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The faithfulness of atomic polymorphism

Abstract. It is known that the full intuitionistic propositional calculus can be embedded into the atomic polymorphic system $\mathbf{F_{at}}$, a calculus with only two connectives: the conditional and the second-order universal quantifier. The embedding uses a translation of formulas due to Prawitz and relies on the so-called property of instantiation overflow. In this paper, we show that the previous embedding is faithful i.e., if a translated formula is derivable in $\mathbf{F_{at}}$, then the original formula is already derivable in the propositional calculus.

Keywords: Predicative polymorphism. Faithfulness. Natural deduction. Kripke models. Intuitionistic propositional calculus.

Introduction

In order to have a decent theory of normalization for the natural deduction calculus in the full language of intuitionistic propositional logic, there is a need to introduce extra conversions pertaining to the connectives $\perp$ and $\vee$: the *commuting conversions*, also known as *permutative conversions*. In [4], Jean-Yves Girard says that "the elimination rules [of $\perp$ and $\vee$] are very bad" and criticizes the commuting conversions by saying that "one tends to think that natural deduction should be modified to correct such atrocities."

In 2006, the first author suggested a way of avoiding the *bad* connectives and, consequently, the commuting conversions of the intuitionistic propositional calculus. The suggestion was to embed the full intuitionistic propositional calculus into $\mathbf{F_{at}}$, a calculus with only two connectives: the conditional and the second-order universal quantifier.[1] The embedding relies on Prawitz's definition of the bad connectives given in [5] and the novelty lies in the fact that the target calculus ($\mathbf{F_{at}}$) is predicative: only atomic instantiations of the second-order quantifiers are allowed. According to the embedding, if A is derivable in the intuitionistic propositional calculus and A^* is its translation into $\mathbf{F_{at}}$, then A^* is derivable in $\mathbf{F_{at}}$. In this paper,

Presented by **Name of Editor**; *Received* December 1, 2005

[1]In [1], the calculus was named *atomic* $\mathsf{PSOL}^{\mathsf{i}}$. The more modern designation of $\mathbf{F_{at}}$ comes from *atomic polymorphism*, a restriction of Girard's system $\mathbf{F}$ to atomic instantiations: cf. title of [3].

we prove the converse of this statement. Our proof uses a simple model-theoretic argument. It is also possible to give a purely proof-theoretic proof, but we do not include it in this paper.

The body of the paper consists of two sections. In Section 1, we briefly describe the atomic polymorphic system $\mathbf{F_{at}}$ and the embedding of the full intuitionistic propositional calculus into $\mathbf{F_{at}}$. Section 2 is dedicated to the proof of the faithfulness of the embedding.

1. An overview of the embedding

We start by briefly describing the atomic polymorphic system $\mathbf{F_{at}}$. The syntax of the calculus consists of propositional constants (denoted by P, Q, R, …), second-order variables (denoted by X, Y, Z, …), two primitive logical connectives, implication and the second-order universal quantifier, and punctuation signs. The formulas of $\mathbf{F_{at}}$ are defined inductively by:

i) Propositional constants and second-order variables are atomic formulas. Atomic formulas are formulas.
ii) If A and B is a formula then $A \to B$ is a formula.
iii) If A is a formula and X is a second-order variable then $\forall X.A$ is a formula.

The logic of $\mathbf{F_{at}}$ is *intuitionistic logic.* We formulate the intuitionistic deduction system by the *natural deduction calculus* with the usual *introduction* rules for the conditional and the second-order universal quantifier:

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \to B}\ {\to}\mathrm{I} \qquad\qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{\forall X.A}\ \forall\mathrm{I}$$

where, in the universal rule, X does not occur free in any undischarged hypothesis. It also has *elimination* rules:

$$\dfrac{\begin{matrix}\vdots \\ A \to B\end{matrix} \qquad \begin{matrix}\vdots \\ A\end{matrix}}{B}\ {\to}\mathrm{E} \qquad\qquad \dfrac{\begin{matrix}\vdots \\ \forall X.A\end{matrix}}{A[C/X]}\ \forall\mathrm{E}$$

where C is an *atomic* formula (free for X in A), and $A[C/X]$ is the result of replacing in A all the free occurrences of X by C.

