A feasible theory for analysis

Fernando Ferreira*

Abstract

We construct a weak second-order theory of arithmetic which includes Weak König's Lemma (WKL) for trees defined by bounded formulae. The provably total functions (with Σ_1^b-graphs) of this theory are the polynomial time computable functions. It is shown that the first-order strength of this version of WKL is exactly that of the scheme of collection for bounded formulae.

1 Introduction

At the end of 1985, during a symposium on Hilbert's Program, Wilfried Sieg posed the following interesting problem : to find a mathematically significant subsystem of analysis whose class of provably recursive functions consists only of the computationally "feasible" ones[1]. (We thank Stephen Simpson for bringing this problem to our attention.) In the present paper we set up a system for analysis – with Sieg's "feasibility" condition fulfilled by the polynomial time computable functions – which permitts induction on notation for

*This work was partially supported by project 6E91 of CMAF (Portugal)

[1]See Sieg's paper "Hilbert's Program Sixty Years Later" in *The Journal of Symbolic Logic*, vol. 53, nº2 (1988).

NP-predicates and which includes the highly nonconstructive axiom asserting that every infinite tree of finite sequences of zeros and ones has an infinite path. (This principle is known as *Weak König's Lemma.*)

The reader will notice that our whole set-up owes much to work of Harvey Friedman, Stephen Simpson and others in the context of Reverse Mathematics with base induction for Σ^0_1-predicates and primitive recursiveness for the provably total functions.[2] There are, of course, differences. Most conspicuous is the special role played by the scheme of collection for bounded formulae, whose strength is exactly the first-order strength of Weak König's Lemma for trees defined by bounded formulae (see Theorem 7). Since this collection scheme is conservative over any bounded theory with respect to Π^0_2-sentences[3], it may be included in our feasible theory for analysis. Hence, contrary to the ordinary framework of Reverse Mathematics, our setting permitts the application of Weak König's Lemma, not only to set trees, but also to some definable trees. A case worth noting is the following: although, in general, bounded formulae do not define sets in our system, they do so provided these formulae happen to define infinite paths through the binary tree of zeros and ones.

[2]For an explanation and achievements (up to 1986) of Reverse Mathematics see Simpson's report "Subsystems of Z_2 and Reverse Mathematics" in the appendix to Gaisi Takeuti's *Proof Theory*, North-Holland 1987.

[3]This result is due to Samuel Buss in "A conservation result concerning bounded theories and the collection axiom", *Proc. Amer. Math. Soc.* 100, pp. 709-715 (1987). A very simple proof of a generalization of this result can be found in Fernando Ferreira's "A note on a result of Buss concerning bounded theories and the collection scheme", submitted to *Portugaliae Mathematica.* This proof is distilled from the proof of a particular case stated in corollary 6 of the present paper.

The question whether our system is *mathematically significant* is not dealt with in this paper. Nonetheless, it seems worthwhile to pin down more precisely what this question amounts to. We take it as requiring an investigation on how much of ordinary mathematics can be formalized within the system (or appropriate conservative extensions). More specifically, we can ask whether our framework is suitable for doing Reverse Mathematics with feasibility taken as basis, i.e., whether our framework provides an adequate vantage point from which to measure the non-feasible contents of ordinary theorems of mathematics. Some tentative work in this direction was done in [F88] and a new report is under way, in which we propose to consider the intermediate value theorem, the Heine-Borel principle, the uniform continuity theorem and the existence (or not) of the maximum of a continuos real function defined on a compact interval.

2 Basic Setup

As a guiding principle, we maintain that in weak systems of arithmetic with computational significance it is more perspicuous to have the class of 0-1 words (set-theoretically, the binary tree $2^{<\omega}$) as the standard model, instead of the more traditional setting of the natural numbers. Additionally, for the present purposes of presenting a second-order theory with WKL, the binary tree setting is ideally transparent. Hence, we shall build upon the binary tree theory Σ^b_1-PIND introduced in [F90].[4] To make the paper relatively self-contained we briefly describe this theory. Its language consists of three constant symbols ε, 0 and 1, two binary function symbols (for *concatenation*, usually omitted)

[4] Henceforth, following [BS90], this theory will be called Σ^b_1-NIA (for *Notation Induction Axiom*).

and $\times$, and a binary relation symbol $\subseteq$ (for *initial subwordness*). There are fourteen basic open axioms:

$$
\begin{array}{rcl}
x\varepsilon = x & & x \times \varepsilon = \varepsilon \\
x(y0) = (xy)0 & & x \times y0 = (x \times y)x \\
x(y1) = (xy)1 & & x \times y1 = (x \times y)x \\
x0 = y0 \rightarrow x = y & & x1 = y1 \rightarrow x = y \\
x \subseteq \varepsilon & \leftrightarrow & x = \varepsilon \\
x \subseteq y0 & \leftrightarrow & x \subseteq y \vee x = y0 \\
x \subseteq y1 & \leftrightarrow & x \subseteq y \vee x = y1 \\
x0 & \neq & y1 \\
x0 & \neq & \varepsilon \\
x1 & \neq & \varepsilon
\end{array}
$$

Note that, in the standard model, $x \times y$ is the word x concatenated with itself length of y times.[5] *Subwordness* of x with respect to y, denoted by $x \subseteq^* y$, is defined by $\exists z \subseteq y(zx \subseteq y)$. The class of *sw.q.-formulae* is the smallest class of formulae of the language containing the atomic formulae and closed under Boolean operations and subword quantification, i.e., quantification of the form $\forall x \subseteq^* t(...)$ or $\exists x \subseteq^* t(...)$, where the variable x does not occur in the term t.[6] The relation of x being of length less than or equal to the length of y,

[5] The growth rate of $\times$ corresponds exactly to the growth rate of Buss' smash function $\sharp$, as defined in [B85].

