

# The finitistic consistency of Heck’s predicative Fregean system

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**Abstract** Frege’s theory is inconsistent (Russell’s paradox). However, the predicative version of Frege’s system is consistent. This was proved by Richard Heck in 1996 using a model theoretic argument. In this paper, we give a finitistic proof of this consistency result. As a consequence, Heck’s predicative theory is rather weak (as was suspected). We also prove the finitistic consistency of the extension of Heck’s theory to  $\Delta_1^1$ -comprehension and of Heck’s ramified predicative second-order system.

## 1 Introduction

Russell’s paradox was a serious blow to Frege’s logicist project. In modern and adapted terminology, we may describe the system of Frege in *Grundgesetze der Arithmetik* [7] as a second-order system with full comprehension and a variable-binding term-forming operator as regulated by the infamous Law V:

$$\hat{x}.\varphi(x) = \hat{x}.\rho(x) \leftrightarrow \forall x(\varphi(x) \leftrightarrow \rho(x)),$$

where  $\varphi(x)$  and  $\rho(x)$  are arbitrary formulas of the language. In the above, the *value-range* operator  $\hat{\phantom{x}}$  yields a first-order term  $\hat{x}.\varphi(x)$  when applied to a formula  $\varphi(x)$ . Full comprehension was left implicit by Frege but it can be brought into the open by the scheme  $\exists F\forall x(Fx \leftrightarrow \varphi(x))$ , where  $\varphi$  is any formula of the language.

Terence Parsons initiated the investigation into consistent subsystems of Frege’s system. He showed in [10] that the “first-order portion of the *Grundgesetze*” is consistent. Of course, in a language in which second-order variables are dropped, comprehension is dropped as well, since it cannot be expressed. Furthermore, the relation of membership – a defined relation in Frege’s *Grundgesetze* – is also not expressible:  $x \in y$  is defined by

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$\exists F(y = \hat{x}.Fx \wedge Fx)$ . Parsons work was pioneering but his system suffers from severe restrictions on expressibility. A few years later, Richard Heck proved in [9] that the predicative fragment of the *Grundgesetze* is consistent. We take the language of Heck's theory H as the language of monadic second-order logic with equality (following [2], we do not admit non-logical constants) together with the value-range operator, but the theory restricts the comprehension scheme (described above) to *predicative formulas*  $\varphi$ . A predicative formula is a formula with no second-order quantifiers (it may have second-order free variables). Note, in particular, that the occurrences of value-range terms  $\hat{x}.\varphi(x)$  in a predicative formula are restricted to formulas  $\varphi$  which do not have second-order quantifiers (the so-called *predicative* value-ranges). Using a model-theoretic argument, Heck showed that his theory is consistent (Heck admits non-logical constants, but nothing is lost by restricting to our case). Heck's result was extended by Kai Wehmeier and the second author of this paper. It is shown in [6] that the extension of Heck's theory to  $\Delta_1^1$ -comprehension is consistent. The proof is also model-theoretic, using the machinery of recursively saturated models.

It is important for the logicist project to investigate how much mathematics can be developed in consistent fragments of Frege's *Grundgesetze*. Heck's theory is able to interpret Robinson arithmetic theory Q. This theory seems too weak to merit serious consideration – it has no induction (e.g., it does not even prove the commutativity of addition) – but, in fact, it is not as plain as one might at first be led to judge. Firstly, it is a classical result that Q is an essentially undecidable theory (cf. [14]). Moreover, Q interprets the theory  $\mathsf{I}\Delta_0$ , viz. Peano arithmetic with the induction scheme restricted to bounded formulas. This beautiful result is the work of many people and the reader can consult the monograph [8] for references. It is even known that a *modicum* of analysis is interpretable in Q (cf. [5]). Can we draw a limit on how much can be interpreted in consistent fragments of Frege's *Grundgesetze*? Part of the importance of the existence of finitistic consistency proofs lies precisely in the fact that they provide good upper bounds for interpretability. By a finitistic proof we mean a proof formalizable in the so-called theory PRA of primitive recursive arithmetic (see [12]), and the reader can study the paper with this aim in mind. However, the claim that the proofs are finitistic can be refined and one can point out to subsystems of PRA where the proofs go through. For instance, John Burgess proved finitistically in [1] (cf. also his book [2]) that the consistency of Parsons theory is provable in the theory  $\mathsf{I}\Delta_0(\text{super}^2\text{exp})$ . Burgess' book is a good reference for the precise definition of this and similar theories. By Gödel's second incompleteness theorem, this result entails that  $\mathsf{I}\Delta_0(\text{super}^2\text{exp})$  is not interpretable in Parsons theory. In the referred book, Burgess asks whether Heck's theory (or even its extension to  $\Delta_1^1$ -comprehension) can be proved finitistically. The present work is dedicated to providing finitistic consistency proofs for these theories.

The paper is organized as follows. In the next section, we prove the extension of the so-called Shoenfield's theorem to theories with the predicative value-range operator, thereby answering positively a question in [2]. Together with the finitistic proof of the consistency of Parsons theory, this entails that there is a finitistic proof of the consistency of Heck's theory restricted to the

predicative value-range operator. In order to deal with the full value-range operator, we first extend our version of Shoenfield's theorem to allow  $\Delta_1^1$ -comprehension. This is done in Section 3. In the next section, having this material available, we finally tackle Heck's theory. A brief Section 5 extends our results to ramified theories. In the last section, we briefly discuss the limits of strict predicativity and raise some technical questions. The paper also includes a small appendix where it is sketched a proof of cut-elimination for predicative second-order logic which is formalizable in the theory  $\text{I}\Delta_0(\text{superexp})$ .

The collaboration between the two authors of this paper can be described as follows. While preparing his address to the Birkbeck conference on "Set Theory and Higher-Order Logic: Foundational Issues and Mathematical Developments" the second author was puzzled by the claim (e.g., in [2]) that the cut-elimination theorem for pure predicative logic is formalizable in the theory  $\text{I}\Delta_0(\text{superexp})$ . It is not that he doubted the result but rather that he could not see how to reduce this result to the usual cut-elimination for first-order logic, nor how to readily formalize in this theory the usual textbook proof of [13]. Moreover, he was unable to locate in print a proof of this claim. So, he posed this problem to the first author. After some tries, a quite simple solution was found by both authors. It is here given in the appendix. The remainder of the paper is due to the second author only.

## 2 Shoenfield's theorem extended

It is well-known how to set up a sequent calculus for pure predicative second-order logic. This is done, for instance, in Takeuti's book on proof theory [13]. In the sequel, we use this calculus but with sequents consisting of sets of formulas, instead of sequences of formulas (this is the inessential variant that is studied in the appendix). The sequent calculus enjoys the property of cut-elimination. It is important that the calculus is *pure*: There are no non-logical axioms in the calculus, not even the equality axioms (but, of course, the equality symbol may be present, inconspicuous among binary relation symbols). The calculus is, nevertheless, set up so that predicative comprehension is provable (to describe the calculus, Takeuti uses the meta-device of *abstracts* given by formulas without second-order quantifiers). In order to simplify notation, our second-order calculus only has unary second order variables even though the natural setting allows any arity. There is no obstacle in extending the cut-elimination theorem to the calculus with a value-range operator. The discussion of the several cases in the cut-elimination proof is unaltered provided that one treats the new terms as plain terms which do not increase the complexity of formulas (even though the value-ranges may apply to complex formulas). The usual proof (see also the appendix) of cut-elimination is independent of the structure of the terms. Of course, it is *crucial* that the calculus remains *pure*, i.e., that there is no Law V present. Even though the next theorem holds for the language with the unrestricted value-range operator, in this and in the next section we are only interested in *predicative* value-range term formation.

**Theorem 2.1** *The sequent calculus of pure predicative logic with the predicative value-range operator enjoys the property of cut-elimination.*

The next result is a Herbrand-like consequence of the above theorem.

**Corollary 2.2** *Suppose that a formula  $\exists F\varphi(F)$ , where  $\varphi$  does not have second-order quantifiers, is provable in the sequent calculus of pure predicative logic with the predicative value-range operator. Then there are abstracts  $\{x : \theta_1(x, \bar{z}_1)\}, \dots, \{x : \theta_n(x, \bar{z}_n)\}$  such that the sequent*

$$\Rightarrow \exists \bar{z}_1 \varphi(\{x : \theta_1(x, \bar{z}_1)\}), \dots, \exists \bar{z}_n \varphi(\{x : \theta_n(x, \bar{z}_n)\})$$

*is provable in the restriction of the above calculus to the language without second-order quantifiers (in this restricted calculus, second-order variables only occur free).*

**Proof** The proof is standard (see, for instance, pp. 174-175 of [13]), but we give it here for completeness and in order to draw attention to the appearance of the variables  $\bar{z}_1, \dots, \bar{z}_n$  and the corresponding existential quantifications. Of course, this appearance makes a lot of sense (semantically speaking), but it is noteworthy to pin down exactly where it is required in the proof-theoretic proof.

