A Note on Finiteness in the Predicative Foundations of Arithmetic

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1 Introduction

In a sequence of two papers, Solomon Feferman and Geoffrey Hellman ([FH95], [FH98]) introduced formal contexts in which they established the existence and categoricity of a natural number structure by predicative means *given* the primitive notion of a finite set of individuals and given also that these individuals have some structure as built up by ordered pairs (Peter Aczel had a crucial contribution in this regard as well). Within these formal contexts, we may define the notion of a *strongly finite* set by saying that it is a finite set which is in one-to-one correspondence with a proper initial segment of a (any) natural number structure. Feferman and Hellman mention that (sic) "we cannot prove that these exhaust the range of finite-set variables" (see [FH98]). In this short paper, we show that it is independent from the theory EFSC of Feferman and Hellman that every finite set is strongly finite. In addition, we propose two new *prima facie* evident principles for finite sets that, when added to the axioms of EFSC, entail that every finite set is strongly finite.

The paper is organized as follows. In the next section, we briefly discuss the theories EFS and EFSC. In section 3, we show that the strongly finite sets assent to a principle of induction. As a corollary, it follows that the union of two strongly finite sets is a strongly finite set. On the other hand, finite sets in EFS or in EFSC do not satisfy some very basic properties. In fact, in section 4, we present a model of EFS in which the union of two finite sets is not a finite set. This entails that there are models of EFSC in which not every finite set is strongly finite. The logical weakness of EFSC is a philosophical strength of the Feferman-Hellman project of grounding the natural number concept on that of "finite set": it shows that the predicative definition of natural numbers due to Aczel et al. does *not* rely (even indirectly) on finite-set induction, thus avoiding the charge that one has escaped the impredicativity of natural-number induction by merely throwing the matter back to the notion of finite set. In the remaining part of the last section, we introduce and discuss the two extra principles referred in the previous paragraph.

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2 Formal background

The language of the theory EFS – an acronym for "Elementary Theory of Finite Sets" – has two sorts of variables. The *individual variables* a, b, x, y, z, ... and the *finite set variables* A, B, C, F, G, ... The latter are intended to vary over finite sets of individuals. There is one binary operation symbol $(,)$ for a *pairing function* on individuals. The individual terms s, t, ... of the language are generated from the individual variables by means of this operation. There are two binary relation

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symbols, $=$ and $\in$ by means of which the atomic formulas of the form $t = s$ and $t \in A$ are obtained. Formulas are generated from these by means of propositional operations and the quantifiers $\forall$ and $\exists$ applied to either sort of variables. Equality between finite sets is *defined* by extensionality, i.e., $A = B$ iff $\forall x(x \in A \leftrightarrow x \in B)$. The axioms of EFS are:

(Sep) $\forall A \exists B \forall x(x \in B \leftrightarrow x \in A \wedge \phi)$, where ϕ is any formula in which the variable B is not free.

(FS-1) $\exists A \forall x(x \notin A)$.

(FS-2) $\forall z \forall A \exists B \forall x(x \in B \leftrightarrow x \in A \vee x = z)$.

(P-1) $(x, y) = (w, z) \leftrightarrow x = w \wedge y = z$.

(P-2) $\exists z \forall x \forall y((x, y) \neq z)$.

The language of EFSC – an acronym for "Elementary Theory of Finite Sets and Classes" – expands the language of EFS by permitting a new sort of variables, written in boldface: the *class variables* $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, ... We permit new atomic formulas of the form $x \in \mathbf{X}$ and the quantifiers are also allowed to apply to the class variables. Equality between classe variables and between a class variable and a finite set variable is *defined* by the principle of extensionality, as before. A class $\mathbf{X}$ is said to be *finite* and we write $Fin(\mathbf{X})$ if $\exists A(A = \mathbf{X})$. A formula of the language of EFSC is said to be *weak second-order* if it contains no bound class variables. The axioms of EFSC are those of EFS – where we now allow weak second-order formulas in the schema for (Sep) – *plus* the following comprehension principle:

(WS-CA) $\exists \mathbf{X} \forall x(x \in \mathbf{X} \leftrightarrow \phi)$, where ϕ is any weak second-order formula in which $\mathbf{X}$ is not free.