[6] These formulae define exactly the FO sets, a notion introduced by N. Immerman: see his paper "Descriptive and Computational Complexity" in *Computational Complexity Theory*, AMS Short Course

denoted by $x \leq y$, is defined by $1 \times x \subseteq 1 \times y$. The class of *bounded formulae*, also named the class of Σ^b_∞-formulae, is the smallest class of formulae containing the sw.q.-formulae and closed under Boolean operations and bounded quantification, i.e., quantification of the form $\forall x \leq t(...)$ or $\exists x \leq t(...)$, where the variable x does not occur in the term t. In the standard model these formulae define exactly the sets of the Meyer-Stockmeyer hierarchy. We are interested in the particular fragment of the bounded formulae consisting of those of the form $\exists x \leq tA$, where A is a sw.q.-formula and t is a term in which the variable x does not occur: these are called the Σ^b_1-formulae. In the standard model these formulae define exactly the sets of the complexity class NP. The theory Σ^b_1-NIA consists of the basic axioms plus the following induction scheme :

$$A(\varepsilon) \wedge \forall x\,(A(x) \rightarrow A(x0) \wedge A(x1)) \;\rightarrow\; \forall x\, A(x)$$

where A is a Σ^b_1-formula, possibly with parameters. This theory is equivalent, in a sense that could be made precise, to Samuel Buss' well-known theory S^1_2 (see [B85] for the definition) and, hence, has the following main property: whenever Σ^b_1-NIA $\vdash \forall x \exists y A(x, y)$, where A is a Σ^b_1-formula, there is a polynomial time computable function f such that $A(\sigma, f(\sigma))$, for all $\sigma \in 2^{<\omega}$.[7] This is the precise sense of saying that the provably total functions of Σ^b_1-NIA are computationally feasible.

The second-order theories that we shall be concerned with are formulated in a two-sorted language with word variables x, y, $z, \ldots$ and set variables X, Y, $Z, \ldots$ (the latter ones intended to vary over subsets of $2^{<\omega}$). The terms of this language are the same as the

Lecture Notes, vol. 38, 1989.

[7] Direct proofs of this result which bypass Buss' formalism can be found in [F90] or [BS90].

terms of the above first-order language ; for atomic formulae, we also allow expressions of the form $t \in X$, where t is a term and X is a set variable. Note that equality between set variables is *not* a basic notion, but rather defined by $\forall x\ (x \in X \leftrightarrow x \in Y)$. Exactly as in the first-order case we define the usual classes of formulae : Σ_1^b, bounded, Π_1^0, Π_2^0, et al. We just have to keep in mind that there are new atomic formulae to start with (in other words, set parameters are permitted).

Our basic second-order theory is Σ_1^b-NIA plus the following *comprehension scheme* :

$$\forall x\ (A(x) \leftrightarrow \neg B(x)) \rightarrow \exists X\ \forall x\ (x \in X \leftrightarrow A(x))$$

where A and B are Σ_1^b-formulae, possibly with parameters, and X is a new set variable. This scheme says that sets in NP$\cap$co-NP exist : call it the ∇_1^b-CA scheme.

Lemma 1. *The second-order theory Σ_1^b-NIA + ∇_1^b-CA is first-order conservative over Σ_1^b-NIA.*

Proof : This follows from the completeness theorems if one shows that for every first-order model M of Σ_1^b-NIA (for convenience we will identify the model with its domain) there is $S \subseteq \mathcal{P}(M)$ such that (M, S) is a second-order model of Σ_1^b-NIA+∇_1^b-CA. Actually a little more is true : if $(M, S) \models \Sigma_1^b$-NIA then $(M, S^*) \models \Sigma_1^b$-NIA + ∇_1^b-CA, where S^* is the class of the subsets of M that are simultaneously definable in (M, S) by a Σ_1^b and a Π_1^b formula. That is, W is in S^* iff there are Σ_1^b-formulae $A(w, \overline{x}, \overline{X})$ and $B(w, \overline{y}, \overline{Y})$ and elements $\overline{a}$, $\overline{b}$ in M and $\overline{U}$, $\overline{V}$ in S such that,

$$W = \{c \in M : (M, S) \models A(c, \overline{a}, \overline{U})\} = \{c \in M : (M, S) \models \neg B(c, \overline{b}, \overline{V})\}.$$

The checking is routine.

