A note on a result of Buss concerning bounded theories and the collection scheme

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Abstract

Samuel Buss showed that, under certain circunstances, adding the collection scheme for bounded formulae to a bounded theory of arithmetic yields a $\forall\Sigma_1$-conservative extension. We present a very simple model theoretic proof of a generalization of this result.

The general form of the collection scheme in the language of first-order Peano Arithmetic is as follows:

$$\forall u \leq x \exists y A(u, y) \rightarrow \exists w \forall u \leq x \exists y \leq w A(u, y)$$

This scheme is obviously true in the standard model and, indeed, it is provable in Peano Arithmetic for any formula A of the language. An early result of Charles Parsons (in [Ps70]) states that the theory $I\Delta_0$ does not prove the collection scheme for bounded formulae (i.e., for formulae A that contain only bounded quantifiers: the so-called Δ_0 or bounded formulae). Recall that the theory $I\Delta_0$ is Robinson's arithmetic Q together with the scheme

$$A(0) \wedge \forall x(A(x) \rightarrow A(x+1)) \;\rightarrow\; \forall x A(x)$$

where $A \in \Delta_0$ (parameters are allowed).

Nevertheless, Jeff Paris showed in [Pr80] that adding the bounded collection scheme to the theory $I\Delta_0$ does not enable the deduction of new $\forall\Sigma_1$-sentences. Paris' proof hinges on the following result:

Theorem. *The scheme of collection for bounded formulae is true in every model of $I\Delta_0$ that has a proper end-extension which is also a model of $I\Delta_0$.*

Proof : Let $M, N \models I\Delta_0$, with N a proper end-extension of M, and suppose that $\forall u \leq a \exists y A(u, y)$, where $a \in M$ and $A \in \Delta_0$ (possibly with parameters from M). Using the absoluteness of bounded formulae between M and N (this fact is an easy consequence of N being an end-extension of M), it is clear that $N \models \forall u \leq a \exists y \leq c A(u, y)$ for all $c \in N \setminus M$. By induction, there is a least such $c \in N$: call it c_0. It is easy to argue that this c_0 is, in fact, in M. Hence, $M \models \forall u \leq a \exists y \leq c_0 A(u, y)$ as required. □

The above proof relies heavily on the least number principle for bounded formulae (equivalently, on the scheme of induction for bounded formulae). Hence, it does not extend readily to fragments of $I\Delta_0$ or to the (now) classical theories S_2^n of Samuel Buss (a good reference for bounded theories is Part C of [HP93]). In [B87] Buss presents two proofs, one proof-theoretic and one model-theoretic, showing that adding the collection scheme for bounded formulae to a bounded *sufficient* theory of arithmetic yields a $\forall\Sigma_1$-conservative extension. Buss' result indicates that this kind of conservation result has nothing to do with bounded induction. In fact, it has nothing to do with the language of arithmetic either.

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In this paper we state a more general condition than Buss', under which it is possible to add to a theory the collection scheme for bounded formulae, without changing the class of provable $\forall\Sigma_1$-sentences. The argument below is based on a very simple and naive model-theoretic construction (compare with Buss' model-theoretic proof, involving resplendent structures). It is distilled from the proof of a theorem in [F94], showing that adding Weak König's lemma for boundedly defined trees to the theory Σ_1^b-NIA (a theory equivalent to Buss' S_2^1) yields a $\forall\Sigma_1$-conservative extension.

We shall be concerned with first-order languages that include a distinguished binary relation symbol $\lhd$. The sintax of these languages is enlarged to permit *bounded quantifiers* of the forms $\forall x \lhd t$ and $\exists x \lhd t$, where t is any term in which x does not occur. The apparatus of deduction is suitably enlarged to convey the obvious meaning of the new quantifiers. A formula is *bounded* just in case it contains no unbounded (i.e., usual) quantifiers. A Σ_1*-formula* is a formula of the form $\exists\vec{y}A$, where A is a bounded formula; their negations are the Π_1*-formulas.* The class of $\forall\Sigma_1$-*sentences* is the set of sentences which are universal closures of Σ_1-formulae. A *nice* $\forall\Sigma_1$*-sentence* is a sentence of the form,

$$\forall\vec{x}\,\exists\vec{w}\,\forall\vec{u} \lhd \vec{x}\,\exists\vec{z} \lhd \vec{w}\,A(\vec{u},\vec{z})$$

where A is a bounded formula in which the variables $\vec{x}$ and $\vec{w}$ do not occur. (We are using some obvious abbreviations; for instance, $\forall\vec{u} \lhd \vec{x}$ abbreviates $\forall u_1 \lhd x_1 \ldots \forall u_k \lhd x_k$, for certain k).

