A Substitutional Framework for Arithmetical Validity

Fernando Ferreira
Universidade de Lisboa

In fond memory of Nina.
(1913-1997)

1 Introduction

A platonist in mathematics believes that arithmetic has a subject matter, i.e., that the statements of arithmetic are about certain objects – the natural numbers. For a platonist, the language of (first-order) arithmetic L_a is referential and he is licensed to speak of true and false sentences of L_a and to endorse Tarski's analysis of truth. It follows from this Tarskian analysis *plus* the fact that every natural number is denoted by some closed term of L_a (a *numeral*, if one insists on canonicity) that the truth values of arithmetical sentences are determined by the truth values of its atomic sentences. Consider now a philosopher who, while not a platonist in any sense, broadly accepts the results of mathematics – however tentatively – and is persuaded that the truths of arithmetic are determined by the truth values of its atomic sentences (whatever may be his reformulation of the notion of arithmetical truth). This article may be viewed as an attempt to frame a position for such a non-platonist philosopher of a non-revisionist bent.

It is well known that certain atomic sentences of arithmetic have a persuasive rendering in terms of schemata of formulas of first-order languages with equality. This rendering is specially persuasive insofar as we focus on the cardinal role of numbers (and leave their ordinal role aside). For instance, the sentence 7+5 =12 can be rendered as

$$(*) \qquad \exists_{\overline{7}} x A(x) \wedge \exists_{\overline{5}} x B(x) \wedge \neg \exists x (A(x) \wedge B(x)) \rightarrow \exists_{\overline{12}} x (A(x) \vee B(x))$$

where A and B are any formulas of a given first-order language and where, for each numeral $\overline{n}$, $\exists_{\overline{n}} x C(x)$ makes the numerical claim that there are exactly n objects x such that $C(x)$. Such numerical claims have straightforward renderings in first-order languages with equality and, thus, expressions of the form $\exists_{\overline{n}} x\, C(x)$ are explicitly eliminable within those languages. I call a first-order scheme like $(*)$ a *checking point* of arithmetic.

In the next section, I introduce a number of checking points of arithmetic, sufficient to determine arithmetical truth (in the platonic sense above). This is done in such a manner that an atomic (or negated atomic, as we will see) sentence of arithmetic is true if, and only if, the corresponding scheme consists of logically valid formulas. This feature should be enough to convince our non-revisionist philosopher that arithmetical truth (better, his reformulation thereof) is determined by logic alone. In section 3, I show how to extend the above correspondence to all sentences of first-order arithmetic. My proposal has the following general features. Given a referential first-order language with equality L enhanced with numerical quantifiers of the form $\exists_{\overline{n}} x$, I firstly extend L to a language L_{sa} in which substitutional quantification is permitted for the substitutional class constituted by the numerals $\overline{n}$ occurring in the expressions $\exists_{\overline{n}} x$ (note that the numerals here are construed as syncategorematic, significant in context but naming nothing). Afterwards, I show how to associate with each first-order sentence S of the language of arithmetic L_a a scheme of sentences $\mathcal{S}$ of the substitutional language L_{sa} such that if S is true then each instance of $\mathcal{S}$ is logically valid. A suitable modification of the converse of this implication also holds. Hence, under this rendering, arithmetical truth is subsumed under a notion of logical validity and the nature of the determination of arithmetic by its checking points is rooted in substitutional quantification. I therefore accomplish a form of reduction of arithmetic to logic, a brand of *logicism* for first-order arithmetic. In the last section, I compare my substitutional approach with Gottlieb's approach as presented in [Got80]. I do not attempt to discuss the ontological issues posed by substitutional quantification. They are too intricate to be discussed in this paper. All the same, I finish the paper with an observation concerning the so-called orthodox interpretation of my substitutional apparatus.

2 Checking points of arithmetic

My insisting that the checking points of arithmetic are schemata of first-order formulas poses some difficulties concerning the proper treatment of the operation of negation. For instance, given natural numbers n, k and r, I render the atomic sentence $\overline{n} + \overline{k} = \overline{r}$ of referential number theory by the scheme,

(1) $$\exists_{\overline{n}} x A(x) \wedge \exists_{\overline{k}} x B(x) \wedge \neg \exists x (A(x) \wedge B(x)) \to \exists_{\overline{r}} x (A(x) \vee B(x)).$$

It is clear that this scheme consists of logically valid sentences if, and only if, $n+k = r$. Consider now the falsity $5 + 3 = 7$. How should its negation, *viz.* $5 + 3 \neq 7$, be rendered? Gottlieb in [Got80] does not have to face *this* problem since he renders

$5+3=7$ by the second-order *sentence*,

$$\forall F \forall G(\exists_{\overline{5}} x Fx \wedge \exists_{\overline{3}} x Gx \wedge \neg \exists x(Fx \wedge Gx) \rightarrow \exists_{\overline{7}} x(Fx \vee Gx)).$$ [1]

Accordingly, Gottlieb renders $5+3 \neq 7$ by,

$$\exists F \exists G(\exists_{\overline{5}} x Fx \wedge \exists_{\overline{3}} x Gx \wedge \neg \exists x(Fx \wedge Gx) \wedge \neg \exists_{\overline{7}} x(Fx \vee Gx)).$$