Given a formula A of the second-order language and x a distinguished (word) variable occuring free in A, we denote by $\mathrm{Tree}_\infty(A_x)$ the following formula,

$$\forall x\,\forall y\,(A(x) \wedge y \subseteq x \rightarrow A(y)) \wedge \forall u\,\exists x \equiv u\,A(x).^8$$

Note that (1) x is a bound variable in the formula $\mathrm{Tree}_\infty(A_x)$ and (2) if A is a bounded formula, then $\mathrm{Tree}_\infty(A_x) \in \Pi^0_1$. Let X be a set variable; $\mathrm{Path}(X)$ is the Π^0_1-formula,

$$\mathrm{Tree}_\infty((x \in X)_x) \wedge \forall x\,\forall y(x \in X \wedge y \in X \rightarrow x \subseteq y \vee y \subseteq x).$$

Weak König's lemma for trees defined by bounded formulae is the following scheme :

$$\mathrm{Tree}_\infty(A_x) \rightarrow \exists X\,(\mathrm{Path}(X) \wedge \forall x\,(x \in X \rightarrow A(x)))$$

where A is a Σ^b_∞-formula and X is a new variable. This principle will be known as Σ^b_∞-WKL.

Theorem 2. *The theory Σ^b_1-NIA $+$ ∇^b_1-CA $+$ Σ^b_∞-WKL is conservative over Σ^b_1-NIA with respect to Π^0_2-formulae.*

Before proving this theorem we need to introduce some new concepts. We say that (M, S) is a *substructure* of (N, T) with *set identification* Φ, and write $(M, S) \subseteq_\Phi (N, T)$[9], if $M \subseteq N$, i.e., M is a first-order substructure of N (henceforth we shall assume that the domain of M is a subset of the domain of N), and Φ is a subset of $S \times T$ such that (1) for each $V \in S$ there is $W \in T$ with $(V, W) \in \Phi$ and (2) for all $V \in S$ and $W \in T$, if $(V, W) \in \Phi$ then $W \cap M = V$. In case $(V, W) \in \Phi$ we say that the set W *already occurs* in

[8] The expression $x \equiv u$ means that x and u have the same length, that is, it abbreviates $x \leq u \wedge u \leq x$.

[9] When there is no confusion, we write $(M, S) \subseteq (N, T)$.

S. The following will also be used: given a third structure (P,R), with $(N,T) \subseteq_\Theta (P,R)$, then $(M,S) \subseteq_{\Phi\circ\Theta} (P,R)$; moreover, if $Z \in R$ already occurs in S then Z already occurs in T.

Consider a chain of structures $(M_0,S_0) \subseteq_{\Phi_0} (M_1,S_1) \subseteq_{\Phi_1} (M_2,S_2) \subseteq_{\Phi_2} \ldots$. The *union* of this chain, represented by (M_∞,S_∞), is defined as follows:

a) $M_\infty = \bigcup_{k=0}^{\infty} M_k$,

b) $S_\infty = \{\bigcup_{i\geq k} V_i : k \in \omega \text{ and, for all } i \geq k,\ (V_i,V_{i+1}) \in \Phi_i\}$.

It is easy to show that, for each $k \in \omega$, $(M_k,S_k) \subseteq_{\Phi_{k,\infty}} (M_\infty,S_\infty)$, where $\Phi_{k,\infty}$ is the relation $\{(V_k,\bigcup_{i\geq k} V_i) : \text{for all } i \geq k,\ (V_i,V_{i+1}) \in \Phi_i\}$. Note that all the sets in S_∞ already occur in some S_k.

We call (M,S) a Σ^b_∞-substructure (resp., a Π^0_1-substructure) of (N,T) if, for every Σ^b_∞-formula (resp., Π^0_1-formula) $A(\overline{x},\overline{X})$ and elements $\overline{a}$ in M, $\overline{V}$ in S and $\overline{W}$ in T such that $\overline{(V,W)}$ is in Φ, then the following holds:

$$(M,S) \models A(\overline{a},\overline{V}) \quad \text{if and only if} \quad (N,T) \models A(\overline{a},\overline{W}).$$

It is clear that the relation of Σ^b_∞-substructureness (resp., Π^0_1-substructureness) is transitive.

Lemma 3. *If $(M_0,S_0) \subseteq_{\Phi_0} (M_1,S_1) \subseteq_{\Phi_1} (M_2,S_2) \subseteq_{\Phi_2} \ldots$ is a Σ^b_∞-chain then, for every $k \in \omega$, (M_k,S_k) is a Σ^b_∞-substructure of (M_∞,S_∞). Moreover, if each (M_k,S_k) is a model of Σ^b_1-NIA then so is (M_∞,S_∞).*

