Yu. I. Manin
(with collaboration by B. Zilber)
A Course in Mathematical Logic for Mathematicians
2nd Edition
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Reviewed by Fernando Ferreira

In the early eighties, when I was a beginning graduate student, I bought the first edition of this book. There it was a book on mathematical logic, written in a lively language and discussing unexpected subjects. It was rather different from the textbooks that I was struggling to understand. It included quantum logic, the exposition of Smullyan's elegant SELF language, a brief on Feferman's transfinite recursive progressions of axiomatic theories, Gödel's result on the length of proofs, Kolmogorov complexity, as well as some interesting digressions (e.g., on the linguistics of Icelandic poetry and on Luria's description of a damaged psyche) and mathematical speculations (on recursive geometry - better called geometry of recursion - for instance). Of course, the main thrust was on standard material (see the discussion ahead), including some advanced material not usually covered in textbooks (chapter VIII is devoted to a full proof of Graham Higman's result on the characterization of the finitely generated subgroups of finitely presented groups as the finitely generated, recursively presented groups). This idiosyncratic view of mathematical logic by a well-known and respected "working mathematician" did not fail to make an impression on a curious young mind.

It is with pleasure that, more than twenty five years later, I have the occasion to review the second edition of Manin's book. This edition includes new material (it has one hundred more pages than the old edition). The preface of the first edition says that the book "is above all addressed to mathematicians [and] it is intended to be a textbook of mathematical logic on a sophisticated level." In the second edition, the author clarifies somehow his statement: he now says that he imagined his readers as "working mathematicians like me." The latter statement is, I believe, more to the point (note also the slight change of title from the first to the second edition, where "for Mathematicians" is now inserted). It is certainly a book that can be read and perused profitably by a graduate student in mathematical logic, but rather as a complement to a more standard textbook. It has also some interesting discussions for the professional logician, but I believe that its natural public is the research mathematician not working in logic. Manin is a very good writer and displays an immense (mathematical and otherwise) culture. It is a pleasure to read him and, for a critical mathematical mind, the book is intellectually very stimulating. Of course, the perenniality and beauty of some of the results expounded also helps in making the book attractive reading for the non specialist.

The book assumes no previous acquaintance with mathematical logic. The first two chapters are devoted to basic material, including Gödel's completeness theorem, the Löwenheim-Skolem theorem (with a discussion of Skolem's "paradox"), and Tarski's theorem on the undefinability of truth. Chapter VII discusses Gödel's first incompleteness theorem in its semantical form: the set of provable sentences of a theory is arithmetically definable whereas the set of truths of arithmetic is not: *ergo*, truth differs from provability. Some reviewers of the first edition observed that the compactness theorem of first-order logic was conspicuously absent. In this edition, however, this is corrected and the compactness theorem is given its proper central place in a new chapter (the last, numbered X) by Boris Zilber: indeed, three different proofs of the theorem are now discussed. In less than fifty pages, Zilber gives the reader a masterful tour in model theory taking us to the frontiers of research. The emphasis of this tour is on the connections of model theory with "tame" geometry. This chapter is certainly one of the highlights of the book and should be of much interest to a mathematician working in real or algebraic geometry.

Set theory is well represented by two chapters (III and IV) on the seminal results of Gödel and Cohen concerning, respectively, the consistency and independence of the continuum hypothesis. The latter result is presented in terms of the Scott-Solovay Boolean-valued model approach within the framework of a second-order theory of real numbers. The more standard approach via forcing, within set theory, is also briefly explained. When the first edition of the book came out, set theory had already embarked on the grand project of relating determinacy assumptions with (very) large cardinals. Work of Martin, Woodin, Steel and others culminated in the late eighties with the result that large cardinal assumptions imply nice regularity properties for the projective sets of the real line, among which are counted Lebesgue measurability and the perfect set property. This is first water mathematical work by any standards. It also brought a nice closure to the despair of the Moscow mathematician Nikolai Luzin when he wrote in 1925 that "there is a family of effective sets (...) such that we do not know *and will never know* if any uncountable set of this family has the power of the continuum (...) nor even if it is measurable." A byproduct of this work is a nice and robust axiomatization of second-order arithmetic which provides the launch pad of a recent attempt of Hugh Woodin for tackling the old chestnut of the continuum hypothesis (yes, there are still some people thinking hard on these issues). In a very brief subsection, Manin discusses this attempt of Woodin. In spite of its briefness (slightly more than half a page), the subsection is informative and appropriate, as it follows a discussion on the philosophy of set theory and Gödel's programme for new axioms. However, one feels that Manin missed the opportunity to call attention to the above mentioned set-theoretic work of the eighties.

The remaining chapters, numbered V, VI and IX, concern computability and complexity. The first two cover the basic material of recursive function theory and include the solution to Hilbert's tenth problem using Pell's equation to produce an example of a function of exponential growth whose graph is Diophantine. Universal (Manin calls them "versal") families of partial recursive functions are constructed using the fact that every recursively enumerable set is Diophantine. Such families can also be constructed using universal Turing machines but these machines are given short shrift in the first edition. Manin's option was very fine but he somehow felt the need to give an explanation saying that "we again recall that we have not at all concerned ourselves with formalizing computational processes, but only with the results of such processes." Be it as it may, Manin wrote a new chapter (numbered IX) for the second edition where models of computations are discussed, including Turing machines and Boolean circuits. Complexity theory is mentioned and the theory of NP-completeness is developed up to Cook's theorem. There are also three sections devoted to quantum computation, including Shor's factoring and Grover's search algorithms. This a felicitous choice of topics for the new volume and in tune with the spirit of the first edition. These topics make up the second half of chapter IX. The first half of the chapter is, in Manin's own words, "a tentative introduction to [the categorical] way of thinking, oriented primarily to some reshuffling of classical computability theory." These sections relate with the speculations on the geometry of recursion already present in the first edition. I cannot but feel that Manin is attempting to fit the untame field of computability theory into the Procrustean bed of category theory.

This is a stimulating and audacious book, with something for everyone: for the logic student, for the professional logician and, specially, for its intended audience of the "working mathematician." The emphasis on the semantical aspects of logic is a good strategy for not alienating the intended reader. Even though the field of mathematical logic is admired for its results, nowadays it is somewhat isolated from the rest of mathematics. This is in part due to the specific preoccupations of the field, ones which result in a perceived entrenchment. However, there are now unmistakable signs of profitable interplay with the wider mathematical community, as can be seen by Zilber's chapter on model theory. I also see Manin's book as fostering an appreciation of mathematical logic within the wider arena of mathematics, not only for the branches of logic which most easily relate to the other parts of mathematics but also for the more foundational aspects of the subject. After all, "foundations" does not seem to be a bad word in Manin's vocabulary. In the preface to the second edition, he advances the view that "mathematics [is] (...) leaving behind old concerns about infinities: a new view of foundations is now emerging" and adds that "much remains to be recognized and said about this [categorical] emerging trend in foundations of mathematics." Let a thousand flowers bloom.

The book has some typos, some of which were already present in the first edition. For instance, it is written twice in the appendix to chapter II that a chain of the form x,y,z,... where x is an element of y, y an element of z, and so on, must terminate. Of course, it is the other way around. I also noticed a typo in the proof of lemma 11.3 of chapter II, one which is absent from the old edition. On page 47, the state of the art concerning Fermat's last theorem is described, for 1977... Curiously, on page 182, a footnote puts the matters aright, citing the work of Wiles. The book would have benefited from a more careful editing.

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