It is easy to see that the theory EFS is interpretable in the theory PA of Peano Arithmetic. In showing the existence of a natural number structure in EFSC (see below), Feferman and Hellman (and Peter Aczel) actually show that PA is interpretable in EFSC. Therefore, it is a consequence of the following theorem (see p. 12 of [FH95]) that the theories PA, EFS and EFSC all have the same proof-theoretic strenght.

Metatheorem. *EFSC is a conservative extension of EFS.*

In the language of EFSC, (binary) relations are identified with classes of ordered pairs, and functions $\mathbf{f}$, $\mathbf{g}$, ... are identified with many-one relations. A triple $\langle \mathbf{M}, a, \mathbf{g} \rangle$ is called an *N-structure* if it satisfies the following three conditions:

(N-1) $\forall x \in \mathbf{M}(\mathbf{g}(x) \neq a)$;

(N-2) $\forall x, y \in \mathbf{M}(\mathbf{g}(x) = \mathbf{g}(y) \rightarrow x = y)$;

(N-3) $\forall \mathbf{X} \subseteq \mathbf{M}(a \in \mathbf{X} \wedge \forall x(x \in \mathbf{X} \rightarrow \mathbf{g}(x) \in \mathbf{X}) \rightarrow \mathbf{X} = \mathbf{M})$, where $\mathbf{X} \subseteq \mathbf{M}$ abbreviates $\forall x(x \in \mathbf{X} \rightarrow x \in \mathbf{M})$.

Notice that (WS-CA) entails that N-structures satisfy the principle of induction for weak second-order formulas (with parameters). It is possible to define functions by primitive recursion on any N-structure. This is a consequence of Theorem 4 of [FH95]. Another consequence of this theorem is the categoricity of natural number structures:

Theorem. *(EFSC) Any two N-structures are isomorphic.*

In their first paper, Feferman and Hellman proved the existence of N-structures from EFSC *plus* the so-called Cardinality Axiom. This axiom says that all finite sets are Dedekind-finite. Peter Aczel ameliorated this existence result and showed that EFSC alone already proves the existence of N-structures (see [FH98]):

Theorem. *(EFSC) There are N-structures.*

3 Strongly finite sets

In this section, we work with an arbitrary but fixed N-structure $\langle \mathbf{M}, 0, \mathbf{s} \rangle$. Following Feferman and Hellman we define

$$Clos^{-}(A) \leftrightarrow \forall x(\mathbf{s}(x) \in A \to x \in A)$$

and

$$y \leq x \leftrightarrow \forall A(x \in A \wedge Clos^{-}(A) \to y \in A).$$

Let $Pd(x) = \{y : y \leq x \wedge y \neq x\}$. The following facts are proved in [FH95]:

Proposition. *(EFSC) For all x, y and z in $\mathbf{M}$:*

1. $x \leq x$;
2. $x \leq y \wedge y \leq z \to x \leq z$;
3. $x \leq \mathbf{s}(y) \leftrightarrow x \leq y \vee x = \mathbf{s}(y)$;
4. $Fin(Pd(x))$. □

Definition. *(EFSC) A class $\mathbf{X}$ is called* strongly finite *if there is an element $n \in \mathbf{M}$ and a bijection $\mathbf{f} : Pd(n) \to \mathbf{X}$.*

