Harrington's Conservation Theorem Redone

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Abstract

Leo Harrington showed that the second-order theory of arithmetic WKL_0 is Π^1_1-conservative over the theory RCA_0. Harrington's proof is model-theoretic, making use of a forcing argument. A purely proof-theoretic proof, avoiding forcing, has been eluding the efforts of researchers. In this short paper, we present a proof of Harrington's result using a cut-elimination argument.

1 Introduction

The language of second-order arithmetic is a two-sorted language, with a numerical sort whose terms are intended to denote natural numbers, and a second-order sort whose variables are intended to range over subsets of the natural numbers. Numerical terms are built up as usual from first-order variables and from function symbols for the primitive recursive functions. The atomic formulas are of the form $t = q$, $t \leq q$ and $t \in X$, where t and q are numerical terms, and X is a second-order variable. RCA_0 denotes the classical theory consisting of quantifier-free axioms regulating the function

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and relation symbols, the induction scheme restricted to Σ^0_1-formulas and the recursive comprehension scheme:

$$\forall x(F(x) \leftrightarrow \neg G(x)) \rightarrow \exists X \forall x(x \in X \leftrightarrow F(x)),$$

where F and G are Σ^0_1-formulas (Σ^0_1-formulas are either bounded formulas or formulas of the form $\exists x F_0$, where F_0 is a bounded formula, and in which parameters of both sorts may occur). This theory, a variation of which is described in detail in [19], plays a prominent role in the studies of *Reverse Mathematics* where it is usually taken as the base theory over which the strength of ordinary theorems of mathematics is gauged. It has also the following conspicuous property: It is Π^0_2-conservative over (the first-order version of) Skolem's primitive recursive arithmetic. This result is originally due to Charles Parsons (see [16]; more information on this matter can be found in [9]). The theory WKL_0 is obtained from RCA_0 by adjoining *weak König's lemma*:

$$\forall T(T \text{ is an infinite binary tree } \rightarrow \exists X(X \text{ is an infinite path through } T)).$$

Even though the above axiom is non-constructive in character (e.g., there are recursive infinite binary trees with no recursive infinite paths through them), Harvey Friedman reported in [12] that WKL_0 is still Π^0_2-conservative over primitive recursive arithmetic. Nowadays there are several proofs of this result in the literature: e.g., [17], [18], [14] (also in [2]), [19] and [10]. Leo Harrington (unpublished result) strengthened Friedman's conservation result by showing that WKL_0 is, in fact, Π^1_1-conservative over RCA_0:

Theorem (Harrington). *Suppose that* WKL_0 *proves the sentence* $\forall X F(X)$*, where* F *is a first-order formula. Then* RCA_0 *already proves* $\forall X F(X)$*.*

Harrington's proof is model-theoretic, using a forcing construction (the proof has now been published in Simpson's book [19]). A purely proof-theoretic proof of Harrington's result has been actively sought and, in the process, some subtle proof-theoretic matters have been clarified (see [15]). One such proof has indeed been found, namely Avigad's proof in [1] where he is able to formalize Harrington's forcing argument within RCA_0 (obtaining, as a result, nice non speed-up results). Our paper presents a proof that bypasses the forcing argument and which is based on a direct analysis of suitable normal proofs of Π^1_1-sentences in WKL_0. It is a subsidy in showing that proof-theoretic methods are a flexible and powerful lot and, also, in fostering an appreciation of Harrington's conservation result for those not familiar with forcing arguments.

The proof of Harrington's conservation result given below uses the Free-Cut Elimination Theorem for an appropriate sequent calculus. In Section 2, we reformulate the theory WKL_0 in the sequent calculus and, conspicuously, state weak König's lemma by way of its contrapositive: the so-called FAN_0 theorem. This is the FAN theorem of intuitionistic mathematics stated in the context of second-order arithmetic for *bounded* matrices:

$$\forall X \exists x F_0(X, x) \to \exists w \forall X \exists x \leq w F_0(X, x),$$

where F_0 is a bounded formula (possibly with parameters of both sorts). Notice two things. Firstly, the FAN theorem is intuitionistically acceptable (at least for Brouwerian intuitionists) while weak König's lemma is not (in our setting we may replace one by the other because our logic is *classic*). Secondly, even though the *general* FAN theorem is not classically true, the above restricted version (where the matrix F_0 is bounded) is classically valid, being classically equivalent (as we pointed) to weak König's lemma. In Section 3, we present the proof of Harrington's result. Finally, in the last section we comment on the role of the general FAN theorem in the intuitionistic setting. The upshot is that a natural formulation of Harrington's theorem in the intuitionistic setting does not hold.