Definition. *Consider* Γ *a theory in a first-order language as above. We shall say that* Γ *is a* nice $\forall\Sigma_1$-theory *if it is axiomatized by a set of nice* $\forall\Sigma_1$*-sentences and,*

(a_1) $\Gamma \vdash \forall x(x \lhd x)$

(a_2) $\Gamma \vdash \forall x\,\forall y\,\exists w(x \lhd w \land y \lhd w)$

(a_3) $\Gamma \vdash \forall x\,\exists w\,\forall y \lhd x\,\forall z \lhd y(z \lhd w)$

(b) For any function symbol $f(\vec{x})$ *of the language,* $\Gamma \vdash \forall\vec{x}\,\exists w\,\forall\vec{y} \lhd \vec{x}(f(\vec{y}) \lhd w)$.

All the usual theories of arithmetic that have Π_1-axiomatizations are nice $\forall\Sigma_1$-theories with respect to $\leq$ (less than or equal to). This includes the usual subsystems I_{open} and IE_n of $I\Delta_0$, as well as Buss' theories S_2^n and T_2^n. It also includes their binary reformulations Σ_n^b-NIA and Σ_n^b-IA, with respect to the relation "…has length less than or equal to …" (see [F90]). As a last example we mention that the theory Th-FO, introduced in [F91], is a nice $\forall\Sigma_1$-theory, both with respect to the relation "…has length less than or equal to …" and to the relation "…is a subword of …".

The *collection scheme* for bounded formulae consists of the following instances,

$$\forall u \lhd x\,\exists y\,A(u,y) \rightarrow \exists w\,\forall u \lhd x\,\exists y \lhd w\,A(u,y)$$

where A is a bounded formula that may contain additional free variables as parameters. This is also called *bounded collection.*

Theorem. *Let* Γ *be a nice* $\forall\Sigma_1$*-theory. Then the theory* Γ' *obtained from* Γ *by adding the collection scheme for bounded formulae is* $\forall\Sigma_1$*-conservative over* Γ *(in other words, every* $\forall\Sigma_1$*-sentence provable in* Γ' *is a theorem of* Γ*).*

In order to argue for this theorem we need the following :

Lemma. *Let* Γ *be a nice* $\forall\Sigma_1$*-theory and* M, N *models of* Γ *with* N *an elementary extension of* M*. Then* $N|_M = \{a \in N : \exists b \in M\; a \lhd b\}$ *is a substructure of* N *that models* Γ*. Additionally,* M *is a* Π_1*-elementary substructure of* $N|_M$.

Proof : Firstly, we check that $N|_M$ is a substructure of N. It is clear that $N|_M$ includes M and, hence, includes the interpretations of all the constants of the language. Let $f(\vec{x})$ be any function symbol of the language and take $\vec{a}$ in $N|_M$. By definition, there are $\vec{b}$ in M with $\vec{a} \lhd \vec{b}$. Using condition (b), pick $c \in M$ such that "$\forall \vec{y} \lhd \vec{b}\, f(\vec{y}) \lhd c$" holds in M. By elementarity, this also holds in N. In particular, $f(\vec{a}) \lhd c$. Hence, $f(\vec{a}) \in N|_M$.

Secondly, we show that N models Γ. The axiom (a_1) is clearly inherited by $N|_M$. The truths of (a_2), (a_3) and (b) in $N|_M$ can be shown through judicious uses of their truth in M, together with the elementarity between M and N. Now, consider

$$(*) \qquad \forall \vec{x}\, \exists \vec{w}\, \forall \vec{u} \lhd \vec{x}\, \exists \vec{z} \lhd \vec{w}\;\; A(\vec{u}, \vec{z})$$

an arbitrary axiom of Γ (A is a bounded formula in which $\vec{x}$ and $\vec{w}$ do not occur). To see that this axiom is true in $N|_M$ we will rely on the fact that bounded formulae are absolute between $N|_M$ and N (this can be shown by a straightforward induction on the complexity of those formulae, since it is quite obvious that if $a \lhd b$ and $b \in N|_M$ then $a \in N|_M$). Let $\vec{a}$ be in $N|_M$; take $\vec{b}$ in M with $\vec{a} \lhd \vec{b}$. According to (a_3), there are elements $\vec{c}$ in M such that "$\forall \vec{y} \lhd \vec{b} \forall \vec{z} \lhd \vec{y}\, (\vec{z} \lhd \vec{c})$" holds in M and, by elementarity, in N. Now, instantiate $(*)$ with $\vec{c}$ to pick $\vec{d}$ in M such that "$\forall \vec{u} \lhd \vec{c}\, \exists \vec{z} \lhd \vec{d}\, A(\vec{u}, \vec{z})$" holds in M and, again by elementarity, in N. It is now easy to conclude that $N|_M \vdash \forall \vec{u} \lhd \vec{a}\, \exists \vec{z} \lhd \vec{d}\, A(\vec{u}, \vec{z})$.