We prove a slightly more general and refined statement. Suppose  $\Gamma$  and  $\Delta$  are sequences of formulae without second-order quantifiers, and let  $\exists F\varphi(F)$  be as in the theorem. We show that if the sequent calculus of pure predicative logic with the predicative value-range operator proves the sequent  $\Gamma \Rightarrow \Delta, \exists F\varphi(F)$ , then there are abstracts  $\{x : \theta_1(x, \bar{z}_1)\}, \dots, \{x : \theta_n(x, \bar{z}_n)\}$  such that the sequent calculus without second-order quantifiers proves

$$\Gamma \Rightarrow \Delta, \exists \bar{z}_1 \varphi(\{x : \theta_1(x, \bar{z}_1)\}), \dots, \exists \bar{z}_n \varphi(\{x : \theta_n(x, \bar{z}_n)\}).$$

*Importantly*, we also require that the free first-order variables of each formula  $\exists \bar{z}_i \varphi(\{x : \theta_i(x, \bar{z}_i)\})$  are exactly the same as the free first-order variables of  $\exists F\varphi(F)$ .

By Theorem 2.1, the sequent  $\Gamma \Rightarrow \Delta, \exists F\varphi(F)$  has a cut-free proof. The theorem is now proved by induction on the number of inferences of this proof. The crucial case is when the sequent is obtained by the  $\exists^2 r$ -rule. In this case, the sequent is inferred from a sequent of the form  $\Gamma \Rightarrow \Delta, \varphi(\{x : \theta(x, \bar{z})\})$  or of the form  $\Gamma \Rightarrow \Delta, \exists F\varphi(F), \varphi(\{x : \theta(x, \bar{z})\})$ , where  $\theta$  is a formula without second-order quantifiers and  $\bar{z}$  are the free first-order variables of  $\theta$  which do not occur among the free first-order variables of  $\exists F\varphi(F)$  (we do not care about second-order variables). In the first case, we can infer  $\Gamma \Rightarrow \Delta, \exists \bar{z} \varphi(\{x : \theta(x, \bar{z})\})$  and, clearly, we are done. In the second case, by the induction hypothesis, there are abstracts  $\{x : \theta_1(x, \bar{z}_1)\}, \dots, \{x : \theta_n(x, \bar{z}_n)\}$  such that the sequent calculus without second-order quantifiers proves the sequent

$$\Gamma \Rightarrow \Delta, \exists \bar{z}_1 \varphi(\{x : \theta_1(x, \bar{z}_1)\}), \dots, \exists \bar{z}_n \varphi(\{x : \theta_n(x, \bar{z}_n)\}), \varphi(\{x : \theta(x, \bar{z})\}).$$

We can apply now the  $\exists r$ -rule to conclude the sequent

$$\Gamma \Rightarrow \Delta, \exists \bar{z}_1 \varphi(\{x : \theta_1(x, \bar{z}_1)\}), \dots, \exists \bar{z}_n \varphi(\{x : \theta_n(x, \bar{z}_n)\}), \exists \bar{z} \varphi(\{x : \theta(x, \bar{z})\}).$$

This is what we want.

The application of the induction hypothesis to the other rules is straightforward (note that the other second-order quantifier rules do not occur in the cut-free proof), but we want to draw attention to the first-order quantifier

rules  $\exists l$  and  $\forall r$ . We must be sure that the eigenvariable condition is still met after applying the induction hypothesis to the top sequent of these rules, in order to be able to apply the very same rule to the very same formula afterwards. Of course, this is guaranteed by the additional requirement described above (thanks to the systematic inclusion of suitable existential first-order quantifications).  $\square$

The appearance of a finite number of abstracts in the conclusion of the above corollary is typical of a Herbrand-like theorem. In such theorems, the finite number of abstracts is usually unavoidable, but sometimes it can be replaced with a single one. This is the case if a procedure for definition by cases is available. In the second-order case, a definition by cases can be simulated. Let us see how this simulation works for two abstracts (the general case is similar). Suppose that we have

$$\Rightarrow \exists \bar{z}_1 \varphi(\{x : \theta_1(x, \bar{z}_1)\}), \exists \bar{z}_2 \varphi(\{x : \theta_2(x, \bar{z}_2)\}).$$

Let us consider the abstract

$$\{x : [\theta_1(x, \bar{z}_1) \wedge \exists \bar{z}_1 \varphi(\{x : \theta_1(x, \bar{z}_1)\})] \vee [\theta_2(x, \bar{z}_2) \wedge \forall \bar{z}_1 \neg \varphi(\{x : \theta_1(x, \bar{z}_1)\})]\}.$$

If we denote this abstract by  $\{x : \theta(x, \bar{z}_1, \bar{z}_2)\}$ , it is clear that

$$\Rightarrow \exists \bar{z}_1 \exists \bar{z}_2 \varphi(\{x : \theta(x, \bar{z}_1, \bar{z}_2)\}).$$

The corollary can be extended in several ways. First of all, we may have a list of second-order quantifiers  $\exists F_1 \dots \exists F_k \varphi(F_1, \dots, F_k)$  instead of just one quantification. The proof is similar. Let us call a formula of the form  $\exists \bar{F} \varphi(\bar{F})$ , with  $\varphi$  without second-order quantifiers, a  $\Sigma_1^1$ -formula. Dually, a  $\Pi_1^1$ -formula is obtained by replacing the existential second-order quantifiers by universal quantifications. (We allow the empty list of second-order quantifiers, thereby including predicative formulas among the  $\Sigma_1^1$  and  $\Pi_1^1$ -formulas.) Let us introduce some more terminology. A *predicative instantiation* of a  $\Sigma_1^1$ -formula (as above) is a formula of the form

$$\exists \bar{y}_1 \dots \bar{y}_k \varphi(\{x : \theta_1(x, \bar{y}_1)\}, \dots, \{x : \theta_k(x, \bar{y}_k)\}),$$

where  $\theta_1, \dots, \theta_k$  are formulas without second-order quantifiers. When  $k = 0$ , there is only one predicative instantiation of the formula: it is the formula itself. A predicative instantiation of a  $\Pi_1^1$ -formula is defined dually, with the first-order existential quantifiers replaced by universal quantifiers. The most general form of the corollary which we will use in the sequel applies when the sequent calculus of pure predicative logic with the predicative value-range operator proves the sequent

$$\forall \bar{G}_1 \rho_1(\bar{G}_1), \dots, \forall \bar{G}_r \rho_r(\bar{G}_r) \Rightarrow \exists \bar{F}_1 \varphi_1(\bar{F}_1), \dots, \exists \bar{F}_m \varphi_m(\bar{F}_m),$$

where the formulas in  $\rho_1, \dots, \rho_r, \varphi_1, \dots, \varphi_m$  are all without second-order quantifiers. Let us denote by  $A_i$  the formula  $\forall \bar{G}_i \rho_i(\bar{G}_i)$  ( $1 \leq i \leq r$ ) and by  $B_j$  the formula  $\exists \bar{F}_j \varphi_j(\bar{F}_j)$  ( $1 \leq j \leq m$ ). Under these circumstances there are predicative instantiations  $\bar{A}_1^*, \dots, \bar{A}_r^*$  and  $\bar{B}_1^*, \dots, \bar{B}_m^*$  of  $A_1, \dots, A_r$  and  $B_1, \dots, B_m$ , respectively, such that the following sequent is provable in the restricted sequent calculus without second-order quantifiers:

$$\bar{A}_1^*, \dots, \bar{A}_r^* \Rightarrow \bar{B}_1^*, \dots, \bar{B}_m^*.$$

This extended case of Corollary 2.2 can be proved directly by the same argument or, else, easily reduced to the corollary itself (modified to allow a list of second-order existential quantifications).

**Definition 2.3** Let us consider a second-order language (with a distinguished symbol for first-order equality) extended with the predicative value-range operator. A  $\Pi_1^1$ -theory in this language is a theory with the first-order axioms of reflexivity, symmetry and transitivity for equality, the further equality axiom:

$$(Eq) \quad \forall F \forall x \forall y (x = y \wedge Fx \rightarrow Fy),$$

the axiom version of Law V:

$$(LV) \quad \forall F \forall G (\hat{x}.Fx = \hat{x}.Gx \leftrightarrow \forall x (Fx \leftrightarrow Gx)),$$

and  $\Pi_1^1$  axioms peculiar to the theory (the so-called *proper* axioms of the theory). We call a  $\Pi_1^1$ -theory with no proper axioms a *pure*  $\Pi_1^1$ -theory.