Proposition. *(EFSC) Every strongly finite class is a finite set.*

Proof : Let $\mathbf{X}$ be a strongly finite class and take $\mathbf{f} : Pd(n) \to \mathbf{X}$ a bijection. Using (FS-1) and (FS-2), it is easy to show

$$x \leq n \to \exists A \forall z(z \in A \leftrightarrow \exists y < x\, \mathbf{f}(y) = z)$$

by induction on x. The proposition follows from the case $x = n$. □

Strongly finite sets are those finite sets which can be counted. To be more exact, given a strongly finite set F there is a unique element $n \in \mathbf{M}$ such that F is in one-to-one correspondence with $Pd(n)$. The uniqueness of n (the *cardinality* of F) is a consequence of the pigeonhole principle for N-structures. The proof of this fact is essentially the same as that of the pigeonhole principle (for class functions) in ACA_0. (The theory ACA_0 is a second-order conservative extension of PA – see, for further information, [Sim87].) Contrary to the mere finite sets, the strongly finite sets in EFSC assent to basic properties of finiteness. For instance, if a class is equipotent (*via* a class function) with a strongly finite set, then that class is also a strongly finite set. On the other hand, we will see in the next section that finite sets may fail to have this property in models of EFSC.

We finish this section with the statement and proof of a principle of induction on strongly finite sets.[1] To ease reading, we write $\forall \tilde{F}(\ldots \tilde{F} \ldots)$ as an abbreviation for $\forall F$(“F is strongly finite” $\rightarrow \ldots F \ldots$). With this notation in mind, we have the following principle of induction on strongly finite sets:

Theorem. *For any weak second-order formula ϕ,*

$$EFSC \vdash \phi(\emptyset) \wedge \forall x \forall \tilde{F}(\phi(\tilde{F}) \rightarrow \phi(\tilde{F} \cup \{x\})) \rightarrow \forall \tilde{F} \phi(\tilde{F})$$

where the symbols $\emptyset$ and $\cup$ have their usual meanings.

Proof : Take F any strongly finite set and fix $\mathbf{f} : Pd(n) \rightarrow F$ a bijection. Assume $\phi(\emptyset)$ and $\forall x \forall \tilde{F}(\phi(\tilde{F}) \rightarrow \phi(\tilde{F} \cup \{x\}))$. It is easy to show by induction on x that

$$x \leq n \rightarrow \forall G(\forall z(z \in G \leftrightarrow \exists y < x\, \mathbf{f}(y) = z) \rightarrow \phi(G)).$$

In particular, for $x = n$ we obtain $\phi(F)$. □

Corollary. *If A and B are strongly finite sets, then so are $A \cup B$ and $A \times B$.*

Proof : Given A a strongly finite set, consider the weak second-order formula

$$\phi_1(G) \leftrightarrow \exists F \forall x(x \in F \leftrightarrow x \in A \vee x \in G).$$

By the principle of induction on strongly finite sets, it is easy to show that $\forall \tilde{G} \phi_1(\tilde{G})$ – the induction step uses the obvious fact that the union of a strongly finite set with a singleton is still a strongly finite set. In particular, putting $G = B$, one concludes that $A \cup B$ is strongly finite.

To deal with the cartesian product, consider the weak second-order formula

$$\phi_2(G) \leftrightarrow \exists F \forall x \forall y\, ((x, y) \in F \leftrightarrow x \in A \wedge y \in G).$$

Using the facts that a class equipotent with a strongly finite set is strongly finite and that the union of two strongly finite sets is still a strongly finite set, it is easy to show by induction on G that $\forall \tilde{G} \phi_2(\tilde{G})$. In particular, putting $G = B$, one concludes that $A \times B$ is strongly finite. □

4 Closing the gap

It is independent from EFSC that every finite set is strongly finite. We prove this result by presenting a model of EFS in which the union of two finite sets is not finite.

Theorem. *There is a model of EFS in which the union of two finite sets is not necessarily finite.*

Proof : Let us define a structure $\mathcal{N}$ in the following way. The first-order domain of $\mathcal{N}$ is the set of natural numbers. The range of the finite set variables are the sets F of natural numbers such that: either the set of odd numbers in F is finite, or the set of even numbers in F is finite. The pairing function is defined by

$$(x, y) = \frac{1}{2}(x + y + 1)(x + y) + x + 1.$$

It is easy to check that $\mathcal{N}$ is a model of EFS. Both the even numbers and the odd numbers constitute $\mathcal{N}$-finite sets while their union clearly does not.