Proof : Let $A(\overline{x}, \overline{X})$ be a Σ^b_∞-formula, $\overline{a}$ in M_k, $\overline{V}$ in S_k and $\overline{W}$ in S_∞, such that $\overline{(V, W)}$ in $\Phi_{k,\infty}$. To prove the first claim of the lemma it suffices to show that, if $(M_\infty, S_\infty) \models A(\overline{a}, \overline{W})$, then $(M_k, S_k) \models A(\overline{a}, \overline{V})$. We show this by induction on the complexity of A. The only interesting case to check is when $A(\overline{x}, \overline{X})$ is $\exists y \leq t(\overline{x}) B(y, \overline{x}, \overline{X})$, where y is a variable not occuring in the term $t(\overline{x})$ and $B \in \Sigma^b_\infty$. By assumption there is $b \in \Sigma^b_\infty$ satisfying $b \leq t(\overline{a})$ and $(M_\infty, S_\infty) \models B(b, \overline{a}, \overline{W})$. Well, $b \in M_n$ for some $n \geq k$. Now, each component W^j of $\overline{W}$ is of the form $\bigcup_{i \geq k} V^j_i$, for some sequence $(V^j_i)_{i \geq k}$ such that V^j_k is the j$^{\text{th}}$-component of $\overline{V}$ and $(V^j_i, V^j_{i+1}) \in \Phi_i$ for all $i \geq k$. Applying the induction hypothesis we get $(M_n, S_n) \models B(b, \overline{a}, \overline{V_n})$. Hence $(M_n, S_n) \models A(\overline{a}, \overline{V_n})$. By Σ^b_∞-absoluteness we conclude $(M_k, S_k) \models A(\overline{a}, \overline{V})$.

The second part of the theorem is a consequence of Σ^b_∞-absoluteness.

Lemma 4. *If $(M_0, S_0) \subseteq_{\Phi_0} (M_1, S_1) \subseteq_{\Phi_1} (M_2, S_2) \subseteq_{\Phi_2}$ ... is a Π^0_1-chain then, for every $k \in \omega$, (M_k, S_k) is a Π^0_1-substructure of (M_∞, S_∞). Moreover, if each (M_k, S_k) is a model of ∇^b_1-CA then so is (M_∞, S_∞).*

Proof : The first part of the lemma can be easily proved with the help of the previous result. To argue for the last claim, suppose that $(M_\infty, S_\infty) \models \forall x(A(x) \leftrightarrow \neg B(x))$, where $A, B \in \Sigma^b_1$, possibly with parameters. Take n large enough so that all parameters already occur in $M_n \cup S_n$. By Π^0_1-absoluteness, $(M_n, S_n) \models \forall x(A(x) \leftrightarrow \neg B(x))$. Hence there is $V \in S_n$ such that $(M_n, S_n) \models \forall x(x \in V \leftrightarrow A(x))$. Again by Π^0_1-absoluteness $(M_\infty, S_\infty) \models \forall x(x \in W \leftrightarrow A(x))$, where W is such that $(V, W) \in \Phi_{n,\infty}$. We are done.

Proof of the Theorem: Using completeness, this follows from the fact that every model of Σ^b_1-NIA + ∇^b_1-CA has a Π^0_1-absolute extension satisfying Σ^b_1-NIA + ∇^b_1-CA + Σ^b_∞-WKL. In order to see this, consider (M,S) a model of Σ^b_1-NIA + ∇^b_1-CA. We build a Π^0_1-absolute chain $(M_0,S_0) \subseteq (M_1,S_1) \subseteq \ldots$ of models of Σ^b_1-NIA + ∇^b_1-CA. Set $M_0 = M$, $S_0 = S$ and suppose that (M_n,S_n) is defined. By compactness take (M'_n,S'_n) an elementary extension of (M_n,S_n) for which there is an element c_n with $M_n < c_n$ (i.e., such that $x \leq c_n$, for all $x \in M_n$). This elementary extension automatically defines a set identification function $\Theta_n \subseteq S_n \times S'_n$. Set M_{n+1} to be $\{c \in M'_n : \exists a \in M_n \;\; c \leq a\}$ and let $S'_n\restriction_{M_{n+1}} = \{W \cap M_{n+1} : W \in S'_n\}$. We get the following situation:

1. $(M_n,S_n) \subseteq_{\Theta_n} (M'_n,S'_n)$
2. $(M_n,S_n) \subseteq_{\Phi_n} (M_{n+1},S'_n\restriction_{M_{n+1}})$
3. $(M_{n+1},S'_n\restriction_{M_{n+1}}) \subseteq_{\Psi_n} (M'_n,S'_n)$

where $\Phi_n = \{(V, W \cap M_{n+1}) : (V,W) \in \Theta_n\}$ and $\Psi_n = \{(W \cap M_{n+1}, W) : W \in S'_n\}$. The facts that the first inclusion is elementary and that the third inclusion is an end-extension (hence preserving Σ^b_∞-statements), readily entail that the second inclusion is Π^0_1-absolute and that $(M_{n+1},S'_{n+1}\restriction_{M_{n+1}})$ is a model of Σ^b_1-NIA. Define $S_{n+1} = (S'_n\restriction_{M_{n+1}})^*$, as in the proof of Lemma 1. The inclusion $(M_n,S_n) \subseteq (M_{n+1},S_{n+1})$ is still Π^0_1-absolute. Let (M_∞,S_∞) be the union of this chain. By the previous lemmas, this union is a model of Σ^b_1-NIA + ∇^b_1-CA and a Π^0_1-absolute extension of each of the models (M_n,S_n).