Given a language as in the definition above, we may consider the restriction of this language to its first-order part: a so-called first-order Parsons language. Let  $T$  be a  $\Pi_1^1$ -theory. The *first-order schematization* of  $T$  is the theory  $T^s$ , formulated in the associated first-order Parsons language, obtained from  $T$  by replacing each  $\Pi_1^1$  axiom, including (LV) and (Eq), by the associated predicative instantiations *which have no second-order variables*.

We are now ready to enunciate and give a finitistic proof of the following extension of Shoenfield's theorem.

**Theorem 2.4** *Let  $T$  be a  $\Pi_1^1$ -theory. If  $T^s$  is consistent then  $T$  with predicative comprehension is also consistent.*

**Proof** Suppose that  $T$  with predicative comprehension proves a first-order contradiction  $\perp$ , e.g.  $\exists x(x \neq x)$ . Let  $A$  be the conjunction of the axioms of equality of reflexivity, symmetry and transitivity. Then there is a finite set of proper axioms  $A_1, \dots, A_n$  of  $T$  such that one can derive the sequent

$$LV, Eq, A, A_1, \dots, A_n \Rightarrow \perp$$

in pure predicative logic with the predicative value-range operator. Since both  $LV$  and  $Eq$  are  $\Pi_1^1$ -sentences, by the discussion following Corollary 2.2, there are sequences of predicative instantiations  $\overline{LV}^*, \overline{Eq}^*, \overline{A}_1^*, \dots, \overline{A}_n^*$  of  $LV, Eq, A_1, \dots, A_n$  (respectively) such that the sequent

$$\overline{LV}^*, \overline{Eq}^*, A, \overline{A}_1^*, \dots, \overline{A}_n^* \Rightarrow \perp$$

is provable in the restriction of the calculus of pure predicative logic with the predicative value-range operator to the language without second-order quantifiers. If in this proof we replace all the second-order variables by (say) the abstract  $\{x : x = x\}$ , we have a first-order proof of the sequent

$$\overline{LV}^\circ, \overline{Eq}^\circ, A, \overline{A}_1^\circ, \dots, \overline{A}_n^\circ \Rightarrow \perp$$

where each  $\circ$ -formula is obtained from the corresponding  $\star$ -formula by the substitution described above. Since all these  $\circ$ -formulas are in  $T^s$ , we have shown that  $T^s$  is inconsistent.  $\square$

The above proof is formalizable in  $\text{ID}_0(\text{superexp})$ . All syntactic manipulations are relatively simple. Technically, they are elementary, i.e., formalizable in

$\text{I}\Delta_0(\text{exp})$ , except for an application of the cut-elimination theorem. But, as noticed, this theorem is formalizable in  $\text{I}\Delta_0(\text{superexp})$ .

Let  $\check{\text{H}}$  be the pure  $\Pi_1^1$ -theory, based on the first-order language of equality, together with predicative comprehension. This theory only differs from Heck's predicative theory in that it does not allow the formation of impredicative value-ranges (along with the corresponding Law V that goes with them).

**Corollary 2.5**  $\check{\text{H}}$  is consistent.

**Proof** The first-order schematization of  $\check{\text{H}}$  is just Parsons theory. As remarked in the introduction, Burgess proved the finitistic consistency of this theory in [1]. Now, use the above theorem.  $\square$

### 3 The extension to $\Delta_1^1$ -comprehension

The scheme of  $\Delta_1^1$ -comprehension is constituted by the formulas

$$\forall x(\varphi(x) \leftrightarrow \rho(x)) \rightarrow \exists F \forall x(Fx \leftrightarrow \varphi(x)),$$

where  $\varphi$  is a  $\Sigma_1^1$ -formula and  $\rho$  is a  $\Pi_1^1$ -formula. The main aim of this section is to extend Theorem 2.4 and prove finitistically the following result:

**Theorem 3.1** Let  $\text{T}$  be a  $\Pi_1^1$ -theory. If  $\text{T}^s$  is consistent then  $\text{T}$  with  $\Delta_1^1$ -comprehension is also consistent.

In a first-order Parsons language, we can define a pairing operation. For instance, the Kuratowski-Wiener ordered pair is defined thus:

$$\langle x, y \rangle := \hat{u}.(u = \hat{v}(v = x \vee v = y) \vee u = \hat{v}(v = x)).$$

In Parsons theory, this pairing operation satisfies the pairing axiom  $P$ :

$$\forall x, y, u, v(\langle x, y \rangle = \langle u, v \rangle \rightarrow x = u \wedge y = v).$$

The presence of pairing simplifies many formulations since there will be no need to speak of tuples. However, the presence of pairing seems to be unavoidable for the efficient formulation of the following principle:

**Definition 3.2** Modified  $\Sigma_1^1$ -choice is the following scheme:

$$\forall x \exists F \varphi(F, x) \rightarrow \exists R \forall x \exists y \varphi(R_{x,y}, x),$$

where  $\varphi$  has no second-order quantifiers and  $R_{x,y}(u)$  stands for  $R(\langle u, \langle x, y \rangle \rangle)$ .

The (seemingly) unavoidability of pairing in the formulation of the above principle lies in the fact that the occurrences of the variable  $y$  above may play the role of a tuple of variables of unspecified arity. As it will be clear by the proof of Lemma 3.4 below, this is important because we are going to rely on Corollary 2.2 and, therefore, on abstracts with (unspecified) tuples of new variables.

Modified  $\Sigma_1^1$ -choice was introduced in [6], where the following result was proved. We repeat the argument here for completeness:

**Lemma 3.3** A pure  $\Pi_1^1$ -theory with predicative comprehension and modified  $\Sigma_1^1$ -choice proves the scheme of  $\Delta_1^1$ -comprehension.

**Proof** We argue informally. Suppose that  $\forall x(\forall G \varphi(G, x) \leftrightarrow \exists F \rho(F, x))$ , where both  $\varphi$  and  $\rho$  have no second-order quantifiers. In particular,

$$\forall x \exists G \exists F (\varphi(G, x) \rightarrow \rho(F, x)).$$

By modified  $\Sigma_1^1$ -choice (and using pairing cleverly), it is not difficult to conclude that  $\exists R \exists Q \forall x \exists y \exists z (\varphi(R_{x,y}, x) \rightarrow \rho(Q_{x,z}, x))$ . Take  $R$  and  $Q$  such that

$$\forall x \exists y \exists z (\varphi(R_{x,y}, x) \rightarrow \rho(Q_{x,z}, x)).$$

We claim that  $\exists F \rho(F, x)$  is equivalent to  $\exists z \rho(Q_{x,z}, x)$ . Note that the latter formula is predicative and, therefore, we can apply predicative comprehension to it. The right-to-left direction of the claim is obvious. Let  $x$  be given and assume that  $\exists F \rho(F, x)$ . Take  $y$  and  $z$  such that  $\varphi(R_{x,y}, x) \rightarrow \rho(Q_{x,z}, x)$ . By hypothesis, we have  $\forall G \varphi(G, x)$ . In particular,  $\varphi(R_{x,y}, x)$ . We get  $\rho(Q_{x,z}, x)$  and, therefore,  $\exists z \rho(Q_{x,z}, x)$ .  $\square$

In order to prove Theorem 3.1, we consider an extension of the calculus of sequents of the pure predicative logic with the predicative value-range operator. The extension is obtained by adding the following rule:

$$(choice) \quad \frac{\Gamma \Rightarrow \Delta, \exists F \varphi(F, a)}{\Gamma \Rightarrow \Delta, \exists R \forall x \exists y \varphi(R_{x,y}, x)}$$

where  $\varphi$  has no second-order quantifiers and  $a$  is an eigenvariable. The following formal deduction shows that modified  $\Sigma_1^1$ -choice is provable in this extended calculus:

$$\begin{array}{l} (\forall l) \\ (choice) \\ (\rightarrow r) \end{array} \quad \frac{\frac{\frac{\exists F \varphi(F, a) \Rightarrow \exists F \varphi(F, a)}{\forall x \exists F \varphi(F, x) \Rightarrow \exists F \varphi(F, a)}}{\forall x \exists F \varphi(F, x) \Rightarrow \exists R \forall x \exists y \varphi(R_{x,y}, x)}}{\Rightarrow \forall x \exists F \varphi(F, x) \rightarrow \exists R \forall x \exists y \varphi(R_{x,y}, x)}$$

The extended calculus does not enjoy the property of cut-elimination. When one tries to eliminate a cut coming from the application of *(choice)* there is no way to proceed. However, it is a known observation (see, for instance, [4] in a slightly different setting) that proofs with *(choice)* enjoy *partial* cut-elimination, in the sense that all cuts – with the exception of those whose cut-formula is a  $\Sigma_1^1$ -formula – can be eliminated. This has a conspicuous consequence for proofs of sequents consisting only of  $\Sigma_1^1$ -formulas. In this case, a proof of such a sequent enjoying partial cut-elimination consists solely of  $\Sigma_1^1$ -formulas (because of the subformula property). In a nutshell: If a sequent consisting only of  $\Sigma_1^1$ -formulas is provable in the extended sequent calculus, then it has a proof, in the same calculus, consisting only of  $\Sigma_1^1$ -formulas.