2 The framework of the sequent calculus

The aim of this section is to reformulate the theory WKL_0 in the sequent calculus. We bypass fine points regarding the precise set-up of the sequent calculus and, rather, direct the reader to [6], [5] and [20] concerning these matters. Following [5], we adopt *bounded quantifications* as a primitive syntactic device and, concurrently, uphold the bounded quantifier rules of the sequent calculus. The non-logical initial sequents consist of the usual (closed under term substitution) quantifier-free sequents regulating the relation $\leq$ and the function symbols associated with the (descriptions of the) primitive recursive functions. Instead of the axioms for Σ^0_1-induction we have the rule:

$$\frac{\Gamma, F(a) \longrightarrow F(a+1)}{\Gamma, F(0) \longrightarrow F(t)} \text{Ind}$$

where F is a Σ^0_1-formula, a is an *Eigenvariable*, and t is a term (see [5] and [20] for the notion of 'Eigenvariable'). In the standard formulation of the above rule, side formulas Δ are also permitted in the right-hand side of the sequents, but it is easy to see that we can do without them (this observation also applies to the formulation of the FAN rule below). Concerning the second-order part of the language, we have the following rules:

$$\frac{\Gamma, F(V) \longrightarrow \Delta}{\Gamma, \forall X F(X) \longrightarrow \Delta}\ \forall^2\text{left} \qquad\qquad \frac{\Gamma \longrightarrow \Delta, F(A)}{\Gamma \longrightarrow \Delta, \forall X F(X)}\ \forall^2\text{right}$$

where V is a *(set) abstract* for a bounded formula and A is a (second-order) Eigenvariable. (See Takeuti's book [20] for the notion of 'abstract'; NB: there are notational differences between our setting and Takeuti's). The sequent calculus also includes similar (dual) rules for the existential second-order quantifier. Since we permit abstracts for bounded formulas in the ($\exists^2$right) rule, it is straightforward to see that sequents of the form $\longrightarrow \exists X \forall x (x \in X \leftrightarrow F(x))$, for F a bounded formula, are derivable. In other words, our rules for second-order quantification allow us to prove comprehension for bounded formulas. Thus, the above sequent calculus is a reformulation of the theory RCA_0^-, obtained from RCA_0 by replacing the Δ^0_1-comprehension scheme by the weaker scheme of bounded comprehension.

The following proposition is clear. It is easily proved by induction on the complexity of the formula F_0:

Proposition. *Let $F_0(A, \underline{b})$ be a bounded formula with a distinguished second-order parameter A and with the first-order parameters $\underline{b}$ as shown. We can effectively associate a term $t_{F_0}(\underline{b})$, with its free variables as shown, such that the theory RCA_0^- proves:*

$$\forall s \in \{0,1\}^{t_{F_0}(\underline{b})} \big(\forall x < t_{F_0}(\underline{b})(x \in A \leftrightarrow s_x = 0) \to (F_0(A, \underline{b}) \leftrightarrow F_0^*(s, \underline{b}))\big),$$

where $s \in \{0,1\}^{t_{F_0}(\underline{b})}$ means that s is a binary sequence of length $t_{F_0}(\underline{b})$, s_x is the value of the sequence s at point x (having default value 0 if x is not less than the length of s), and F_0^* is obtained from F_0 by replacing its atomic subformulas of the form $q \in A$ by the expression $s_q = 0$.