Finally, we check that Σ^b_∞-WKL holds in (M_∞,S_∞). Assume that $(M_\infty,S_\infty) \models \mathrm{Tree}_\infty(A_x)$, where A is a Σ^b_∞-formula. Take n large enough so that all parameters from A already occur in $M_n \cup S_n$. Then, by Π^0_1-absoluteness, $(M_n,S_n) \models \mathrm{Tree}_\infty(A_x)$. By elementarity, $(M'_n, S'_n) \models \mathrm{Tree}_\infty(A_x)$. So, there is $c \in M'_n$ with $c \equiv c_n$ and $(M'_n, S'_n) \models A(c)$. Consider $W = \{a \in M'_n : a \subseteq c\}$. Clearly $W \in S'_n$ and, hence, $W \cap M_{n+1}$ is in S_{n+1}. We get

$$(M_{n+1}, S_{n+1}) \models \mathrm{Path}(W \cap M_{n+1}) \wedge \forall x(x \in W \cap M_{n+1} \rightarrow A(x)).$$

Hence, by Π^0_1-absoluteness,

$$(M_\infty, S_\infty) \models \mathrm{Path}(Z) \wedge \forall x(x \in Z \rightarrow A(x)),$$

where $(W \cap M_{n+1}, Z) \in \Phi_{n+1,\infty}$.

Observe that the above conservation result also holds for sentences of the form $\forall X \forall x \exists y$ $A(X, x, y)$, with A a bounded formula.

3 Bounded collection and WKL

The principle of bounded collection, denoted in our setting by $B\Sigma^b_\infty$, is the following scheme :

$$(S_1) \qquad \forall x \leq a\ \exists y\ A(x, y) \rightarrow \exists z\ \forall x \leq a\ \exists y \leq z\ A(x, y)$$

where A is a bounded formula and z is a new variable (parameters are allowed). Within Σ^b_1-NIA this scheme is equivalent to the following slight modification :

$$(S_2) \qquad \forall x \equiv a\ \exists y\ A(x, y) \rightarrow \exists z\ \forall x \equiv a\ \exists y \leq z\ A(x, y).$$

Clearly, $(S_1) \Rightarrow (S_2)$. To argue for the other direction, consider the linear ordering $<_\ell$ of $2^{<\omega}$ defined first according to length and then, within the same length, lexicographically.

More formally,

$$x <_\ell y \Leftrightarrow (x \leq y \wedge x \not\equiv y) \vee (x \equiv y \wedge \exists z \subseteq x(z0 \subseteq x \wedge z1 \subseteq y)).$$

Now, in models of Σ_1^b-NIA it is possible to introduce the natural operation of addition "+" that stems from this linear ordering. Moreover, the following properties hold:

(P_1) $\qquad \Sigma_1^b\text{-NIA} \vdash x \leq a \;\rightarrow\; (0 \times a1) + x \equiv a1 \wedge (0 \times a1) + x \neq 1 \times a1$

(P_2) $\qquad \Sigma_1^b\text{-NIA} \vdash w \equiv a1 \;\rightarrow\; w = 1 \times a1 \vee \exists^1 x \leq a\, (0 \times a1) + x = w$

(Note that in the arithmetic setting the value $0 \times a1$ correspondes to the number $2^{u+1} - 1$, where u is the length of a, and the value $1 \times a1$ corresponds to the number $2^{u+2} - 2$.)[10] Assume (S_2) and the left hand side of (S_1). Then, by property (P_2), $\forall w \equiv a1\, (w \neq 1 \times a1 \;\rightarrow\; \exists y\, \exists x \leq a\, (0 \times a1) + x = w \wedge A(x, y))$. It follows, from (S_2), that $\exists z\, \forall w \equiv a1 (w \neq 1 \times a1 \;\rightarrow\; \exists y \leq z\, \exists x \leq a\, (0 \times a1) + x = w \wedge A(x, y))$. By (P_1) and the uniqueness part of (P_2), we may conclude the right hand side of (S_1).

Proposition 5. *Σ_1^b-NIA $+ \Sigma_\infty^b$-WKL $\vdash B\Sigma_\infty^b$.*

Proof : Let (M, S) be a model of Σ_1^b-NIA $+ \Sigma_\infty^b$-WKL and assume that $(M, S) \models \forall x \equiv a\; \exists y\; A(x, y)$, with A a Σ_∞^b-formula and a in M. Define $T = \{c \in M : a \leq c \rightarrow \forall y \leq c\, \neg A(c\,|_a, y)\}$[11]. Clearly T is a Σ_∞^b-tree ; we claim that T is not infinite. If it were,

[10] The possibility of introducing the operation of addition in Σ_1^b-NIA and of proving the above two properties needs, of course, careful work. However, this work would not be appropriate for the present paper, being more effective in a study of the precise relationship between our binary framework and Buss' setting. We plan to effect such a study soon.

[11] We are using the notation of [F90] : $x\,|_y$ is the word x truncated at the length of y ; if X is a path, $X\,|_y$ is the initial segment of X with the same length as y.

there would be a path X through T. Now consider y such that $A(X\mid_a, y)$; by definition of T we get that $X\mid_{\widehat{ay}} \notin T$, which contradicts the definition of X. T being finite, there is b with $\forall x \in T\ (x \leq b)$. This clearly entails that $\forall x \equiv a\ \exists y \leq b1\ A(x, y)$. [12]

An easy consequence of the above proposition and Theorem 2 is the following result mentioned in note 3,

Corollary 6. *The theory Σ_1^b-NIA + $B\Sigma_\infty^b$ is conservative over the theory Σ_1^b-NIA with respect to Π_2^0-sentences.*