□

[1] With some extra work, it is also possible to state and prove a principle of recursion on (strongly) finite sets. A finite-set recursion principle allowing the definition of a function g may be based on the following equations: $g(a, \emptyset) = \psi(a)$ and $g(a, F \cup \{b\}) = \chi(a, F, b, g(a, F))$ *provided* that the condition $\chi(a, F \cup \{b\}, c, \chi(a, F, b, d)) = \chi(a, F \cup \{c\}, b, \chi(a, F, c, d))$ holds universally. This definedness condition is an adaptation of one such statement occurring in principle N5 of the so-called Naturalistic Set Theory of Garcia in [Gar80].

Using the results of the last section and the conservativity of EFSC over EFS, we get the following corollary:

Corollary. *The theory EFSC does not prove that every finite set is strongly finite.* □

The fact that EFSC cannot prove that the union of two finite sets is a finite set shows that the structure of finite sets in models of EFSC may fail to have some rather basic properties. For instance, there are models of EFSC in which the principle of induction on finite sets for weak second-order formulas fails (the model presented above does the job, since such a principle of induction entails that the union of two finite sets is a finite set). Another example of a basic failure is the following: a class in one-to-one correspondence with a finite set need not be a finite set. More specifically, the following sentence is false in $\mathcal{N}$:

$$\forall F \exists R \forall x (x \in F \leftrightarrow (x, 0) \in R),$$

where 0 denotes the (unique) element of the domain of $\mathcal{N}$ which is not a pair. For instance, the falsity of the above sentence is witnessed by the set F of even numbers since there are infinitely many even numbers and infinitely many odd numbers of the form $(2n, 0)$.

These basic failures in EFS and EFSC have an interesting positive reading. The failures show that it is possible to give a predicative definition of the natural numbers, given the primitive notion of a finite set of individuals (and given also that these individuals have some structure as built up by ordered pairs), *without* relying on finite-set induction. Thus, the logical weakness of EFSC is a philosophical strength of the Feferman-Hellman project of grounding the natural number concept on that of "finite set". It is important for Feferman and Hellman to avoid any appeal to a principle of finite-set induction in their constructions, as that would subject their procedure to a charge of vicious circularity. This sustains their point in favor of the view that natural number induction can be derived from a very weak fragment of finite-set theory, so weak that the very circularity that had, quite reasonably, been suspected (by Parsons in [Par92] and others) has been avoided.

On the other hand, the strongly finite sets assent to the usual properties of finiteness, e.g., to a principle of induction. Therefore, it is pertinent to ask whether there are *prima facie* evident principles for finite sets which entail that every finite set is strongly finite. According to Feferman and Hellman (see [FH98]), these principles should not depend on a grasp of the infinite natural number structure, nor on an explicit understanding of the even more complex infinite structure of its finite subsets ordered by inclusion. To motivate the acceptable principles for finite sets, they suggest that one takes the notion of "finite list of quasi-concrete objects" as understood. With this is mind, we propose the following two principles:

(FS-3) Every finite set can be linearly ordered.

(FS-4) Every linear ordering on a (non-empty) finite set has a first and a last element.[2]

Clearly, (FS-3) is a necessary consequence of the principle that every finite set is strongly finite. The same is true of (FS-4), although for this case a small argument is in order. Let F be a non-empty finite set linearly ordered by R. By assumption, F is strongly finite and, thus, there is an N-structure $\langle \mathbf{M}, 0, \mathbf{s} \rangle$, an element $n \in \mathbf{M}$ and a bijection $\mathbf{g} : Pd(\mathbf{s}(n)) \to F$. By Theorem 4 of [FH95], let $\mathbf{f}^+ : \mathbf{M} \to F$ be defined by $\mathbf{f}^+(0) = \mathbf{g}(0)$ and

$$\mathbf{f}^+(\mathbf{s}(x)) = \begin{cases} \mathbf{g}(\mathbf{s}(x)) & \text{if } x < n \wedge (\mathbf{f}^+(x), \mathbf{g}(\mathbf{s}(x))) \in R \\ \mathbf{f}^+(x) & \text{otherwise} \end{cases}$$

[2]These two principles are related to characterizations of finiteness presented in §6 of [Tar86].