Note that what distinguishes $\mathbf{F_{at}}$ from Girard's well-known polymorphic system $\mathbf{F}$ (see [4]) is the restriction to atomic formulas in the $\forall$E rule.

Whereas in system **F** the instantiation of X can be done by any formula of the system, in $\mathbf{F_{at}}$ the instantiation is *restricted* to the atomic formulas. This feature explains the predicativity of the system and, as opposed to Girard's system, embodies $\mathbf{F_{at}}$ with a natural notion of subformula. For more on $\mathbf{F_{at}}$, including the subformula property and the disjunction property, see [1]. The reader can find in [3] a proof of the strong normalization property of the system $\mathbf{F_{at}}$.

Although $\mathbf{F_{at}}$ is a severe restriction of system **F**, it is not as weak as we might at first be led to think. As remarked, it is flexible enough to embed the full intuitionistic propositional calculus. The reason lies in a phenomenon, dubbed *instantiation overflow*, which ensures that for formulas A with a certain structure, we can instantiate $\forall X.A$ by *any* formula of the language whatsoever.

More precisely, from formulas of the form

$$\forall X.X$$
$$\forall X((A \to (B \to X)) \to X)$$
$$\forall X((A \to X) \to ((B \to X) \to X)),$$

it is possible to deduce in $\mathbf{F_{at}}$ (respectively)

$$C$$
$$(A \to (B \to C)) \to C$$
$$(A \to C) \to ((B \to C) \to C),$$

for *any* (not necessarily atomic) formula C. The proof of instantiation overflow was given in [2] and, in fact, yields algorithmic methods for obtaining the three kinds of deductions above.

With the property of instantiation overflow in place, it is easy to see that the full intuitionistic propositional calculus can be embedded into $\mathbf{F_{at}}$. For a detailed proof of the embedding, see [1, 2]. The embedding uses a well-known translation of formulas due to Prawitz [5]: namely, for every formula A of the full propositional calculus, we define its translation A^* into $\mathbf{F_{at}}$ inductively as follows:

$(P)^* :\equiv P$, for P a propositional constant

$$(\bot)^* :\equiv \forall X.X$$
$$(A \to B)^* :\equiv A^* \to B^*$$
$$(A \wedge B)^* :\equiv \forall X((A^* \to (B^* \to X)) \to X)$$

$(A \vee B)^* :\equiv \forall X((A^* \to X) \to ((B^* \to X) \to X)),$

where X is a second-order variable which does not occur in A^* or B^*. As remarked, the embedding is sound, i.e., we have:

If $\vdash_i A$ then $\vdash_{\mathbf{F_{at}}} A^*$.

In the next section we prove the converse of this result, viz. that the embedding is faithful:

If $\vdash_{\mathbf{F_{at}}} A^*$ then $\vdash_i A$.

2. Faithfulness

Definition 1. *A* Kripke structure *is a triple* $\mathcal{K} = (K, \leq, \Vdash)$ *where:*

i) $(K, \leq)$ *is a non empty partial order,*

ii) $\Vdash \subseteq K \times \mathcal{P}$ *(with* $\mathcal{P}$ *the set of propositional constants) is such that*

if $(k, P) \in \Vdash$ *and* $k \leq k'$ *then* $(k', P) \in \Vdash$ *(monotonicity)*

Instead of $(k, P) \in \Vdash$, it is usual to use infix notation and write $k \Vdash P$, which we read as "k forces P." It is well-known that the forcing relation $\Vdash$ can be extended to all formulas of the intuitionistic propositional calculus:

Definition 2. *Let* $\mathcal{K} = (K, \leq, \Vdash)$ *be a Kripke structure,* $k \in K$ *and* A *a formula of propositional calculus. We define* $k \Vdash A$ *(read "k forces A") in the following inductive way:*

a) $k \Vdash P$ *is, by definition,* $(k, P) \in \Vdash$

b) $k \Vdash A \wedge B$ *is* $k \Vdash A$ *and* $k \Vdash B$

c) $k \Vdash A \vee B$ *is* $k \Vdash A$ *or* $k \Vdash B$

d) $k \Vdash A \to B$ *is: for all* $k' \in K$*, if* $k \leq k'$ *and* $k' \Vdash A$*, then* $k' \Vdash B$

e) No $k \in K$ *forces* $\perp$*.*

As it is well-known, the monotonicity of the forcing relation extends to any formula A of the propositional calculus, i.e.,

$$\forall k, k' \in K \ (k \Vdash A \wedge k \leq k' \Rightarrow k' \Vdash A).$$

The following definition is standard:

DEFINITION 3. *Let $\mathcal{K} = (K, \leq, \Vdash)$ be a Kripke structure, Γ a set of formulas of the propositional calculus and A a formula of the propositional calculus.*

- *We say that A* is forced by (or valid in) $\mathcal{K}$ *(denoted by $\mathcal{K} \Vdash A$) if*

$$\forall k \in K \ (k \Vdash A);$$

- *We say that $\Gamma \Vdash A$ if all Kripke structures which force the formulas in Γ also force the formula A.*

The next result is widely known and its proof can be found in many textbooks in Mathematical Logic/Proof Theory. See, for example, [6].

PROPOSITION 1 (Soundness and completeness). *Let Γ, A be respectively a set of formulas, a formula, of the propositional calculus. Then,*

$$\Gamma \vdash_i A \ \text{ if, and only if, } \ \Gamma \Vdash A.$$

We are going to *extend* the definition of Kripke structure in order to include the second-order universal quantifier. It is very important to notice that, although the extended definition covers the language of $\mathbf{F_{at}}$, it goes beyond it because it includes the *primitive symbols* of the propositional calculus, viz. $\bot$, $\wedge$ and $\vee$.

f) $k \Vdash \forall X.A$ is: for all formulas F of the full propositional calculus (i.e., formulas obtained by means of the propositional constants and the *primitive* connectives $\bot$, $\wedge$, $\vee$, $\to$) and for all $k' \in K$, if $k \leq k'$ then $k' \Vdash A[F/X]$.

The previous forcing relation, which incorporates second-order universal quantifications, is denoted by $\Vdash^2$ in order to distinguish it from the usual forcing relation applied just to the formulas of the propositional calculus. Notice that the above inductive definition is in good standing since it is defined in terms of less complex sentences. The measure of complexity of a formula can be taken to be the ordinal

$$(\text{number of } 2^{\text{nd}}\text{-order quantifiers})\omega + (\text{number of } 1^{\text{st}}\text{-order connectives}).$$

It is clear that the monotonicity property is preserved by $\Vdash^2$ and that, for sentences in which the second-order quantifier does not occur, the relations $\Vdash^2$ and $\Vdash$ coincide.

We now mimick Definition 3:

Definition 4. *Let $\mathcal{K} = (K, \leq, \Vdash^2)$ be a Kripke structure, Γ a set of sentences of $\mathbf{F_{at}}$ and A a sentence of $\mathbf{F_{at}}$.*

- *We say that A* is forced by (or valid in) $\mathcal{K}$ *(denoted by $\mathcal{K} \Vdash^2 A$) if*

$$\forall k \in K \ (k \Vdash^2 A);$$

- *We say that $\Gamma \Vdash^2 A$, if all Kripke structures which force the sentences in Γ also force the sentence A.*

Note that the previous definition can be extended to sentences in the extended language with the primitive connectives $\bot$, $\wedge$, $\vee$, $\rightarrow$, second-order $\forall$. We will, in fact, use this extension.

Lemma 1. *Let $\mathcal{K} = (K, \leq, \Vdash)$ be a Kripke structure, A a formula of the propositional calculus and A^* its translation into $\mathbf{F_{at}}$. Then,*

$$\mathcal{K} \Vdash^2 A^* \text{ if, and only if, } \mathcal{K} \Vdash A.$$

Proof. We are going to prove, by induction on the complexity of the formula A, that for all $k \in K$, $k \Vdash^2 A^*$ iff $k \Vdash A$.

If A is a propositional constant, then $A^* \equiv A$. Thus, the result is immediate.

Case $A \equiv \bot$. In this case $A^* \equiv \forall X.X$. It is easy to see that no node of $\mathcal{K}$ forces $\forall X.X$ (otherwise, it would force $\bot$, which is impossible). Therefore, the equivalence is true, since both sides are false.