The next result says that the first-order strength of Σ_∞^b-WKL is exactly $B\Sigma_\infty^b$ (over the base theory Σ_1^b-NIA) :

Theorem 7. *The first-order part of the theory Σ_1^b-NIA + ∇_1^b-CA + Σ_∞^b-WKL is Σ_1^b-NIA + $B\Sigma_\infty^b$.*

Proof : One half of this result is Proposition 5. To argue for the other half, we show that for every countable model M of Σ_1^b-NIA + $B\Sigma_\infty^b$ there is $S \subseteq \mathcal{P}(M)$ such that (M, S) is a countable model of Σ_1^b-NIA + ∇_1^b-CA + Σ_∞^b-WKL. The construction of S hinges on the following lemma :

Lemma 8. *Let (M, S) be a countable second-order model of Σ_1^b-NIA + $B\Sigma_\infty^b$. Consider $A(x)$ a Σ_∞^b-formula, with parameters in $M \cup S$, such that $(M, S) \models Tree_\infty(A_x)$. Then there is a countable second-order model (M, S') of Σ_1^b-NIA + $B\Sigma_\infty^b$ such that $S \subseteq S'$ and $(M, S') \models \exists X\ (Path(X) \wedge \forall x\ (x \in X \rightarrow A(x)))$.*

[12] A close inspection of the above proof actually yields Σ_1^b-NIA + Π_i^b-WKL $\vdash B\Sigma_i^b$, for $i \geq 1$.

Proof of the lemma[13] : Let $\mathcal{C}$ be the class of all boundedly defined subsets of M with parameters in $M \cup S$ and put $\mathcal{T} = \{T : T \in \mathcal{C} \;\&\; (M,S) \models \mathrm{Tree}_\infty(T)\}$. We say that $\mathcal{D} \subseteq \mathcal{T}$ is *dense* if $\forall T \in \mathcal{T}\, \exists T' \in \mathcal{D}(T' \subseteq T)$. Moreover $\mathcal{D}$ is *definable* if it is definable over (M,S) allowing parameters in $M \cup S$. We say that $X \subseteq M$ is a *generic* path if X is a path in M and for each definable dense set $\mathcal{D} \subseteq \mathcal{T}$ there exists $T \in \mathcal{D}$ with $X \subseteq T$.

We claim that there is a generic path X through $T_0 = \{c \in M : (M,S) \models A(c)\}$. To prove this consider an enumeration $\mathcal{D}_1,\ \mathcal{D}_2,\ \mathcal{D}_3, \ldots$ of all definable dense subsets of $\mathcal{T}$. It is easy to define recursively a sequence $T_0,\ T_1,\ T_2,\ T_3, \ldots$ of elements of $\mathcal{T}$ with $T_{i+2} \subseteq T_{i+1} \in \mathcal{D}_{i+1}$, for all $i \in \omega$. Let $X = \bigcap_{i\in\omega} T_i$. It is clear that $X \subset T_0$ and that X is a tree. To show that X is an *infinite path* it is enough to argue that for each $\ell \in M$ the set

$$\mathcal{D}(\ell) = \{T \in \mathcal{T} : \forall x, y \in T\ (\ell \leq x \wedge \ell \leq y \rightarrow x\,|_\ell = y\,|_\ell)\}$$

is dense. Take any $T \in \mathcal{T}$; if we show that

$$(*) \qquad (M,S) \models \exists x \equiv \ell\ (x \in T \wedge \forall w\ (x \leq w \rightarrow \exists z \equiv w\ (x \subseteq z \wedge z \in T)))$$

we are done, because we just have to consider $T' = \{z \in T : z \subseteq x_0 \vee x_0 \subseteq z\}$, where x_0 witnesses $(*)$. Assume, to obtain a contradiction, that $\neg(*)$. Then,

$$\forall x \equiv \ell\ \exists w\ (x \in T \rightarrow \forall z \equiv w\ (x \subseteq z \rightarrow z \notin T)).$$

Using <u>bounded collection</u> we get,

$$\exists b\ \forall x \equiv \ell\ \exists w \leq b\ (x \in T \rightarrow \forall z \equiv w\ (x \subseteq z \rightarrow z \notin T)).$$

[13]The argument below is based on a forcing construction due to Jockusch and Soare [JS72]. We will be careful in pointing out the exact places where bounded collection is used.

This contradicts the infinitude of T.