We now add the following FAN rule to our calculus:

$$\frac{\Gamma \longrightarrow \exists x F_0(A, x, \underline{b})}{\Gamma \longrightarrow \exists v \forall s \in \{0,1\}^{t(v,\underline{b})} \exists x \leq v F_0^*(s, x, \underline{b})} \text{ Fan}_0$$

where F_0 is a bounded formula, A is a Eigenvariable and the term $t(v, \underline{b})$ is the term associated (according to the previous proposition) to the bounded formula $\exists x \leq v F_0(A, x, \underline{b})$. Note that the formula following the quantification $\exists v$ above is a bounded formula. The rule entails that under the supposition that $\forall X \exists x F_0(X, x, \underline{b})$ we may conclude $\exists v \forall X \exists x \leq v F_0(X, x, \underline{b})$, for bounded formulas F_0. We are using the above proposition at this point. In sum, the above rule entails FAN_0.

Proposition. *The theories* $\mathsf{RCA}_0^- + \mathsf{FAN}_0$ *and* WKL_0 *are the same.*

Proof. It is sufficient to show that $\mathsf{RCA}_0^- + \mathsf{FAN}_0$ proves the Δ_1^0-comprehension scheme because FAN_0 entails (classically) weak König's lemma (over RCA_0^-), and vice-versa. Suppose that $\forall u (\exists y F_0(u, y) \leftrightarrow \forall z G_0(u, z))$, where F_0 and G_0 are bounded formulas. We claim that

$$\forall w \exists X \forall x \leq w \forall u, y, z \leq x ((F_0(u, y) \to u \in X) \land (u \in X \to G_0(u, z))).$$

Given w we just take, by bounded comprehension,

$$X := \{u : \exists y \leq w \, F_0(u, y)\}.$$

Now, by FAN_0, we may conclude that

$$\exists X \forall x \forall u, y, z \leq x ((F_0(u, y) \to u \in X) \land (u \in X \to G_0(u, z))),$$

and this entails the desired result. □

3 The new proof

Suppose that WKL_0 proves the sentence $\forall X F(X)$, with F a first-order formula. Then there is a proof of the sequent $\longrightarrow F(X)$ in the sequent calculus $\mathsf{LK}_{\mathsf{FAN}}$ described in the previous section. By the Free-Cut Elimination Theorem (see [6], [5] and also Chapter 9 of [4] and [20] for second-order systems) there is a free-cut free proof of $\longrightarrow F(X)$ in this sequent calculus (the notion of 'free-cut' must be adapted by also declaring that every direct descendent of the principal formula of a (Fan_0) inference is anchored). Since the principal formulas of the (Ind) and (Fan_0) rules are Σ^0_1 *and* since the abstracts are given by *bounded* formulas, we conclude that there is a proof of $\longrightarrow F(X)$ in $\mathsf{LK}_{\mathsf{FAN}}$ in which the cut rule applies only to Σ^0_1-formulas. As a consequence, this proof has no occurrences of second-order quantifiers. Disregarding the order of the formulas, every sequent in the proof has the form

$$(\star)\quad \left\{ \begin{array}{l} \Gamma, \exists w_1 H_1(w_1, \underline{A}), \ldots, \exists w_n H_n(w_n, \underline{A}) \longrightarrow \\ \qquad\quad \Delta, \exists y_1 G_1(y_1, \underline{A}), \ldots, \exists y_m G_m(y_m, \underline{A}) \end{array} \right.$$

where:

a. the Hs and the Gs are bounded formulas (we admit the absence of the existential quantifiers $\exists w_i$ or $\exists y_j$ in order to accommodate plain bounded formulas in the above sequent);

b. there are no Σ^0_1-formulas in Γ or Δ;

c. the tuple $\underline{A}$ displays exactly the second-order parameters which occur in the Hs or in the Gs *without* occuring neither in Γ nor in Δ. These are called the *special* parameters of the sequent;

d. we are not displaying other (first or second order) parameters. In particular, we are not displaying second-order parameters that occur in Γ or in Δ (and which may concurrently occur in the Hs or in the Gs).