**Lemma 3.4** *If a sequent consisting only of  $\Sigma_1^1$ -formulas can be proved in the sequent calculus of pure predicative logic with the predicative value-range operator extended with the rule (choice), then it can be proved without this rule in the presence of the pairing axiom.*

**Proof** As we have discussed, such a sequent has a proof consisting only of  $\Sigma_1^1$ -formulas. We prove by induction on the number of inferences of this proof that the sequent can be proved without *(choice)* in the presence of the pairing axiom  $P$ . The only case that must be discussed is when the proof ends with an application of *(choice)*. So, let us consider an application of *(choice)* in which



the sequents  $\Gamma$  and  $\Delta$  consist solely of  $\Sigma_1^1$ -formulas. By induction hypothesis, there is a proof in the restricted system – i.e., without (*choice*) – of the sequent  $P, \Gamma \Rightarrow \Delta, \exists F\varphi(F, a)$ . If  $\Gamma$  has the form  $\exists \overline{G}_1 \rho_1(\overline{G}_1), \dots, \exists \overline{G}_r \rho_r(\overline{G}_r)$ , where the formulas  $\rho_1(\overline{G}_1), \dots, \rho_r(\overline{G}_r)$  are all without second-order quantifications, then the restricted system proves

$$P, \rho_1(\overline{G}_1), \dots, \rho_r(\overline{G}_r) \Rightarrow \Delta, \exists F\varphi(F, a).$$

By the discussion following Theorem 2.2, we can replace the  $\Sigma_1^1$ -formulas in  $\Delta$  and the very formula  $\exists F\varphi(F, a)$  by suitable predicative instantiations. We actually only care for the instantiations of the last formula. As discussed, these instantiations can be reduced to only one. Let this instantiation be given by the abstract  $\{u : \theta(u, y, a)\}$ . Note that we may assume that  $y$  is a single variable because of the availability of the pairing axiom.

It is clear that we may conclude that the restricted theory proves

$$P, \rho_1(\overline{G}_1), \dots, \rho_r(\overline{G}_r) \Rightarrow \Delta, \exists y\varphi(\{u : \theta(u, y, a)\}, a).$$

Let  $\tilde{\theta}(z)$  be the formula  $\exists u, y, x (\theta(u, y, x) \wedge z = \langle u, \langle x, y \rangle \rangle)$ . It is easy to see that  $\theta(u, y, x)$  and  $\tilde{\theta}(\langle u, \langle x, y \rangle \rangle)$  are equivalent (provably in the restricted system *with the pairing axiom*). Therefore, the restricted theory proves

$$P, \rho_1(\overline{G}_1), \dots, \rho_r(\overline{G}_r) \Rightarrow \Delta, \exists y\varphi(\{u : \tilde{\theta}(\langle u, \langle a, y \rangle \rangle)\}, a).$$

Using the  $\forall r$ -rule, it also proves

$$P, \rho_1(\overline{G}_1), \dots, \rho_r(\overline{G}_r) \Rightarrow \Delta, \forall x \exists y \varphi(\{u : \tilde{\theta}(\langle u, \langle x, y \rangle \rangle)\}, x),$$

and, hence,  $P, \rho_1(\overline{G}_1), \dots, \rho_r(\overline{G}_r) \Rightarrow \Delta, \exists R \forall x \exists y \varphi(R_{x,y}, x)$ . We conclude that the sequent

$$P, \exists \overline{G}_1 \rho_1(\overline{G}_1), \dots, \exists \overline{G}_r \rho_r(\overline{G}_r) \Rightarrow \Delta, \exists R \forall x \exists y \varphi(R_{x,y}, x)$$

is provable in the restricted system. In other words, the restricted system does indeed prove the sequent  $P, \Gamma \Rightarrow \Delta, \exists R \forall x \exists y \varphi(R_{x,y}, x)$ , which corresponds to the conclusion of the rule (*choice*).  $\square$

We are now ready to prove Theorem 3.1. We will actually show that the theory  $\mathbb{T}$  with  $\Delta_1^1$ -comprehension is  $\Sigma_1^1$ -conservative over  $\mathbb{T}$  with predicative comprehension (then apply Theorem 2.4). Suppose that  $B$  is a  $\Sigma_1^1$ -sentence and that the theory  $\mathbb{T}$  with  $\Delta_1^1$ -comprehension proves  $B$ . Let  $A$  be the conjunction of the axioms of reflexivity, symmetry and transitivity for equality. Then there is a finite set of proper axioms  $A_1, \dots, A_n$  of  $\mathbb{T}$  such that the sequent

$$LV, Eq, A, A_1, \dots, A_n \Rightarrow B$$

is provable in the extension, with (*choice*), of the calculus of sequents of the pure predicative logic with the predicative value-range operator. All the formulas in the antecedent of the above sequent are  $\Pi_1^1$ -formulas. Consider their negations  $\widetilde{LV}, \widetilde{Eq}, \widetilde{A}_1, \dots, \widetilde{A}_n$  in the form of  $\Sigma_1^1$ -formulas by using the De Morgan laws. Of course, we can prove the sequent

$$A \Rightarrow \widetilde{LV}, \widetilde{Eq}, \widetilde{A}_1, \dots, \widetilde{A}_n, B$$

in the extended calculus. By the previous lemma, the sequent

$$P, A \Rightarrow \widetilde{LV}, \widetilde{Eq}, \widetilde{A}_1, \dots, \widetilde{A}_n, B$$

can be proved in the calculus of sequents of the pure predicative logic with the predicative value-range operator. Therefore, so it happens with the sequent

$$LV, Eq, P, A, A_1, \dots, A_n \Rightarrow B.$$

It follows that the theory  $\mathbb{T}$  with predicative comprehension proves  $B$ .

The proof of Theorem 3.1 is finished and it is plain that it can be formalized in  $\mathbb{I}\Delta_0(\text{superexp})$ . It is also worth noting that the proof delivers more than  $\Delta_1^1$ -comprehension: It even delivers modified  $\Sigma_1^1$ -choice.

#### 4 The consistency of Heck's predicative second-order system

It was observed in the introduction that the consistency of Parsons theory has a finitistic proof, formalizable in  $\mathbb{I}\Delta_0(\text{super}^2\text{exp})$ . Hence, by Corollary 2.5, the theory  $\mathbb{H}$  (the modification of Heck's predicative theory that does not allow the formation of impredicative value-range terms) has a consistency proof in  $\mathbb{I}\Delta_0(\text{super}^2\text{exp})$ . In this section, we show that Heck's theory  $\mathbb{H}$ , with the full value-range operator as regulated by schematic Law V, has also a consistency proof in the theory  $\mathbb{I}\Delta_0(\text{super}^2\text{exp})$ .