Case $A \equiv A_1 \wedge A_2$. Let us prove the left-to-right implication. Fix $k \in K$ such that $k \Vdash^2 A^*$, i.e., $k \Vdash^2 \forall X((A_1^* \rightarrow (A_2^* \rightarrow X)) \rightarrow X)$. In particular $k \Vdash^2 (A_1^* \rightarrow (A_2^* \rightarrow A_1 \wedge A_2)) \rightarrow A_1 \wedge A_2$. In order to prove that $k \Vdash A_1 \wedge A_2$, it suffices to show that $k \Vdash^2 A_1^* \rightarrow (A_2^* \rightarrow A_1 \wedge A_2)$. This is clear using the induction hypothesis and the monotonicity of the forcing relation.

For the right-to-left implication, fix $k \in K$ such that $k \Vdash A_1 \wedge A_2$, i.e. $k \Vdash A_1$ and $k \Vdash A_2$. By induction hypothesis we know that $k \Vdash^2 A_1^*$ and $k \Vdash^2 A_2^*$. We want to prove that $k \Vdash^2 \forall X((A_1^* \rightarrow (A_2^* \rightarrow X)) \rightarrow X)$, i.e., that for all $k' \geq k$ and for all formula F of the propositional calculus we have $k' \Vdash^2 (A_1^* \rightarrow (A_2^* \rightarrow F)) \rightarrow F$. This is now clear, using the monotonicity of the forcing relation.

Case $A \equiv A_1 \vee A_2$. Let us first consider the left-to-right implication. Suppose that $k \Vdash^2 \forall X((A_1^* \rightarrow X) \rightarrow ((A_2^* \rightarrow X) \rightarrow X))$. We want to show that $k \Vdash A_1 \vee A_2$. By the supposition, we have

$$k \Vdash^2 (A_1^* \rightarrow A_1 \vee A_2) \rightarrow ((A_2^* \rightarrow A_1 \vee A_2) \rightarrow A_1 \vee A_2).$$

Using the induction hypothesis, it is clear that $k \Vdash^2 A_1^* \to A_1 \vee A_2$ and $k \Vdash^2 A_2^* \to A_1 \vee A_2$. It readily follows the desired conclusion.

For the right-to-left implication, suppose that $k \Vdash A_1 \vee A_2$. Then, $k \Vdash A_1$ or $k \Vdash A_2$. Suppose, without loss of generality, that $k \Vdash A_1$. By induction hypothesis $k \Vdash^2 A_1^*$. We want to show that

$$k \Vdash^2 \forall X((A_1^* \to X) \to ((A_2^* \to X) \to X)),$$

i.e. that $k \Vdash^2 (A_1^* \to F) \to ((A_2^* \to F) \to F)$ for all formulas F of the propositional calculus. This is clear using the induction hypothesis (i.e., that $k \Vdash^2 A_1^*$) and the monotonicity of the forcing relation.

Case $A \equiv A_1 \to A_2$.

$$\begin{aligned} k \Vdash^2 A_1^* \to A_2^* &\equiv \forall k' \geq k\ (k' \Vdash^2 A_1^* \Rightarrow k' \Vdash^2 A_2^*) \\ &\overset{H.I.}{\Leftrightarrow} \forall k' \geq k\ (k' \Vdash A_1 \Rightarrow k' \Vdash A_2) \\ &\equiv k \Vdash A_1 \to A_2. \end{aligned}$$

■

PROPOSITION 2 (Soundness). *Let* Γ *be a set of formulas in* $\mathbf{F_{at}}$ *and* A *be a formula in* $\mathbf{F_{at}}$ *with the free-variables among the variables in* $\bar{X}$.

If $\Gamma[\bar{X}] \vdash_{\mathbf{F_{at}}} A[\bar{X}]$ then $\Gamma[\bar{F}/\bar{X}] \Vdash^2 A[\bar{F}/\bar{X}]$,

for any tuple of formulas $\bar{F}$ *of the propositional calculus.*

PROOF. The proof is by induction on the length of the derivation.

• The base case is $A[\bar{X}] \vdash_{\mathbf{F_{at}}} A[\bar{X}]$, for a certain formula A. There is nothing to argue in this case.