If the (Fan_0) rule is not applied in the normal proof then, of course, $\forall X F(X)$ is a theorem of RCA_0^-. Otherwise, it occurs for a first time in some branch of the proof tree. At this point we need a lemma. Let $\mathsf{LK}_{\mathsf{RCA}_0^-}$ be the sequent calculus $\mathsf{LK}_{\mathsf{FAN}}$ minus the (Fan_0) rule:

Lemma. *Let be given a proof of a sequent of the form ($\star$) in the sequent calculus $\mathsf{LK}_{\mathsf{RCA}_0^-}$. Suppose further that this proof is normal in the following sense: every cut formula is a Σ^0_1-formula; and, no formula of the proof has second-order quantifiers. Under these conditions, the theory RCA_0^- proves*

$$(\$)\quad \left\{ \begin{array}{l} \Gamma \wedge \neg\Delta \rightarrow \big(\forall w_1 \ldots \forall w_n \exists v \forall \underline{A} (H_1(w_1, \underline{A}) \wedge \ldots \wedge H_n(w_n, \underline{A}) \rightarrow \\ \qquad\quad \exists y_1 \leq v G_1(y_1, \underline{A}) \vee \ldots \vee \exists y_m \leq v G_m(y_m, \underline{A}))\big). \end{array} \right.$$

If this is shown then, when we arrive at the top sequent of a (first) application of the (Fan_0) rule, RCA_0^- proves

$$\Gamma(\underline{b}) \to \forall \underline{w} \exists v \forall A (H_1(w_1, \underline{b}) \wedge \ldots \wedge H_n(w_n, \underline{b}) \to \exists x \leq v F_0(A, x, \underline{b})),$$

where we are not showing any special parameters besides A. Note that A is an Eigenvariable and only shows up in the auxiliary formula of the (Fan_0) rule. Note, also, that the universal quantifications over the *other* special parameters (variables) can safely cross over the quantifier $\exists v$. We are displaying the first-order parameters that appear in the auxiliary formula (and which may appear elsewhere). Hence, RCA_0^- proves

$$\Gamma(\underline{b}) \wedge \exists w_1 H_1(w_1, \underline{b}) \wedge \ldots \wedge \exists w_n H_n(w_n, \underline{b}) \to \exists v \forall A \exists x \leq v F_0(A, x, \underline{b}).$$

As a consequence, the theory RCA_0^- proves the conditional whose antecedent is $\Gamma(\underline{b}) \wedge \exists w_1 H_1(w_1, \underline{b}) \wedge \ldots \wedge \exists w_n H_n(w_n, \underline{b})$ and whose consequent is $\exists v \forall s \in \{0,1\}^{t(v,\underline{b})} \exists x \leq v F_0^*(s, x, \underline{b})$, where the term t is as in the (Fan_0) rule.

We have arrived at the conclusion of a first application of the (Fan_0) rule in a normal proof in $\mathsf{LK}_{\mathsf{FAN}}$ via a proof in $\mathsf{LK}_{\mathsf{RCA}_0^-}$ (of course, we may take the latter as a normal proof, in the sense of Lemma 3). If we repeat this procedure enough times, we arrive at a (normal) proof of $\longrightarrow F(X)$ in $\mathsf{LK}_{\mathsf{RCA}_0^-}$. Hence, the theory RCA_0 (actually, RCA_0^-) already proves the sentence $\forall X F(X)$.

It remains to prove the lemma. At various points, the proof of the lemma makes appeal to the so-called *bounded collection* scheme $\mathsf{B}\Sigma_1^0$:

$$\forall x \leq a \exists y\, F_0(x, y) \to \exists z \forall x \leq a \exists y \leq z\, F_0(x, y),$$

where F_0 is a bounded formula (*vide* comments on this issue after the proof below). As it is well known, bounded collection is a consequence of Σ_1^0-induction.

Proof of the Lemma. The proof is by induction on the depth of the sequents appearing in the given normal proof. There is nothing to prove regarding initial sequents, since they are quantifier-free. One must check that the induction hypothesis is carried over by every rule of $\mathsf{LK}_{\mathsf{RCA}_0^-}$. We will do this for some distinguished cases, namely for the ($\exists^{\leq}$left), ($\forall$left), (Ind) and the cut rules. A complete checking can be found in [11] (note that the rules for the second-order quantifiers never show up in a normal proof).