In order to take care of impredicative value-range terms, we chose to work on the firm and well-studied ground of theories without a variable-binding term-forming operator. We define a (consistent) theory  $\text{PV}_\omega^+$  and show that  $\mathbb{H}$  is interpretable in it. The language of this theory is the language of  $\text{PV}_\omega$  of Burgess (cf. [2]) together with a pairing apparatus (pairing objects into objects). Briefly put, there is a style of variables  $x, y, z, \dots$  (first-order variables) for objects and, for each natural number  $n$ , a style of variable for  $n$ -th round concepts variables  $F^n, G^n, H^n, \dots$  (we usually omit the superscript of zeroth round concept variables and write  $F, G, H, \dots$  instead). We have the identity symbol  $=$  for objects, a binary function symbol  $\langle \cdot, \cdot \rangle$  for the pairing of objects and, for each  $n$ , an extension symbol  $\ddagger^n$  which can be applied to  $n$ -th round concept variables  $F^n$  in order to form a first-order term  $\ddagger^n F^n$  (when  $n = 0$ , we usually omit the superscripts and simply write  $\ddagger F$ ). The part of the language restricted to variables of round at most  $n$  is denoted by  $\mathcal{L}_n$  (hence, only extension symbols  $\ddagger^k$ , with  $k \leq n$ , appear in  $\mathcal{L}_n$ ). The full language is denoted by  $\mathcal{L}_\omega$ . There are four kinds of axioms:

- (1)  $\langle x, y \rangle = \langle u, v \rangle \rightarrow x = u \wedge y = v$ .
- (2)  $\forall x \exists F \varphi(F, x) \rightarrow \exists R \forall x \exists y \varphi(R_{x,y}, x)$ , where  $\varphi$  is a formula of  $\mathcal{L}_0$  without second-order quantifiers.
- (3)  $\exists F^n \forall x (F^n x \leftrightarrow \varphi(x))$ , for  $n \geq 1$  and  $\varphi$  a formula of  $\mathcal{L}_n$  without second-order quantifiers of variables of round  $n$ .
- (4)  $\ddagger^n F^n = \ddagger^m G^m \leftrightarrow \forall x (F^n x \leftrightarrow G^m x)$ , for natural numbers  $n$  and  $m$ .

**Lemma 4.1** *Heck's theory  $\mathbb{H}$  is interpretable in  $\text{PV}_\omega^+$ .*

**Proof** The first-order domain of  $\mathbb{H}$  is interpreted by the first-order domain of  $\text{PV}_\omega^+$ , and the second-order domain of  $\mathbb{H}$  is interpreted by the *zeroth* round domain of  $\text{PV}_\omega^+$ . We must interpret the value-range operator of  $\mathbb{H}$ . The treatment is different depending on whether the term is predicative or impredicative.

Let us first consider the predicative case, concerning terms of Heck's language of the form  $\hat{x}.\varphi(x)$ , where  $\varphi$  has no second-order quantifiers. The interpretation of these terms relies crucially on the availability of modified

$\Sigma_1^1$ -choice, as stated in axiom (2). The class of *extended*  $\Sigma_1^1$ -formulas (respectively, *extended*  $\Pi_1^1$ -formulas) is the smallest class of formulas of  $\mathcal{L}_0$  which contains the formulas without second-order quantifiers and is closed for conjunction, disjunction, first-order quantifications and existential second-order quantifiers (respectively, universal second-order quantifiers). Since we have modified  $\Sigma_1^1$ -choice in  $\mathcal{L}_0$ , every extended  $\Sigma_1^1$ -formula (respectively, extended  $\Pi_1^1$ -formula) is equivalent to a  $\Sigma_1^1$ -formula (respectively, a  $\Pi_1^1$ -formula) of the language  $\mathcal{L}_0$ .

We are now ready to interpret  $\check{\mathbb{H}}$  in the fragment of  $\text{PV}_\omega^+$  restricted to  $\mathcal{L}_0$ . The terms of  $\check{\mathbb{H}}$  of the form  $\hat{x}.\varphi(x)$  can be ranked according to the depth of nesting of these terms. If the rank is zero, this means that there are no value-range operators in  $\varphi$ . Let  $\mathcal{L}_{\check{\mathbb{H}}^0}$  be the fragment of the language  $\mathcal{L}_{\check{\mathbb{H}}}$  of  $\check{\mathbb{H}}$  in which only terms of rank zero occur. We interpret a formula of the form  $\hat{x}.\varphi(x) = y$ , where  $\hat{x}.\varphi(x)$  has rank zero, by the formula  $\exists F(y = \dagger F \wedge \forall x(Fx \leftrightarrow \varphi(x)))$ . By predicative comprehension (a consequence of axiom 2) and axiom (4) restricted to  $m = n = 0$ , this formula is equivalent to the  $\Pi_1^1$ -formula:  $\forall F(y = \dagger F \leftrightarrow \forall x(Fx \leftrightarrow \varphi(x)))$ . Of course, a negation of the form  $\hat{x}.\varphi(x) \neq y$ , for  $\hat{x}.\varphi(x)$  of rank zero, can also be put in  $\Sigma_1^1$  and  $\Pi_1^1$ -form. With this base case discussed, it is now standard to translate (by induction on the complexity) every formula  $\varphi$  of  $\mathcal{L}_{\check{\mathbb{H}}^0}$  *without second-order quantifiers* into equivalent extended  $\Sigma_1^1$ -formulas  $\varphi^\exists$  and extended  $\Pi_1^1$ -formulas  $\varphi^\forall$  of  $\mathcal{L}_0$ . By  $\Delta_1^1$ -comprehension (again, a consequence of axiom 2), we have (\*)  $\text{PV}_\omega^+ \vdash \exists F \forall x((Fx \leftrightarrow \varphi^\exists(x)) \wedge (Fx \leftrightarrow \varphi^\forall(x)))$  for such  $\varphi$  of  $\mathcal{L}_{\check{\mathbb{H}}^0}$ . Suppose now that  $\hat{x}.\varphi(x)$  is a term of rank 1. Let us interpret the formula  $\hat{x}.\varphi(x) = y$ . We translate this equality by the equivalent formulas  $\exists F(y = \dagger F \wedge \forall x(Fx \leftrightarrow \varphi^\exists(x)))$  and  $\forall F(y = \dagger F \leftrightarrow \forall x(Fx \leftrightarrow \varphi^\forall(x)))$ . This time the equivalence holds because we have (\*), as well as the already mentioned restriction of (4). Note that the above formulas are *equivalent* to  $\Sigma_1^1$ -formulas and  $\Pi_1^1$ -formulas, respectively. This can be easily seen by replacing  $\varphi^\exists$  by  $\varphi^\forall$  (and vice-versa) in appropriate places. The translation standardly extends to all formulas of  $\mathcal{L}_{\check{\mathbb{H}}^1}$  (the fragment of  $\mathcal{L}_{\check{\mathbb{H}}}$  in which only terms of rank zero and one occur) *without second-order quantifiers*, and the comprehension scheme (\*) extends to these formulas. It is clear that the iteration of this process provides a translation of the formulas of  $\mathcal{L}_{\check{\mathbb{H}}}$  *without second-order quantifiers* into the fragment of  $\text{PV}_\omega^+$  restricted to  $\mathcal{L}_0$ . By construction, the comprehension principle (\*) holds for these formulas. The extension of the translation to all formulas of  $\mathcal{L}_{\check{\mathbb{H}}}$  is automatic: Translate second-order quantifiers by corresponding (zero round) second-order quantifiers. Since the base case without second-order quantifiers has two (equivalent) translations, we fix one such translation and extend it – as was just described – to all formulas of  $\check{\mathbb{H}}$ :  $\varphi \rightsquigarrow \varphi^T$ .

It remains to extend the interpretation to the full language of  $\mathbb{H}$ , i.e., to formulas which also include terms of the form  $\hat{x}.\varphi(x)$ , where  $\varphi$  may have second-order quantifiers. We can give a (impredicative) rank to terms of this form. If  $\varphi$  has no second-order quantifiers, the term  $\hat{x}.\varphi(x)$  has zero (impredicative) rank. If  $\varphi$  has only terms of (impredicative) rank  $\leq n$ , then  $\hat{x}.\varphi(x)$  has (impredicative) rank  $\leq n + 1$ . We have already shown how to interpret formulas of the language of  $\check{\mathbb{H}}$ , i.e., formulas of  $\mathbb{H}$  which only have

terms of zero (impredicative) rank. Given one such formula  $\varphi(x)$ , we translate the equality  $\hat{x}.\varphi(x) = y$  by  $\exists F^1(y = \dagger F^1 \wedge \forall x(F^1 x \leftrightarrow \varphi^T(x)))$ . Note that, due to axiom (3),  $\text{PV}_\omega^+ \vdash \exists F^1 \forall x(F^1 x \leftrightarrow \varphi^T(x))$ . It is now standard to translate formulas of  $\mathbf{H}$  with terms of at most impredicative rank 1 by formulas of  $\mathcal{L}_1$ . This procedure can be iterated and we end up translating the formulas of  $\mathbf{H}$  which only include terms of at most (impredicative) rank  $n$  by formulas of  $\mathcal{L}_n$ . The translation of the Law V of the theory  $\mathbf{H}$  is provable in  $\text{PV}_\omega^+$  because of axiom scheme (4).

We have obtained an interpretation of  $\mathbf{H}$  into  $\text{PV}_\omega^+$ .  $\square$

Let us call  $\mathsf{T}_0$  the restriction of the theory  $\text{PV}_\omega^+$  to  $\mathcal{L}_0$ , together with the sentences  $\exists_k x \forall F(x \neq \dagger F)$  (one for each natural number  $k$ , saying that there are at least  $k$  objects outside of the range of  $\dagger$ ).