It is easy to see by induction on x that

$$x \leq n \rightarrow \mathbf{f}^+(x) \in F \wedge \forall z \leq x\, R_=(\mathbf{g}(z), \mathbf{f}^+(x))$$

where $R_=(x,y)$ abbreviates $(x,y) \in R \vee x = y$. In particular, we conclude that $R_=(\mathbf{g}(z), \mathbf{f}^+(n))$ for all $z \leq n$. Hence, $\mathbf{f}^+(n)$ is the last element of F with respect to R. A similar argument shows that F has a first element.

In the remaining part of this section, we show that EFSC *plus* (FS-3) and (FS-4) entails that every finite set is strongly finite.

Let $\mathcal{L}(R,F)$ abbreviate the conjunction of the following three sentences:

(i) $\forall x\, (x,x) \notin R$;

(ii) $\forall x \forall y \forall z ((x,y) \in R \wedge (y,z) \in R \rightarrow (x,z) \in R)$;

(iii) $\forall x, y \in F\, (x \neq y \rightarrow (x,y) \in R \vee (y,x) \in R)$.

Note that if $\mathcal{L}(R,F)$, then the (finite) set $\{(x,y) \in R : x \in F \wedge y \in F\}$ exists by (Sep) and it is a (strict) linear ordering of F. Thus, the two above extra principles may be formally rendered by:

(FS-3) $\forall F \exists R\, \mathcal{L}(R,F)$.

(FS-4) $\forall F \forall R (F \neq \emptyset \wedge \mathcal{L}(R,F) \rightarrow \exists x, y \in F\, \forall z \in F (z \neq x \wedge z \neq y \rightarrow (x,z) \in R \wedge (z,y) \in R))$.

We denote by EFS^+ (resp., EFSC^+) the theory EFS (resp., EFSC) *plus* the above two principles.

Lemma. *(EFS^+) Every linear ordering R of a finite set F has the following property: all non-empty subsets X of F have a first and a last element.*

Proof : Just observe that $\{(x,y) \in R : x \in X \wedge y \in X\}$ exists by (Sep) and it is a linear ordering of X. By (FS-4), this linear ordering has a first and a last element. These are the elements we want. □

Theorem. *($EFSC^+$) Every finite set is strongly finite.*

Proof : Let $\langle \mathbf{M}, 0, \mathbf{s} \rangle$ be an N-structure and let F be a (non-empty, non-singular) finite set. Using (FS-3), fix a linear ordering R of F. Let a be the first element of F with respect to this linear ordering. By (WS-CA), we define the (class) function $\mathbf{g} : F \rightarrow F$ as follows:

$$\mathbf{g}(x) = \begin{cases} \text{least element greater than } x & \text{if } \exists y\, (x,y) \in R \\ x & \text{otherwise} \end{cases}$$

Firstly, observe that $\mathbf{g}(x) \neq a$, for all $x \in F$ (remember that F is non-empty, non-singular). Secondly, let us see that if z is an element of F different from a, then there is $z^- \in F$ such that $\mathbf{g}(z^-) = z$. Take z^- the last element of the non-empty (finite) set $\{x \in F : (x,z) \in R\}$. In particular, $(z^-, z) \in R$. Therefore, by definition of $\mathbf{g}$, either $z = \mathbf{g}(z^-)$ or $(\mathbf{g}(z^-), z) \in R$. The latter case contradicts the definition of z^-.

Now, let b be the last element of F with respect to the ordering R. We claim that, for all $x, y \in F$,

$$(*) \qquad (x,b) \in R \wedge (y,b) \in R \wedge x \neq y \rightarrow \mathbf{g}(x) \neq \mathbf{g}(y).$$

By trichotomy, either $(x,y) \in R$ or $(y,x) \in R$. Without loss of generality, assume the earlier possibility. By definition of $\mathbf{g}$, either $\mathbf{g}(x) = y$ or $(\mathbf{g}(x), y) \in R$. Since $(y, \mathbf{g}(y)) \in R$ we get, in both cases, that $(\mathbf{g}(x), \mathbf{g}(y)) \in R$. Thus, $\mathbf{g}(x) \neq \mathbf{g}(y)$.