The ($\exists^{\leq}$left) rule has the form

$$\frac{\Gamma, a \leq t, F(a, \underline{A}), \exists w H(w, \underline{A}) \longrightarrow \Delta, \exists y G(y, \underline{A})}{\Gamma, \exists x \leq t\, F(x, \underline{A}), \exists w H(w, \underline{A}) \longrightarrow \Delta, \exists y G(y, \underline{A})}$$

where, for simplicity, we consider only one (bounded) formula H and one bounded formula G, and where a is an Eigenvariable and t is a term. The interesting case is when F is a bounded formula. In this case, $\underline{A}$ are the special parameters (of both sequents). By induction hypothesis, the theory RCA_0^- proves the conditional whose antecedent is $\Gamma \wedge \neg\Delta$ and whose consequent is

$$\forall a \forall w \exists v \forall \underline{A}(a \leq t \wedge F(a, \underline{A}) \wedge H(w, \underline{A}) \rightarrow \exists y \leq v\, G(y, \underline{A})).$$

Fix w. Then $\forall x \leq t \exists v \forall \underline{A}(F(x, \underline{A}) \wedge H(w, \underline{A}) \rightarrow \exists y \leq v\, G(y, \underline{A}))$. By Proposition 2, the subformula of the previous formula that begins with $\exists v$ is equivalent to a Σ_1^0-formula. Hence, by bounded collection $\mathsf{B}\Sigma_1^0$,

$$\exists v \forall x \leq t \forall \underline{A}(F(x, \underline{A}) \wedge H(w, \underline{A}) \rightarrow \exists y \leq v G(y, \underline{A})).$$

The conclusion of the induction step is an immediate consequence of the above.

The study of the ($\forall$left) rule is only interesting when the auxiliary formula is Σ_1^0. In this case we have an inference of the form:

$$\frac{\Gamma, \exists w H(w, \underline{A}, \underline{B}), \exists x F(x, t, \underline{B}) \longrightarrow \Delta, \exists y G(y, \underline{A}, \underline{B})}{\Gamma, \exists w H(w, \underline{A}, \underline{B}), \forall z \exists x F(x, z, \underline{B}) \longrightarrow \Delta, \exists y G(y, \underline{A}, \underline{B})}$$

under the usual conditions. Here t is a term, and we are distinguishing between the special parameters which occur in the auxiliary formula (the parameters $\underline{B}$) and those that do not occur there (the parameters $\underline{A}$). Note that the former are no longer special parameters of the lower sequent. By induction hypothesis, the theory RCA_0^- proves the conditional whose antecedent is $\Gamma \wedge \neg\Delta$ and whose consequent is

$$\forall w \forall x \exists v \forall \underline{A} \forall \underline{B}(H(w, \underline{A}, \underline{B}) \wedge F(x, t, \underline{B}) \rightarrow \exists y \leq v\, G(y, \underline{A}, \underline{B})).$$

We reason inside RCA_0^-. Fix $\underline{B}'$ and assume the conjunction of Γ with $\neg\Delta$ *and with* $\forall z \exists x F(x, z, \underline{B}')$. Take x' such that $F(x', t, \underline{B}')$. Fix w. It is now clear that there is v such that $\forall \underline{A}(H(w, \underline{A}, \underline{B}') \rightarrow \exists y \leq v\, G(y, \underline{A}, \underline{B}'))$.

Consider the (Ind) rule:

$$\frac{\Gamma, \exists w H(w, \underline{A}), \exists x F(x, a, \underline{A}) \longrightarrow \exists x' F(x', a+1, \underline{A})}{\Gamma, \exists w H(w, \underline{A}), \exists x F(x, 0, \underline{A}) \longrightarrow \exists x' F(x', t, \underline{A})}$$

under the usual conditions, and where a is an Eigenvariable and t is an arbitrary term. By induction hypothesis, the theory RCA_0^- proves the conditional whose antecedent is Γ and whose consequent is

$$(\$)\ \ \forall a \forall w \forall x \exists v \forall \underline{A}(H(w, \underline{A}) \wedge F(x, a, \underline{A}) \rightarrow \exists x' \leq v\, F(x', a+1, \underline{A})).$$