**Lemma 4.2** *The theory  $\mathsf{T}_0$  is consistent.*

**Proof** The proof consists of three steps. In the first step, we observe that Burgess' consistency proof of Parsons theory in pp. 136-137 and p. 140 of [2] can be easily adapted so that it contains an auxiliary unary predicate symbol  $R$  such that: (i) the scheme  $\neg R(\hat{x}.\varphi(x))$  holds for all formulas  $\varphi$  of the language; (ii)  $\exists_k x R(x)$ , for all natural numbers  $k$ . In the second step, we use the results of Section 3 to conclude that the second-order version of this theory with modified  $\Sigma_1^1$ -choice and the axiom  $\forall F \neg R(\hat{x}.Fx)$  is consistent. (This is the *only* step in proving the finitistic consistency of  $\mathbf{H}$  where we use the results of the previous sections.) In the third step, we interpret  $\mathsf{T}_0$  in the theory of the previous step by translating  $\dagger F$  by  $\hat{x}.Fx$  and pairing via the Kuratowski-Wiener definition. It is clear that this argument does the job.  $\square$

We are working with the (two-sorted) first-order theory  $\mathsf{T}_0$  instead of the more natural theory described in the first step of the above proof because, in order to deal with impredicative value-range terms, we need to rely on some results typical of first-order logic: the splitting lemma, the injection lemma, etc (cf. [2]). These results have not yet been considered in the framework of a language with value-ranges.

**Lemma 4.3** *The theory  $\text{PV}_\omega^+$  is consistent.*

**Proof** We have shown that the theory  $\mathsf{T}_0$  is consistent. Notice that there are infinitely many elements outside of the range of  $\dagger$  (the intuitive idea is that there is enough room left for interpreting the impredicative value-ranges). The construction at the turn of pages 136-137 of [2] (which uses the above mentioned splitting lemma, injection lemma, etc) shows that the theory  $\mathsf{T}_1$ , the restriction of  $\text{PV}_\omega^+$  to the language  $\mathcal{L}_1$ , is consistent. Moreover, we can ensure that  $\exists_k x \forall F^1(x \neq \dagger^1 F^1)$ . Of course, this process iterates to all the restrictions  $\mathsf{T}_n$  of  $\text{PV}_\omega^+$  to the language  $\mathcal{L}_n$ . Therefore, the consistency of their union, i.e., of  $\text{PV}_\omega^+$ , is established.  $\square$

As Burgess remarks, the relative consistency proofs of  $\mathsf{T}_{n+1}$  with respect to  $\mathsf{T}_n$  are formalizable in  $\mathbf{ID}_0(\text{superexp})$ . In fact, the theory  $\mathbf{ID}_0(\text{superexp})$  proves  $\forall n (\text{Con}_{\mathsf{T}_n} \rightarrow \text{Con}_{\mathsf{T}_{n+1}})$ , where  $\text{Con}_{\mathsf{T}}$  formalizes the consistency of the theory  $\mathsf{T}$ . As a consequence, we get  $\mathbf{ID}_0(\text{super}^2\text{exp}) \vdash \text{Con}_{\mathsf{T}_0} \rightarrow \text{Con}_{\text{PV}_\omega^+}$ . On the other

hand, the consistency of  $\mathsf{T}_0$  hinges upon the consistency of Parsons theory. It is now clear that  $\mathsf{I}\Delta_0(\text{super}^2\text{exp}) \vdash \text{Con}_{\mathsf{T}_0}$ . Hence,  $\mathsf{I}\Delta_0(\text{super}^2\text{exp}) \vdash \text{Con}_{\text{PV}_{\downarrow}^+}$ . Now, by Lemma 4.1, we may conclude that the consistency of  $\mathsf{H}$  is provable in  $\mathsf{I}\Delta_0(\text{super}^2\text{exp})$ . Of course, our proof even shows that this is also true for Heck's theory with the  $\Delta_1^1$ -comprehension scheme.

### 5 The consistency of Heck's ramified predicative system

Let  $\mathsf{T}$  be a theory in a first-order Parsons language and consider  $\mathsf{T}^{\mathsf{H}}$  its extension to the second-order language with predicative comprehension and with the full value-range operator regulated by schematic Law V. The arguments of the previous section can be adapted to show the following:

**Theorem 5.1** *If  $\mathsf{T}$  is consistent, then  $\mathsf{T}^{\mathsf{H}}$  is also consistent.*

Although the checking is a bit tiresome (one must go through theorems 2.4, 3.1 and the constructions of the previous section), it should be clear that the above theorem is also true for first-order Parsons languages with finitely many sorts. Therefore, we can apply the theorem to the theory  $\mathsf{T}^{\mathsf{H}}$  itself and get that  $(\mathsf{T}^{\mathsf{H}})^{\mathsf{H}}$  is consistent, if  $\mathsf{T}$  is. We can iterate this procedure and define  $\mathsf{R}_0 = \mathsf{T}$  and  $\mathsf{R}_{n+1} = (\mathsf{R}_n)^{\mathsf{H}}$ . Of course, the union of these theories is consistent. Do notice that, when we start with Parsons "first-order portion of the *Grundgesetze*," the union of these theories is essentially Heck's ramified predicative fragment of Frege's arithmetic (cf. section 4 of [9]). Therefore:

**Theorem 5.2** *Heck's ramified predicative second-order system is consistent.*

This result was proved by Heck in [9], using model theory. We have proved more. Theorem 5.1 is true even if we allow  $\Delta_1^1$ -comprehension (or modified  $\Sigma_1^1$ -choice) instead of just predicative comprehension. As a consequence, Theorem 5.2 also holds if, at each round, one has  $\Delta_1^1$ -comprehension (or even modified  $\Sigma_1^1$ -choice).

Our consistency proofs were designed to be finitistic. What finitistic theory proves Theorem 5.2 (and its extension mentioned in the previous paragraph)? The consistency of Parsons first-order theory is provable in  $\mathsf{I}\Delta_0(\text{super}^2\text{exp})$ . Theorem 5.1 is also provable in this theory. Therefore, the above theorem is provable in  $\mathsf{I}\Delta_0(\text{super}^3\text{exp})$ .

### 6 Some observations, some questions

Burgess comments in [2] that "technical issues have not been wholly resolved" for predicative Fregean theories with the value-range operator. In this paper, we addressed the problems described by Burgess. Nevertheless, there do remain some unresolved technical issues, although ones that are not solely typical of Fregean theories.

We have observed in the introduction that the bounded theory  $\mathsf{I}\Delta_0$  is interpretable in Heck's predicative second-order system. One can do much better for Heck's ramified predicative second-order system: This theory interprets the bounded theory  $\mathsf{I}\Delta_0(\text{exp})$ . This is a consequence of a result of Burgess and Allen Hazen in [3]. What Burgess and Hazen actually show is that  $\mathsf{I}\Delta_0(\text{exp})$  is interpretable in the ramified predicative arithmetic built on top of an infinite Dedekind domain (axiomatized via a first-order theory with successor and zero

symbols, and axioms saying that successor is injective and that zero is outside the range of the injection). A close inspection of the proof shows that only two-rounds of variables are necessary to interpret  $\mathsf{I}\Delta_0(\text{exp})$ . Since the consistency of this two-round fragment is known to be provable in  $\mathsf{I}\Delta_0(\text{superexp})$ , the result is optimal for these two rounds. However, the consistency of the full ramified theory (with all finite rounds) is only known to be provable in  $\mathsf{I}\Delta_0(\text{super}^2\text{exp})$ . There is a gap between what is known to be interpretable in the Burgess-Hazen ramified theory (compare with the last section of [3]) and the theory in which it is known that its consistency proof can be formalized. Is it possible to close this gap? The gap is even wider for Heck's ramified predicative second-order system because our consistency proof is only formalizable in  $\mathsf{I}\Delta_0(\text{super}^3\text{exp})$ . Of course, Heck's ramified second-order system has more resources than the Burgess-Hazen ramified theory, due to the presence of the value-range operator. Can the presence of this operator be explored to obtain stronger interpretations? Can the gaps be bridged?