Lastly, we show that M is a Π_1-elementary substructure of $N|_M$. Let $A(\vec{x})$ be a Π_1-formulae and take $\vec{a} \in M$ such that $M \models A(\vec{a})$. By elementarity, $N \models A(\vec{a})$. Now, by downward absoluteness (due to the absoluteness of bounded formulae between N and $N|_M$), we may conclude that $N|_M \models A(\vec{a})$. □

Proof of the theorem : We claim that given any model M of Γ there is a model N of Γ' such that M is a Π_1-elementary substructure of N.

The theorem follows easily from this claim. In fact, if $\Gamma \nvdash \forall \vec{x}\, \exists \vec{y}\, A(\vec{x}, \vec{y})$, with A bounded, then there exists a model M of Γ and elements $\vec{a}$ in M so that $M \models \forall \vec{y} \neg A(\vec{a}, \vec{y})$; according to the claim, "$\forall \vec{y} \neg A(\vec{a}, \vec{y})$" holds in N and, hence, $\Gamma' \nvdash \forall \vec{x}\, \exists \vec{y}\, A(\vec{x}, \vec{y})$.

Build a chain $(M_i)_{i \in \omega}$ of models of Γ according to the following specifications : M_0 is M ; given M_i, take M'_i an elementary extension of M_i with an element c_i such that $M'_i \models \forall x \lhd c(x \lhd c_i)$, for all $c \in M_i$ (this is possible by compactness, using properties (a_1), (a_2) and (a_3)). We let M_{i+1} be $M'_i|_{M_i}$ and take N the limit of this chain. Clearly, the inclusions $M_i \subseteq N$ are Π_1-elementary (because the inclusions $M_i \subseteq M_{i+1}$ are). It is also clear that $N \models \Gamma$.

It remains to show that the collection scheme for bounded formulae holds in N. Suppose that for a bounded formula A and a certain a in N,

$$N \models \forall w\, \exists u \lhd a\, \forall y \lhd w\, \neg A(u, y)$$

Take $i \in \omega$ large enough so that a and all parameters in A occur in M_i. We successively get :

$$\begin{array}{lcl} M_i & \models & \forall w\, \exists u \lhd a\, \forall y \lhd w\;\; \neg A(u, y) \\ M'_i & \models & \forall w\, \exists u \lhd a\, \forall y \lhd w\;\; \neg A(u, y) \\ M'_i & \models & \exists u \lhd a\, \forall y \lhd c_i\;\; \neg A(u, y) \\ M'_i & \models & \forall y \lhd c_i\;\; \neg A(u_0, y) \end{array}$$

for a certain $u_0 \in M'_i$, with $u_0 \lhd a$ (hence, $u_0 \in M_{i+1}$). Let b be an arbitrary element of M_{i+1}. Clearly $b \lhd c_i$, and so $M'_i \models \neg A(u_0, b)$. By absoluteness, and the arbitrariness of b, $M_{i+1} \models \forall y \neg A(u_0, y)$, concluding that $N \models \forall y \neg A(u_0, y)$. It follows that $N \models \exists u \lhd a\, \forall y \neg A(u, y)$. □

Buss' theorem in [B87] seems to apply to certain situations not covered by the above result. This is just apparent. Define (following Buss) the class of *extended* Σ_1*-formulae* as the set of those formulae in which every unbounded quantifier is either existential and in the scope of an even number of negations or universal and in the scope of an odd number of negations. The theorem above still holds if the collection scheme for bounded formulae is enlarged to permit extended Σ_1-formulas.

In order to see this, remark that in the presence of bounded collection every extended Σ_1-formula F is equivalent to a Σ_1-formula F'. Moreover, the implication "$F' \to F$" is provable using logic alone. Hence, it suffices to argue that the colection scheme for Σ_1-formulae follows from bounded collection in a nice $\forall\Sigma_1$-theory.

This can be shown using a well-known trick. Consider $A(u,y) := \exists v\, B(u,v,y)$, with B a bounded formula (the restriction to a single variable v is made to simplify the argument ; alternatively, we could use induction on the number of variables). We reason in an arbitrary model M of a nice $\forall\Sigma_1$-theory in which bounded collection holds. Suppose that $\forall u \lhd a\, \exists y\, A(u,y)$; by (a_2), $\forall u \lhd a\, \exists y'\, (\exists y \lhd y'\, \exists v \lhd y'\, B(u,v,y))$. Hence, by bounded collection, there is w such that $\forall u \lhd a\, \exists y' \lhd w\, (\exists y \lhd y'\, \exists v \lhd y'\, B(u,v,y))$. Use ($a_3$) to pick w' satisfying $\forall y \lhd w\, \forall z \lhd y\, (z \lhd w')$. The statement "$\forall u \lhd a\, \exists y \lhd w'\, A(u,y)$" follows.

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