Let us reason inside RCA_0^-. Assume Γ. Fix elements w and x. We claim that, for all elements a,

$$\exists v \forall \underline{A}(H(w, \underline{A}) \wedge F(x, 0, \underline{A}) \rightarrow \exists\, x' \leq v F(x', a, \underline{A})).$$

This solves our problem (instantiate a by t). The claim is proved by induction on a. Note that this induction is permissible because, by Proposition 2, the above formula is equivalent to a Σ^0_1-formula. The base case $a = 0$ is clear. To argue for the induction step, assume that there is v such that

$$\forall \underline{A}(H(w, \underline{A}) \wedge F(x, 0, \underline{A}) \rightarrow \exists x'' \leq v\, F(x'', a, \underline{A})).$$

By ($) we have:

$$\forall x'' \leq v \exists v' \forall \underline{A}(H(w, \underline{A}) \wedge F(x'', a, \underline{A}) \rightarrow \exists x' \leq v' F(x', a+1, \underline{A})).$$

By bounded collection $\mathsf{B}\Sigma^0_1$, there is v' such that,

$$\forall x'' \leq v \forall \underline{A}(H(w, \underline{A}) \wedge F(x'', a, \underline{A}) \rightarrow \exists x' \leq v'\, F(x', a+1, \underline{A})).$$

It clearly follows that

$$\forall \underline{A}(H(w, \underline{A}) \wedge F(x, 0, \underline{A}) \rightarrow \exists x' \leq v' F(x', a+1, \underline{A})).$$

Finally, we study the cut rule (with a Σ^0_1 cut-formula). This rule says that from the two sequents

$$\Gamma, \exists w H(w, \underline{A}) \longrightarrow \Delta, \exists y G(y, \underline{A}), \exists x F(x, \underline{A}, \underline{B}) \text{ and}$$

$$\exists x F(x, \underline{A}, \underline{B}), \Gamma, \exists w H(w, \underline{A}) \longrightarrow \Delta, \exists y G(y, \underline{A})$$

one can infer the sequent $\Gamma, \exists w H(w, \underline{A}) \longrightarrow \Delta, \exists y G(y, \underline{A})$. We are distinguishing the special parameters which only occur in the cut-formula (the parameters $\underline{B}$). By induction hypothesis, the theory RCA_0^- proves that $\Gamma \wedge \neg\Delta$ implies both

$$\forall w \exists v_1 \forall \underline{A}(H(w,\underline{A}) \to \exists y \leq v_1\, G(y,\underline{A}) \vee \forall \underline{B} \exists x \leq v_1\, F(x,\underline{A},\underline{B})) \text{ and}$$

$$\forall w \forall x \exists v_2 \forall \underline{A} \forall \underline{B}(F(x,\underline{A},\underline{B}) \wedge H(w,\underline{A}) \to \exists y \leq v_2\, G(y,\underline{A})).$$

Let us fix w. Take v_1 according to the first assertion above. An application of bounded collection $\mathsf{B}\Sigma^0_1$ to the second assertion above yields v_2 such that

$$\forall \underline{B} \forall x \leq v_1 \forall \underline{A}(F(x,\underline{A},\underline{B}) \wedge H(w,\underline{A}) \to \exists y \leq v_2\, G(y,\underline{A})).$$

It is now clear that $\forall \underline{A}(H(w,\underline{A}) \to \exists y \leq \max(v_1,v_2)\, G(y,\underline{A}))$ follows, as wanted. $\square$

Let PRA^2 be the theory obtained from RCA_0^- by replacing the Σ^0_1-induction rule by the *set induction* axiom:

$$\forall X(0 \in X \wedge \forall x(x \in X \to x+1 \in X) \to \forall x(x \in X)).$$

In the sequent calculus, this axiom is replaced by the rule:

$$\frac{\Gamma, a \in X \longrightarrow a+1 \in X}{\Gamma, 0 \in X \longrightarrow t \in X}\,\text{SetInd}$$

where a is an Eigenvariable and t is a term. An inspection on the above proof shows that the following theorem is true (the above rule poses no difficulty):