## 7 Appendix on predicative cut-elimination

The result that the theorem of cut-elimination for pure predicative second-order logic is formalizable in the theory  $\mathsf{I}\Delta_0(\text{superexp})$  seems to be folklore. The proof of Takeuti in [13] does not seem to be readily formalizable in this theory. The problem hinges on the fact that a (predicative) instantiation of a second-order quantification can arise through first-order formulas of arbitrary complexity. A new measure of complexity of formulas is needed. Below, in order to avoid a transfinite measure, we opt for a two stage cut-elimination. In the first stage, we describe a cut-elimination procedure for cut-formulas with second-order bounded quantifiers (second-order cuts). This procedure *does not* eliminate predicative cuts (i.e., cuts of first-order formulas, possibly with second-order parameters). We claim that this cut-elimination result is formalizable in  $\mathsf{I}\Delta_0(\text{superexp})$ . Therefore, this theory proves that every theorem of pure predicative logic has a derivation without second-order cuts. The attentive reader will notice that this amount of cut-elimination is sufficient for the proofs of this paper, namely for Corollary 2.2 and (suitably adapted for the partial cut-elimination used) for the remark before Lemma 3.4. However, having now a proof without second-order cuts, it is a well trodden path to get the full cut-elimination result. We duly discuss this further stage also.

*Rules of the sequent calculus.* We work within the context of a sequent calculus where sequents are of the form  $\Gamma \Rightarrow \Delta$ , with  $\Gamma$  and  $\Delta$  *sets* of formulas (unlike in Takeuti [13]). As usual, we separate the elements in these sets by commas, and write e.g.  $\Gamma, \varphi$  for  $\Gamma \cup \{\varphi\}$ . The logical rules are the same as in [13], with two exceptions. Firstly, the only structural rule is Cut. Secondly, axioms are of the form  $\Gamma, \varphi \Rightarrow \Delta, \varphi$ , where  $\varphi$  is a predicative formula (this choice of initial sequents is specially adapted for the proof below). We do without the syntactic distinction between free and bound variables and adopt the conventions of [11] (suitably adapted to our setting) regarding this issue.

*Second-order size of a formula.* Since we are only interested in eliminating second-order cuts, we use a special measure for the size of a formula. In this measure, predicative formulas have measure zero. The *second-order size*  $|\varphi|_2$

of a formula  $\varphi$  is defined as follows:  $|\varphi|_2 = 0$  if  $\varphi$  is a predicative formula; in the *remaining cases*  $|\neg\varphi|_2 = |\forall x\varphi|_2 = |\exists x\varphi|_2 = |\forall^2 R\varphi|_2 = |\exists^2 R\varphi|_2 = |\varphi|_2 + 1$  and  $|\varphi \vee \psi|_2 = |\varphi \wedge \psi|_2 = |\varphi \rightarrow \psi|_2 = \max(|\varphi|_2, |\psi|_2) + 1$ .

A straightforward induction shows that  $|\varphi[\theta/R]|_2 = |\varphi|_2$  for every formula  $\varphi$  and first-order abstract  $\theta$ . Here,  $\varphi[\theta/R]$  denotes the formula obtained by effecting the substitution of the abstract  $\theta$  for the second-order variable  $R$  in  $\varphi$ .

*Length of a derivation.* This is defined as in first-order logic, with axioms as derivations of length zero (see [11]). We denote the length of a derivation  $d$  by  $|d|$ .

*Second-order cut-rank of a derivation.* If  $d$  is a derivation, then the *second-order cut-rank* of  $d$  is  $\rho_2(d) = \max\{|\varphi|_2: \varphi \text{ is a cut-formula of } d\}$ .

Our strategy is an adaption of the argument in [11].

*Weakening lemma.* A derivation  $d$  of  $\Gamma \Rightarrow \Delta$  can be weakened into a derivation  $d_{\Gamma, \Delta}^{\Gamma', \Delta'}$  of  $\Gamma' \Rightarrow \Delta'$  by adding  $\Gamma' \setminus \Gamma$  to the left-hand side of each sequent and  $\Delta' \setminus \Delta$  to the right-hand side of each sequent, assuming  $\Gamma \subseteq \Gamma'$  and  $\Delta \subseteq \Delta'$ . The length and second-order cut-rank of  $d$  and  $d_{\Gamma, \Delta}^{\Gamma', \Delta'}$  are the same.

*First-order substitution lemma.* If  $d$  is a derivation of  $\Gamma \Rightarrow \Delta$  and  $x$  is not the eigenvariable of any application of  $\forall r$  or  $\exists l$  in  $d$  then, by replacing every occurrence of  $x$  by a term  $s$  in  $d$ , one obtains a derivation  $d[s/x]$  of  $\Gamma[s/x] \Rightarrow \Delta[s/x]$  with the same length and second-order cut-rank as  $d$ .

*Second-order substitution lemma.* If  $d$  is a derivation of  $\Gamma \Rightarrow \Delta$  and  $R$  is not the eigenvariable of any application of  $\forall^2 r$  or  $\exists^2 l$  in  $d$  then, by effecting the substitution of every occurrence of  $R$  by the predicative abstract  $\theta$  in  $d$ , one obtains a derivation  $d[\theta/R]$  of  $\Gamma[\theta/R] \Rightarrow \Delta[\theta/R]$  with the same length and second-order cut-rank as  $d$ . Both the remark after the definition of second-order size of a formula and our statement of the axioms with *first-order formulas*  $\varphi$  are essential to the proof of this result.

*Second-order inversion lemma.* Suppose that the formulas  $\neg\varphi$ ,  $\varphi \wedge \psi$ ,  $\varphi \vee \psi$ ,  $\varphi \rightarrow \psi$ ,  $\forall x\varphi$ ,  $\exists x\varphi$ ,  $\forall R\varphi$  and  $\exists R\varphi$  below are *not* predicative formulas. Then

- (1) If  $d$  is a derivation of  $\Gamma, \neg\varphi \Rightarrow \Delta$ , then there exists a derivation  $d_\varphi$  of  $\Gamma \Rightarrow \Delta, \varphi$ .
- (2) If  $d$  is a derivation of  $\Gamma \Rightarrow \Delta, \varphi \wedge \psi$ , then there exist a derivation  $d_\varphi$  of  $\Gamma \Rightarrow \Delta, \varphi$  and a derivation  $d_\psi$  of  $\Gamma \Rightarrow \Delta, \psi$ .
- (3) If  $d$  is a derivation of  $\Gamma, \varphi \vee \psi \Rightarrow \Delta$ , then there exist a derivation  $d_\varphi$  of  $\Gamma, \varphi \Rightarrow \Delta$  and a derivation  $d_\psi$  of  $\Gamma, \psi \Rightarrow \Delta$ .
- (4) If  $d$  is a derivation of  $\Gamma, \varphi \rightarrow \psi \Rightarrow \Delta$ , then there exist a derivation  $d_\varphi$  of  $\Gamma \Rightarrow \Delta, \varphi$  and a derivation  $d_\psi$  of  $\Gamma, \psi \Rightarrow \Delta$ .
- (5) If  $d$  is a derivation of  $\Gamma \Rightarrow \Delta, \forall x\varphi$ , then there exists a derivation  $d_\varphi$  of  $\Gamma \Rightarrow \Delta, \varphi$ .
- (6) If  $d$  is a derivation of  $\Gamma, \exists x\varphi \Rightarrow \Delta$ , then there exists a derivation  $d_\varphi$  of  $\Gamma, \varphi \Rightarrow \Delta$ .
- (7) If  $d$  is a derivation of  $\Gamma \Rightarrow \Delta, \forall R\varphi$ , then there exists a derivation  $d_\varphi$  of  $\Gamma \Rightarrow \Delta, \varphi$ .

- (8) If  $d$  is a derivation of  $\Gamma, \exists R\varphi \Rightarrow \Delta$ , then there exists a derivation  $d_\varphi$  of  $\Gamma, \varphi \Rightarrow \Delta$ .

Furthermore, in all cases  $|d_\theta| \leq |d|$  and  $\rho_2(d_\theta) \leq \rho_2(d)$ , for  $\theta = \varphi, \psi$ .

The above result is an adaptation of the inversion lemma of [11] to our setting. The reader might be puzzled by the restriction to non-predicative formulas in the above lemma. In fact, points (1) to (8) of the inversion lemma are always true. It is the bound  $|d_\theta| \leq |d|$ , with  $\theta = \varphi, \psi$ , that does not hold anymore. The reason lies in the fact that we are admitting axioms of the form of  $\Gamma, \varphi \Rightarrow \Delta, \varphi$ , with  $\varphi$  is a predicative formula (as opposed to atomic formulas  $\varphi$  in [11]). For instance, consider in case (1) the situation in which  $\Gamma, \neg\varphi \Rightarrow \Delta$  is an axiom. If  $\neg\varphi$  is non-predicative then  $\Gamma \cap \Delta$  must have a common formula and, therefore,  $\Gamma \Rightarrow \Delta, \varphi$  is also an axiom. However, if  $\neg\varphi$  were predicative, then the sequent  $\Gamma \Rightarrow \Delta, \varphi$  need not be an axiom. In this case,  $\neg\varphi \in \Delta$  and the argument is as follows:  $\Gamma, \varphi \Rightarrow \Delta, \varphi$  is an axiom and, from this sequent, we may conclude  $\Gamma \Rightarrow \Delta, \varphi$  by the rule  $\neg r$ . Note that, in this situation,  $|d_\varphi| = |d| + 1$ . A unit must be added. Actually, we could have formulated the above lemma for all formulas (not only the predicative formulas) as long as we were content with the bound  $|d_\theta| \leq |d| + 1$ , with  $\theta = \varphi, \psi$ . This is a perfectly good option. The only consequence is that the bounds in the results below would have to be slightly increased (but still quite acceptable for our formalizations).