• Case where the last rule is a $\to$I:

$$\dfrac{\begin{array}{cc} \langle A[\bar{X}] \rangle & \Gamma[\bar{X}] \\ & \vdots \\ & B[\bar{X}] \end{array}}{A[\bar{X}] \to B[\bar{X}]}$$

where the angle-brackets surrounding the formula $A[\bar{X}]$ mean that this formula is discharged.

Fix $\bar{F}$ a tuple of formulas of the propositional calculus and let us prove that $\Gamma[\bar{F}/\bar{X}] \Vdash^2 A[\bar{F}/\bar{X}] \to B[\bar{F}/\bar{X}]$. Take $\mathcal{K} = (K, \leq, \Vdash)$ a Kripke structure such that $\mathcal{K} \Vdash^2 \Gamma[\bar{F}/\bar{X}]$. We need to prove that $k \Vdash^2 A[\bar{F}/\bar{X}] \to B[\bar{F}/\bar{X}]$, for all $k \in K$. Fix $k \in K$ and $k' \in K$ such that $k' \geq k$ and $k' \Vdash^2 A[\bar{F}/\bar{X}]$ and let us prove that $k' \Vdash^2 B[\bar{F}/\bar{X}]$. Given that $A[\bar{X}], \Gamma[\bar{X}] \vdash_{\mathbf{F_{at}}} B[\bar{X}]$ then, by induction hypothesis, we have $A[\bar{F}/\bar{X}], \Gamma[\bar{F}/\bar{X}] \Vdash^2 B[\bar{F}/\bar{X}]$. Consider $\mathcal{K}' = (K', \leq, \Vdash)$ the Kripke structure that results from $\mathcal{K}$ with $K' \subseteq K$ the set of nodes of K which are greater than or equal to k'. Since $k' \Vdash^2 \Gamma[\bar{F}/\bar{X}]$ and $k' \Vdash^2 A[\bar{F}/\bar{X}]$, by monotonicity, $\mathcal{K}'$ forces $\Gamma[\bar{F}/\bar{X}]$, $A[\bar{F}/\bar{X}]$. By induction hypothesis $\mathcal{K}' \Vdash^2 B[\bar{F}/\bar{X}]$. Therefore $k' \Vdash^2 B[\bar{F}/\bar{X}]$.

• Case where the last rule is a $\to$E:

$$\dfrac{\begin{array}{c}\Gamma[\bar{X}]\\ \vdots \\ A[\bar{X}]\end{array} \qquad \begin{array}{c}\Gamma[\bar{X}]\\ \vdots \\ A[\bar{X}] \to B[\bar{X}]\end{array}}{B[\bar{X}]}$$

We must show that if $\bar{F}$ is a tuple of formulas of the propositional calculus and $\mathcal{K}$ is a Kripke structure which forces $\Gamma[\bar{F}/\bar{X}]$, then $\mathcal{K} \Vdash^2 B[\bar{F}/\bar{X}]$. Well, by induction hypothesis, $\Gamma[\bar{F}/\bar{X}] \Vdash^2 A[\bar{F}/\bar{X}]$ and $\Gamma[\bar{F}/\bar{X}] \Vdash^2 A[\bar{F}/\bar{X}] \to B[\bar{F}/\bar{X}]$. Hence, $\mathcal{K} \Vdash^2 A[\bar{F}/\bar{X}]$ and $\mathcal{K} \Vdash^2 A[\bar{F}/\bar{X}] \to B[\bar{F}/\bar{X}]$. Therefore $\mathcal{K} \Vdash^2 B[\bar{F}/\bar{X}]$.

• Case where the last rule is a $\forall$I:

$$\dfrac{\begin{array}{c}\Gamma[\bar{Y}]\\ \vdots \\ A[\bar{Y}, X]\end{array}}{\forall X.A[\bar{Y}, X]}$$

where X is not one of the $\bar{Y}$.