According to Theorem 4 of [FH98], there is a (class) function $\mathbf{f} : \mathbf{M} \to F$ such that $\mathbf{f}(0) = a$ and $\mathbf{f}(\mathbf{s}(x)) = \mathbf{g}(\mathbf{f}(x))$ for all $x \in \mathbf{M}$. We argue that there is an element $n \in \mathbf{M}$ such that $\mathbf{f}(n) = b$. By the previous lemma, let b' be the last element of the range of $\mathbf{f}$. In particular, $b' = \mathbf{f}(n)$ for some $n \in \mathbf{M}$. Since $\mathbf{f}(\mathbf{s}(n)) = \mathbf{g}(\mathbf{f}(n)) = \mathbf{g}(b')$, we conclude that $\mathbf{g}(b') = b'$ and , therefore, that b' is the last element b of F.

Take $n \in \mathbf{M}$ the least element of $\mathbf{M}$ such that $\mathbf{f}(n) = b$. We show that the function $\mathbf{f}$ restricted to the finite set $Pd(\mathbf{s}(n))$ is a bijection between $Pd(\mathbf{s}(n))$ and F. Let us first check injectivity, i.e., that $\forall y \leq n(\mathbf{f}(x) = \mathbf{f}(y) \to x = y)$, for all $x \leq n$. We proceed by induction on x. If $x = 0$, we must show that $\forall y \leq n(y \neq 0 \to \mathbf{f}(y) \neq a)$. Given that $y \neq 0$, take $y' \in \mathbf{M}$ such that $\mathbf{s}(y') = y$. We get, $\mathbf{f}(y) = \mathbf{f}(\mathbf{s}(y')) = \mathbf{g}(\mathbf{f}(y')) \neq a$. Now, let $\mathbf{s}(x) \leq n$. We must show that $\forall y \leq n(\mathbf{g}(\mathbf{f}(x)) = \mathbf{f}(y) \to \mathbf{s}(x) = y)$. This trivially holds for $y = 0$. If $y \neq 0$, then for some $y' \in \mathbf{M}$, $y = \mathbf{s}(y')$ and $\mathbf{f}(y) = \mathbf{f}(\mathbf{s}(y')) = \mathbf{g}(\mathbf{f}(y'))$. By $(*)$, from $\mathbf{g}(\mathbf{f}(x)) = \mathbf{g}(\mathbf{f}(y'))$ we conclude that $\mathbf{f}(x) = \mathbf{f}(y')$. By induction hypothesis, we get $x = y'$. Thus, $\mathbf{s}(x) = y$. It remains to show surjectivity, i.e., that $\forall z \in F \exists x \leq n(\mathbf{f}(x) = z)$. In order to obtain a contradiction, assume that the finite set $X = \{z \in F : \forall x \leq n\, \mathbf{f}(x) \neq z\}$ is non-empty. By the previous lemma, take z the least element of X with respect to the ordering R. Clearly, $z \neq a$. Take $x' \leq n$ such that $\mathbf{f}(x') = z^-$. Clearly, $x' \neq n$. Hence, $\mathbf{s}(x') \leq n$ and $\mathbf{f}(\mathbf{s}(x')) = \mathbf{g}(\mathbf{f}(x')) = \mathbf{g}(z^-) = z$, which contradicts the fact that $z \in X$. □

By the results of the previous section, the above theorem shows that in EFSC^+ we are allowed to use basic forms of reasoning concerning finite sets, *viz.* a principle of induction on finite sets. All the same, the theories EFSC and EFSC^+ have equal proof-theoretic strength since it is clear that EFSC^+ is interpretable in ACA_0 (remember that ACA_0 has the same proof-theoretic strength as PA).

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