**Lemma 7.1 (Second-order reduction lemma)** *Suppose that  $d_1$  is a derivation of  $\Gamma_1 \Rightarrow \varphi, \Delta_1$  and  $d_2$  is a derivation of  $\Gamma_2, \varphi \Rightarrow \Delta_2$  such that  $\rho_2(d_1) < |\varphi|_2$  and  $\rho_2(d_2) < |\varphi|_2$ . Then there exists a derivation  $d$  such that:*

- $d$  is a derivation of  $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ ;
- $|d| \leq |d_1| + |d_2|$ ;
- $\rho_2(d) < |\varphi|_2$ .

**Proof** To start with, observe that  $\varphi$  is not a predicative formula (because  $|\varphi|_2 > 0$ ). The proof distinguishes several cases, according to the form of  $\varphi$ . In each case, the proof is by induction on  $|d_1| + |d_2|$  and a corresponding case of the second-order inversion lemma is used in the argument (the proof follows the blueprint of the reduction lemma of [11]). Here, we analyze only the case where  $\varphi$  is the formula  $\forall^2 R\psi$ .

We begin with the situation in which  $\varphi$  is not the principal formula in at least one of the derivations  $d_1$  or  $d_2$ . Suppose it is not the principal formula in  $d_1$  (the  $d_2$  case is similar). A possibility is that  $\Gamma_1 \Rightarrow \varphi, \Delta_1$  is an axiom. In this case we do not have to use the induction hypothesis: there must be a common (predicative) formula in  $\Gamma_1$  and  $\Delta_1$  and, therefore,  $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$  is also an axiom. In case  $\Gamma_1 \Rightarrow \varphi, \Delta_1$  is the conclusion of a rule in  $d_1$ , one just applies the induction hypothesis to the derivation of each premise together with  $d_2$  and, with the resulting sequence(s), apply the very same rule (see [11]).

The interesting situation is when  $\varphi$  is the principal formula of the last step of both  $d_1$  and  $d_2$ . Then the rules  $\forall^2 r$  and  $\forall^2 l$  must have been applied in the last inferences of  $d_1$  and  $d_2$ , respectively. Without loss of generality



(eventually applying the weakening lemma), assume that  $\varphi$  is a side formula in the last step of  $d_2$ . Then  $d_2$  has the form

$$(\forall^2 l) \quad \frac{\frac{d'_2}{\Gamma_2, \varphi, \psi[\theta/R] \Rightarrow \Delta_2}}{\Gamma_2, \varphi \Rightarrow \Delta_2}$$

where  $\theta$  is a predicative abstract. (In our notation,  $d'_2$  is a derivation of the sequent  $\Gamma_2, \varphi, \psi[\theta/R] \Rightarrow \Delta_2$ . Therefore, it *includes* this sequent.) Applying the induction hypothesis to  $d_1$  and  $d'_2$ , we find a derivation  $d'$  of  $\Gamma_1, \Gamma_2, \psi[\theta/R] \Rightarrow \Delta_1, \Delta_2$  such that

$$|d'| \leq |d_1| + |d'_2| < |d_1| + |d_2| \quad \text{and} \quad \rho_2(d') < |\varphi|_2.$$

Applying the inversion lemma to  $d_1$ , we find a derivation  $d_\psi$  of  $\Gamma_1 \Rightarrow \Delta_1, \psi$  such that

$$|d_\psi| \leq |d_1| \quad \text{and} \quad \rho_2(d_\psi) \leq \rho_2(d_1).$$

By the second-order substitution lemma, the derivation  $d_\psi[\theta/R]$  has the same length and second-order cut-rank as  $d_\psi$ , and furthermore  $d_\psi[\theta/R]$  is a derivation of  $\Gamma_1 \Rightarrow \Delta_1, \psi[\theta/R]$ . Take  $d$  to be the following derivation.

$$(cut) \quad \frac{\frac{d_\psi[\theta/R]}{\Gamma_1 \Rightarrow \Delta_1, \psi[\theta/R]} \quad \frac{d'}{\Gamma_1, \Gamma_2, \psi[\theta/R] \Rightarrow \Delta_1, \Delta_2}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Then:

$$|d| = \max(|d_\psi[\theta/R]|, |d'|) + 1 \leq \max(|d_1| + 1, |d_1| + |d_2|) \leq |d_1| + |d_2|$$

$$\rho_2(d) = \max(\rho_2(d_\psi[\theta/R]), \rho_2(d'), |\psi[\theta/R]|_2) < |\varphi|_2$$

since  $|\psi[\theta/R]|_2 = |\psi|_2$ , as observed before.  $\square$

**Lemma 7.2 (Second-order cut-elimination)** *Suppose that  $d$  is a derivation of  $\Gamma \Rightarrow \Delta$ . If  $\rho_2(d) > 0$ , then there exists a derivation  $d'$  of  $\Gamma \Rightarrow \Delta$  such that  $\rho_2(d') < \rho_2(d)$  and  $|d'| \leq 2^{|d|}$ .*

**Proof** The proof is by induction on  $|d|$ . If the last inference is not a cut with second-order cut-rank  $\rho_2(d)$ , the result follows easily by the induction hypothesis. So, assume that the last inference is

$$(cut) \quad \frac{\frac{d_1}{\Gamma_1 \Rightarrow \Delta_1, \varphi} \quad \frac{d_2}{\Gamma_2, \varphi \Rightarrow \Delta_2}}{\Gamma \Rightarrow \Delta}$$

where  $|\varphi|_2 = \rho_2(d)$ . By the induction hypothesis, there are derivations  $d'_1$  and  $d'_2$  of  $\Gamma_1 \Rightarrow \Delta_1, \varphi$  and  $\Gamma_2, \varphi \Rightarrow \Delta_2$ , respectively, such that  $|d'_1| \leq 2^{|d_1|}$  and  $|d'_2| \leq 2^{|d_2|}$ , both with second-order cut-rank strictly less than  $\rho_2(d)$ .

By the reduction lemma there is a derivation  $d'$  of  $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$  such that

$$|d'| \leq |d'_1| + |d'_2| \leq 2^{|d_1|} + 2^{|d_2|} \leq 2^{\max(|d_1|, |d_2|) + 1} = 2^{|d|}$$

and  $\rho_2(d') < |\varphi|_2 = \rho_2(d)$ .  $\square$

As in [11], let  $2_0^a = a$  and  $2_{k+1}^a = 2^{2_k^a}$ . The following is now straightforward by induction on  $n$ :

**Theorem 7.3** *Assume that  $d$  is a derivation of  $\Gamma \Rightarrow \Delta$ . Then there exists a derivation  $d'$  of the same sequent such that  $\rho_2(d') = 0$  and  $|d'| \leq 2^{|d|}_{\rho_2(d)}$ . Therefore,  $d'$  has only predicative cuts.*

Notice that the above results also hold if we allow for value-range terms in the syntax of the language (predicative or impredicative, it does not matter), as long as the calculus remains pure (without Law V), because the above proofs are independent of the structure of the terms.

As discussed in the first paragraph of this appendix, the above results are enough for the arguments of this paper. However, we can go further and obtain full cut-elimination. We only have to remove the remaining predicative cuts. This can be done following the blueprint of cut-elimination for first-order logic (the extra second-order rules pose no problems in the analysis since a cut formula can never be the principal formula of such a rule). If we follow [11], it would be very convenient to have only axioms of the form  $\Gamma, \varphi \Rightarrow \Delta, \varphi$ , with  $\varphi$  is an *atomic* formula, in the proof without second-order cuts (in order to have an inversion lemma similar to the one in [11]: see the discussion before Lemma 7.1). But, of course, we can suppose this. Just replace all the axioms of the form  $\Gamma, \varphi \Rightarrow \Delta, \varphi$ , with  $\varphi$  a predicative formula, by derivations starting with axioms for atomic formulas. There are such simple enough (even cut-free) derivations.

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