The remainder of the proof is like in Lemma 4.5 of [SS86] : if X is a generic path then the structure (M, S'), where $S' = S \cup \{X\}$, satisfies Σ_1^b-NIA $+ B\Sigma_\infty^b$. We give a very brief sketch of this. Clearly Σ_1^b-NIA holds in (M, S'), due to the fact that each initial segment of X is in M. To prove $B\Sigma_\infty^b$ let $B(x, y)$ be a Σ_∞^b-formula with parameters in $M \cup S \cup \{X\}$ and suppose that (M, S') satisfies $\forall x \leq a\ \exists y\ B(x, y)$. It is easy to write $B(x, y)$ in *normal form* as $\exists \ell\ C(x, y, X\ |_\ell)$, with $C(x, y, z)$ a Σ_∞^b-formula with parameters from $M \cup S$. Define $\mathcal{E}$ to be the set of all T in $\mathcal{T}$ such that (M, S) satisfies $\exists x \leq a\ \forall v \in T\ \forall y \leq v\ \forall u \subseteq v\ \neg C(x, y, u)$ and let $\mathcal{D}$ be the set of all $T \in \mathcal{T}$ such that $T \in \mathcal{E} \vee (\neg \exists T' \in \mathcal{E})(T' \subseteq T)$. $\mathcal{D}$ is definable and dense ; so let $T \in \mathcal{D}$ with $X \subseteq T$. We can not have $T \in \mathcal{E}$. Hence there is no $T' \in \mathcal{E}$ with $T' \subseteq T$. So, for each $x \leq a$ the tree $T_x = \{v \in T : \forall y \leq v\ \forall u \subseteq v\ \neg C(x, y, u)\}$ must be finite. That is, $\forall x \leq a\ \exists c$ "T_x is bounded by c", i.e., $\forall x \leq a\ \exists c\ \forall v \equiv c1\ v \notin T_x$. Using bounded collection we conclude that $\exists b\ \forall x \leq a\ \exists c \leq b\ \forall v \equiv c1\ v \notin T_x$. This entails that $\exists b\ \forall x \leq a\ \exists y \leq b1\ B(x, y)$.

(of the lemma)

The proof of the theorem proceeds easily. From the previous lemma and the construction of lemma 1, define an increasing sequence $(S_i)_{i \in \omega}$ of subsets of M satisfying

a) (M, S_i) is a model of Σ_1^b-NIA $+ B\Sigma_\infty^b + \nabla_1^b$-CA

b) for each boundedly defined infinite tree T from (M, S_i) there is $j > i$ such that T is satisfied in (M, S_j) to have a path.

(Notice that the first requirement can be accomplished because the construction in lemma

1 preserves $B\Sigma^b_\infty$; this can be checked routinely.)

The limit of the first-order absolute chain $(M, S_i)_{i\in\omega}$ does the job.

(of the theorem)

Corollary 9. *The theory Σ^b_1-NIA$+\nabla^b_1$-CA$+\Sigma^b_\infty$-WKL is Π^1_1-conservative over Σ^b_1-NIA$+$ ∇^b_1-CA$+B\Sigma^b_\infty$.*

4 A base theory for feasible analysis

In this section we propose a *Base Theory for Feasible Analysis*, which we abbreviate by the acronym BTFA. This theory consists of Σ^b_1-NIA+$B\Sigma^b_\infty$ plus the following strengthening of ∇^b_1-CA:

$$(\$) \qquad \forall x\, (\exists y\, A(x,y) \leftrightarrow \forall z\, \neg B(x,z)) \rightarrow \exists X\; \forall x\, (x \in X \leftrightarrow \exists y\; A(x,y))$$

where A and B are Σ^b_1-formulae (possibly with parameters). Notice that the structure ($2^{<\omega}$, NP$\cap$co-NP) is a model of Σ^b_1-NIA+∇^b_1-CA+$B\Sigma^b_\infty$, while $(2^{<\omega}, \Delta^0_1)$ is the smallest model of BTFA with the standard model $2^{<\omega}$ for first-order part. This shows that ($) is, indeed, stronger than ∇^b_1-CA. The scheme ($) is often useful: for instance, it ensures that the composition of two total functions (given by sets of ordered pairs) is still a total function.

Theorem 10.

i. The theory BTFA is conservative over Σ^b_1-NIA with respect to Π^0_2-formulae.

ii. The theory BTFA $+\Sigma^b_\infty$-WKL is conservative over BTFA with respect to Π^1_1-formulae.

Proof : By Corollary 6, the first statement follows from the fact that BTFA is a first-order conservative extension of Σ^b_1-NIA $+ B\Sigma^b_\infty$. To show this, let (M, S) be a model of the latter theory and consider $S^\star$ the class of all subsets X of M that are simultaneously definable in (M, S) by formulae of the form $\exists y\ A(x, y)$ and $\forall z\ \neg B(x, z)$, with $A, B \in \Sigma^b_1$ and allowing parameters from $M \cup S$. The checking that $(M, S^\star)$ is a model of BTFA follows closely the proof of lemma 4.2 of [SS86]. We argue that to each sw.q-formula C, with parameters in $M \cup S^\star$ and no free set variables, it is possible to associate two formulae C_Σ and C_Π such that,

a. C_Σ is of the form $\exists y\ A$, with parameters in $M \cup S$ and $A \in \Sigma^b_1$

b. C_Π is of the form $\forall z\ B$, with parameters in $M \cup S$ and $B \in \Pi^b_1$

c. C_Σ and C_Π have the same free variables as C

d. C_Σ and C_Π are equivalent over (M, S) and equivalent to C over $(M, S^\star)$.