Fix $\bar{F}$ a tuple of formulas of the propositional calculus. We argue that $\Gamma[\bar{F}/\bar{Y}] \Vdash^2 \forall X.A[\bar{F}/\bar{Y}, X]$. Let $\mathcal{K} = (K, \leq, \Vdash)$ be a Kripke structure with $\mathcal{K} \Vdash^2 \Gamma[\bar{F}/\bar{Y}]$ and fix $k \in K$. We must show that $k \Vdash^2 \forall X.A[\bar{F}/\bar{Y}, X]$, i.e., that $k' \Vdash^2 A[\bar{F}/\bar{Y}, G/X]$ for all $k' \geq k$ and G a formula of the propositional calculus. By induction hypothesis, $\Gamma[\bar{F}/\bar{Y}] \Vdash^2 A[\bar{F}/\bar{Y}, G/X]$. Therefore $\mathcal{K} \Vdash^2 A[\bar{F}/\bar{Y}, G/X]$, which implies $k' \Vdash^2 A[\bar{F}/\bar{Y}, G/X]$.

• Case where the last rule is a $\forall$E:

$$\dfrac{\begin{array}{c}\Gamma[\bar{Y}]\\ \vdots \\ \forall X.A[X, \bar{Y}]\end{array}}{A[C/X, \bar{Y}]}$$

with C an atomic formula. Note that C is either a propositional constant or a propositional variable. In the second case, we may suppose (without loss of generality) that C is among the variables $\bar{Y}$.

Fix $\bar{F}$ formulas of the propositional calculus. We must show that i) $\Gamma[\bar{F}/\bar{Y}] \Vdash^2 A[C/X, \bar{F}/\bar{Y}]$ if C is a propositional constant, and ii) $\Gamma[\bar{F}/\bar{Y}] \Vdash^2 A[G/X, \bar{F}/\bar{Y}]$ if C is a second-order variable. Note that, in the latter case, G is the relevant propositional formula of the tuple $\bar{F}$. Let $\mathcal{K} = (K, \leq, \Vdash)$ be a Kripke structure such that $\mathcal{K} \Vdash^2 \Gamma[\bar{F}/\bar{Y}]$. Take $k \in K$ and let us see that in i) $k \Vdash^2 A[C/X, \bar{F}/\bar{Y}]$, and in ii) $k \Vdash^2 A[G/X, \bar{F}/\bar{Y}]$. By induction hypothesis, $\Gamma[\bar{F}/\bar{Y}] \Vdash^2 \forall X.A[X, \bar{F}/\bar{Y}]$. So $\mathcal{K} \Vdash^2 \forall X.A[X, \bar{F}/\bar{Y}]$ which implies that $k \Vdash^2 \forall X.A[X, \bar{F}/\bar{Y}]$. In particular, i) if C is a propositional constant (and so, a formula of the propositional calculus) we have $k \Vdash^2 A[C/X, \bar{F}/\bar{Y}]$ and ii) if C is a second-order variable, we have $k \Vdash^2 A[G/X, \bar{F}/\bar{Y}]$. ■

THEOREM 1 (Faithfulness). *Let $\Gamma = A_1, \ldots, A_n$ and A be formulas of the full propositional calculus ($\bot$, $\wedge$, $\vee$, $\to$) and consider their translations $\Gamma^* :\equiv A_1^*, \ldots, A_n^*$ and A^* into $\mathbf{F_{at}}$.*

If $\Gamma^* \vdash_{\mathbf{F_{at}}} A^*$ then $\Gamma \vdash_i A$.

PROOF. Suppose that $\Gamma \nvdash_i A$. By completeness (Proposition 1), there is a Kripke structure $\mathcal{K} = (K, \leq, \Vdash)$ for the intuitionistic propositional calculus such that $\mathcal{K}$ forces all formulas in Γ but $\mathcal{K} \nVdash A$. Extend the forcing relation $\Vdash$ as shown in the beginning of this section (to $\Vdash^2$) in order to include the formulas of $\mathbf{F_{at}}$. By Lemma 1, $\mathcal{K}$ forces all formulas in Γ^* and $\mathcal{K} \nVdash^2 A^*$. Therefore, by Proposition 2, $\Gamma^* \nvdash_{\mathbf{F_{at}}} A^*$. ■

Acknowledgements. Both authors acknowledge the support of Centro de Matemática e Aplicações Fundamentais of Universidade de Lisboa [PEst-OE/MAT/UI0209/2011]. The second author is also grateful to Fundação para a Ciência e a Tecnologia [grant SFRH/BPD/34527/2006] and Núcleo de Investigação em Matemática (Universidade Lusófona).

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