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i C[F_i/Y_i]$.

In case (ii), we have the following proof in $\mathbf{F_{at}}$

$$\dfrac{\begin{array}{c}\Gamma[\bar{Y}]\\ \vdots \\ \forall X((H^\star \to (E^\star \to X)) \to X)\end{array}}{(H^\star \to (E^\star \to C)) \to C}$$

We want to prove that $\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i (H \to (E \to C[F_i/Y_i])) \to C[F_i/Y_i]$, for any tuple $\bar{F}$ of propositional formulas. Fix $\bar{F}$. By induction hypothesis we know that $\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i H \wedge E$. Thus, we have the following proof in the intuitionistic propositional calculus

$$
\dfrac{\dfrac{\langle H \to (E \to C[F_i/Y_i])\rangle \qquad \dfrac{\begin{array}{c}\tilde{\Gamma}[\bar{F}/\bar{Y}]\\ \vdots \\ H \wedge E\end{array}}{H}}{E \to C[F_i/Y_i]} \qquad \dfrac{\begin{array}{c}\tilde{\Gamma}[\bar{F}/\bar{Y}]\\ \vdots \\ H \wedge E\end{array}}{E}}{\dfrac{C[F_i/Y_i]}{(H \to (E \to C[F_i/Y_i])) \to C[F_i/Y_i]}}
$$

This is what we want.

In case (iii), we have the following proof in $\mathbf{F_{at}}$

$$
\dfrac{\begin{array}{c}\Gamma[\bar{Y}]\\ \vdots \\ \forall X((H^\star \to X) \to ((E^\star \to X) \to X))\end{array}}{(H^\star \to C) \to ((E^\star \to C) \to C)}
$$

Given any tuple $\bar{F}$ of propositional formulas, the aim is to show that $\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i (H \to C[F_i/Y_i]) \to ((E \to C[F_i/Y_i]) \to C[F_i/Y_i])$. Fix $\bar{F}$. By induction hypothesis, $\tilde{\Gamma}[\bar{F}/\bar{Y}] \vdash_i H \vee E$. Thus, we have the following proof in the intuitionistic propositional calculus

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\tilde{\Gamma}[\bar{F}/\bar{Y}]\\ \vdots \\ H \vee E\end{array} \qquad \dfrac{\langle H \to C[F_i/Y_i]\rangle \qquad \langle H\rangle}{C[F_i/Y_i]} \qquad \dfrac{\langle E \to C[F_i/Y_i]\rangle \qquad \langle E\rangle}{C[F_i/Y_i]}}{C[F_i/Y_i]}}{(E \to C[F_i/Y_i]) \to C[F_i/Y_i]}}{(H \to C[F_i/Y_i]) \to ((E \to C[F_i/Y_i]) \to C[F_i/Y_i])}
$$

We are done. □

Theorem 2.3 (Faithfulness)**.** *Let* $\Gamma :\equiv A_1, \ldots, A_n$ *and* A *be propositional formulas and consider their translations* $\Gamma^\star :\equiv A_1^\star, \ldots, A_n^\star$ *and* $A^\star$ *into* $\mathbf{F_{at}}$.

If $\Gamma^\star \vdash_{\mathbf{F_{at}}} A^\star$ then $\Gamma \vdash_i A$.

Proof. Suppose that $\Gamma^\star \vdash_{\mathbf{F_{at}}} A^\star$. Since $\mathbf{F_{at}}$ has the normalization property (see [3]), we know that there is a proof, say $\mathcal{D}$, in normal form of $A^\star$ with premises $\Gamma^\star$. By the subformula property (see [1], page 5), all formulas that occur in $\mathcal{D}$ are subformulas of $A^\star$ or are subformulas of formulas in $\Gamma^\star$. Therefore, we are in the conditions of application of Lemma 2.2. Applying such lemma, we conclude that $\widetilde{\Gamma^\star} \vdash_i \widetilde{A^\star}$, i.e., $\Gamma \vdash_i A$. □

3 Application

An advantage of having a sound and faithful embedding between two systems is the possibility to transfer certain results from one system to the other. In this section, as an application of the (proof-theoretic proof of the) faithfulness of $\mathbf{F_{at}}$, we give a new proof of the disjunction property of the intuitionistic propositional calculus. Note that the usual proof-theoretic proof of the disjunction property requires the introduction of extra conversions associated with the connectives $\perp$ and $\vee$: the so called *commuting conversions* or *permutative conversions*. They are needed to ensure that a proof in normal form has the subformula property. The proof-theoretic proof that we present below does not rely on commuting conversions.

Theorem 3.1. *If* $\vdash_i A \vee B$ *then* $\vdash_i A$ *or* $\vdash_i B$*.*

Proof. Suppose that $\vdash_i A \vee B$. Since the embedding of the full intuitionistic propositional calculus into $\mathbf{F_{at}}$ is sound, we have $\vdash_{\mathbf{F_{at}}} (A \vee B)^\star$. Applying the disjunction property of $\mathbf{F_{at}}$ (see [1], pages 5-7), we know that $\vdash_{\mathbf{F_{at}}} A^\star$ or $\vdash_{\mathbf{F_{at}}} B^\star$. By Theorem 2.3 (faithfulness), we conclude $\vdash_i A$ or $\vdash_i B$. □

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