Proposition. *Suppose that* $\mathsf{PRA}^2 + \mathsf{FAN}_0$ *proves the sentence* $\forall X F(X)$*, where* F *is a first-order formula. Then* $\mathsf{PRA}^2 + \mathsf{B}\Sigma^0_1$ *already proves it.*

The inclusion of the bounded collection principle in the second theory above is unavoidable because the theory $\mathsf{PRA}^2 + \mathsf{FAN}_0$ proves $\mathsf{B}\Sigma^0_1$. This was noticed in [8] (the precise setting was different). Observe that it is known that $\mathsf{B}\Sigma^0_1$ is *not* a consequence of PRA^2 (cf. [13]). For weak theories of arithmetic and analysis (i.e., in which the totality of exponentiation is not provable) one has to be careful in formulating weak König's lemma (or the FAN principle), and tight relationships emerge between these formulations and various formulations of bounded collection. On this regard, one should pay attention to the exact formulation of weak König's lemma in the feasible setting (see [8]), and to the results in [7]. We may say with confidence that Harrington type theorems for these weaker settings can also be proved with arguments based on the one presented in this paper.

4 Final considerations

Let FAN be the second-order principle

$$\forall X \exists x\, F(X,x) \to \exists w \forall X \exists x \leq w\, F(X,x),$$

where F is an *arbitrary* second-order formula. It is easy to see that the theory $\mathsf{RCA}_0^- + \mathsf{FAN}$ is classically inconsistent. We repeat here a well known argument to the effect that one can prove intuitionistically in $\mathsf{RCA}_0^- + \mathsf{FAN}$ the *negation* of the classical truth $\forall X(\forall y(y \in X) \vee \exists x(x \notin X))$. Suppose that this classical truth holds. Then, $\forall X \exists x(\forall y(y \in X) \vee x \notin X)$. By FAN, there exists w such that $\forall X \exists x \leq w(\forall y(y \in X) \vee x \notin X)$. This is clearly a contradiction: just consider $X = \{x : x \leq w\}$.

The theory $\mathsf{RCA}_0^- + \mathsf{FAN}$ is, nevertheless, intuitionistically consistent. As a matter of fact, a Friedman type conservation result holds for this theory:

Theorem. *The intuitionistic version of* $\mathsf{RCA}_0^- + \mathsf{FAN}$ *is* Π^0_2*-conservative over primitive recursive arithmetic.*

This result follows directly from work on a newly found functional interpretation in [10]. In fact, the work on the new interpretation shows that one can even join to the intuitionistic version of $\mathsf{RCA}_0^- + \mathsf{FAN}$ the following classical principles: Markov's principle, a form of independence of premises for universal antecedents and (surprisingly) both the lesser limited principle of omniscience (cf. [3]) and weak König's lemma.

It is natural to ask whether a Harrington type conservation result holds in the intuitionistic setting. More precisely: Is *intuitionistic* $\mathsf{RCA}_0^- + \mathsf{FAN}$ a Π^1_1-conservative theory over RCA_0? The answer is negative, even when RCA_0 is conceived classically. The reason is simple:

Proposition. Intuitionistic $\mathsf{RCA}_0^- + \mathsf{FAN}$ *proves bounded collection for all formulas of the language.*

Now, if *intuitionistic* $\mathsf{RCA}_0^- + \mathsf{FAN}$ were Π^1_1-conservative over RCA_0 then the latter theory would prove bounded collection for all arithmetical formulas, a well-known falsity. In order to prove the proposition, assume that $\forall x \leq a \exists y F(x,y)$. Let $F^\dagger(X,y,a)$ be the formula $\exists x(x = \mathrm{card}\{z \in X : z < a\} \wedge F(x,y))$. It is clear that $\forall X \exists y F^\dagger(X,y,a)$. By FAN, there is w such that $\forall X \exists y \leq w\, F^\dagger(X,y,a)$. This easily entails $\forall x \leq a \exists y \leq w F(x,y)$.

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