The construction of the formulae C_Σ and C_Π is done by induction on the complexity of C. It is only worth commenting on those cases for which $C = t \in X$ or $C = \forall x \subseteq^* t\ D$. In the first case $C_\Sigma = \exists y\ A(t, y)$ and $C_\Pi = \forall z\ \neg B(t, z)$, where A and B are as in the definition of the parameter $X \in S^\star$. In the second case the definition of C_Π is clear, while C_Σ can be defined using $B\Sigma^b_\infty$ and the following result of Σ^b_1-NIA:

$$\forall x \subseteq^* a\ \exists y \leq c\ F \quad \leftrightarrow \quad \exists b \leq (c \times a1 \times a1)\ \forall x \subseteq^* a\ \exists y \subseteq^* b\ (y \leq c \wedge F)$$

where F is any Σ^b_1-formula (see [F90] for a proof of this). The idea is to use the above schemes to pull out the existential quantifiers (both the unbounded and the bounded).

It is now clear that ($) holds in $(M, S^\star)$. It is also easy to argue that $(M, S^\star) \models \Sigma_1^b$-NIA. Take $a \in M$ and $A \in \Sigma_1^b$ such that $(M, S^\star) \models A(\emptyset) \wedge \neg A(a)$, in order to find $c \in M$ with $c \subseteq a$, $c0 \subseteq a$ (say) and $(M, S^\star) \models A(c) \wedge \neg A(c0)$. Well, if $A(x) = \exists w \leq t\ B(x, w)$, with B a sw.q.-formula, we successively get

$$\begin{aligned} (M, S) &\models \forall x\, \forall w\ (B_\Sigma(x, w) \leftrightarrow B_\Pi(x, w)) \\ (M, S) &\models \forall x \subseteq a\ \forall w \leq t\ (\exists y\ B_1(x, w, y) \leftrightarrow \forall z\ B_2(x, w, z)) \end{aligned}$$

where $B_\Sigma = \exists y B_1, B_\Pi = \forall z B_2$, B_1 and B_2 sw.q.-formulae. By $B\Sigma_\infty^b$ there is $b \in M$ so that,

$$(M, S) \models \forall x \subseteq a\ \forall w \leq t\ (\exists y \leq b\ B_1(x, w, y) \leftrightarrow \forall z \leq b\ B_2(x, w, z)).$$

Hence, $(M, S) \models \forall x \subseteq a\ (\exists w \leq t\ B_\Sigma(x, w) \leftrightarrow \exists w \leq t\ \exists y \leq b\ B_1(x, w, y))$. The element c can now be found using the scheme of notation on induction to the Σ_1^b-formula $\exists w \leq t\ \exists y \leq b\ B_1(x, w, y)$.

To finish the proof of *i.*, we still need to argue that $B\Sigma_\infty^b$ holds in $(M, S^\star)$. This is a straightforward consequence of the following fact : the mapping $C \to (C_\Sigma, C_\Pi)$ can be extended to all bounded formulae C if the formulae A and B of requirements a and b are bounded. Of course, extending this map uses $B\Sigma_\infty^b$ heavily.

The proof of *ii.* proceeds as the proof of Theorem 7 and Corollary 9, the only difference being that we use the map $S \to S^\star$ (instead of the map $S \to S^*$) to build an increasing sequence $(S_i)_{i \in \omega}$ of subsets of M, with $S_0 = S$, satisfying :

a) (M, S_i) is a model of BTFA

b) for each boundedly defined infinite tree T from (M, S_i) there is $j > i$ such that T has a path in (M, S_j).

We claim that the limit (M, S_∞) of the chain $(M, S_i)_{i\in\omega}$ is a model of $BTFA + \Sigma^b_\infty$-WKL. To see this we only have to check that (\$) holds in (M, S_∞). Suppose $(M, S_\infty) \models \forall x\ (\exists y\ A(x,y) \leftrightarrow \forall z\ \neg B(x,z))$, with A and B Σ^b_1-formulae. Take $n \in \omega$ large enough so that all parameters already occur in (M_n, S_n). Now, notice that this is a first-order absolute chain. Hence,

$$(M, S_n) \models \forall x\ (\exists y\ A(x,y) \leftrightarrow \forall z\ \neg B(x,z)).$$

So, there is $V \in S_n$ such that

$$(M, S_n) \models \forall x\ (x \in V \leftrightarrow \exists y\ A(x,y)).$$

The conclusion follows, again, by first-order absoluteness.

References

[B85] S. Buss, *Bounded Arithmetic*, Ph. D. Dissertation, Princeton University, June 1985. A revision of this thesis was published by Bibliopolis in 1986.

[BS90] W. Buchholz & W. Sieg, "A note on polynomial time computable arithmetic", *Contemporary Mathematics* 106, pp. 51-55 (1990)

[F88] F. Ferreira, *Polynomial Time Computable Arithmetic and Conservative Extensions*, Ph. D. Dissertation, Pennsylvania State University, December 1988.

[F90] F. Ferreira, "Polynomial Time Computable Arithmetic", *Contemporary Mathematics* 106, pp. 137-156 (1990).

[JS72] C. Jockusch, Jr. & R. Soare, "Π^0_1 classes and degrees of theories, "*Trans. Amer. Math. Soc.*" 173, pp. 35-56 (1972).

[SS86] S. Simpson & R. Smith, "Factorization of polynomials and Σ^0_1 induction", *Annals of Pure and Applied Logic* 31, pp. 289-306 (1986).

Universidade de Lisboa

Departamento de Matemática

Rua Ernesto de Vasconcelos, Bloco C1, 3

1700 Lisboa

PORTUGAL

(mferferr@ptearn.bitnet)