This rendering is problematic. In a universe with less than eight elements $5+3=7$ is rendered true, while $5+3 \neq 7$ is rendered false. A curious inversion of truth-values. Gottlieb welcomes the first type of situations in which, due to the finiteness of the universe, certain falsities of arithmetic are rendered true. He adds that that 'is precisely what is to be expected if we found our account of arithmetic upon its application to multiplicity attributions.'[2] I agree with Gottlieb's remark but I add a *correctness* constraint: in a proper account of arithmetic, the truths of arithmetic should yield truths in applications to objectual languages. In this light, the fact that some falsities of arithmetic may – in certain situations – yield truths is beside the point. Gottlieb could not have subscribed to *this* constraint because, in his framework, the *negations* of the above falsities (which are *bona fide* arithmetical truths) would, in the very same situations, yield falsities. That is what happens with his rendering of $5+3 \neq 7$. Gottlieb rightly sees a problem here and he proposes a revision of the axioms of arithmetic. I resist revisionism. Therefore, I deal with this problem differently.

Given numerals $\overline{n}$, $\overline{k}$ and $\overline{r}$, I render the atomic sentence $\overline{n}+\overline{k} \neq \overline{r}$ of referential number theory by the scheme,

(2) $$\exists_{\overline{n}} x A(x) \wedge \exists_{\overline{k}} x B(x) \wedge \neg \exists x(A(x) \wedge B(x)) \rightarrow \neg \exists_{\overline{r}} x(A(x) \vee B(x)).$$

Each pair of corresponding instances of the schematic renderings of $\overline{n}+\overline{k}=\overline{r}$ and $\overline{n}+\overline{k} \neq \overline{r}$ is constituted by *subcontrary* sentences, i.e., by two sentences that cannot be both false but which can be both true. As a matter of fact, there are situations in which all the instances of both schemes $\overline{n}+\overline{k}=\overline{r}$ and $\overline{n}+\overline{k} \neq \overline{r}$ are true. This seems to be a fatal blow for a proper treatment of the non-atomic sentences of arithmetic, namely for a proper treatment of true arithmetic sentences of the form $\neg(\overline{n}+\overline{k}=\overline{r} \wedge \overline{n}+\overline{k} \neq \overline{r})$. However, the no go situation is apparent. The heart of the solution to this problem will be presented in the following paragraphs and, in the next section, a mathematical theorem will dispel any remaining doubts concerning the soundness of my solution.

[1] In Gottlieb's account, the second-order quantifiers are interpreted substitutionally. To be more exact, he uses *full* relative substitutional quantification, as in the terminology of Parsons [Par82].

[2] In p. 103 of [Got80].

It is convenient to work with a language L_a of first-order arithmetic which has no function symbols and which has a constant symbol c_n for each natural number n.[3] Instead of function symbols for *successor*, *addition* and *multiplication* I have, respectively, a binary relation symbol $Suc(x, y)$ and ternary relation symbols $Add(x, y, z)$ and $Mul(x, y, z)$. These relation symbols have the intended meaning of being the graphs of the corresponding arithmetical operations. It is well known that this reformulation of the standard language of first-order arithmetic gains and loses nothing in terms of expressive power. I extend the vocabulary of this language by associating with each relation symbol Suc, Add, Mul a corresponding relation symbol $\widetilde{Suc}$, $\widetilde{Add}$, $\widetilde{Mul}$. These new relation symbols stand for the negations of the corresponding earlier relations. Suc, $\widetilde{Suc}$, Add, $\widetilde{Add}$, Mul and $\widetilde{Mul}$ are called *literals* and Suc and $\widetilde{Suc}$ (resp., Add and $\widetilde{Add}$, and Mul and $\widetilde{Mul}$) are *opposite* literals. The equality sign $=$ and the inequality sign $\neq$ are also called opposite literals. The formulas of L_a are built up from the atomic formulas (of the extended vocabulary) by means of conjunction, disjunction, universal quantification and existential quantification. The negation $\neg A$ of a formula A is *defined* to be the formula obtained from A by

i. replacing each literal by its opposite and

ii. replacing $\wedge$, $\vee$, $\forall$, $\exists$ by $\vee$, $\wedge$, $\exists$, $\forall$, respectively.

This treatment of negation for classical logic is well-known (see [Tai68]). Note that $\neg\neg A$ is the *same* formula as A. As usual, we define $A \to B$ to be $\neg A \vee B$ and $A \leftrightarrow B$ to be $(A \to B) \wedge (B \to A)$. According to this set-up, the sentence $\neg(\overline{n} + \overline{k} = \overline{r} \wedge \overline{n} + \overline{k} \neq \overline{r})$ is firstly rendered as $\neg(Add(\overline{n}, \overline{k}, \overline{r}) \wedge \widetilde{Add}(\overline{n}, \overline{k}, \overline{r}))$ and finally takes the official form $\widetilde{Add}(\overline{n}, \overline{k}, \overline{r}) \vee Add(\overline{n}, \overline{k}, \overline{r})$. Now, if we translate Add and $\widetilde{Add}$ according to (1) and (2) above and if we suitably conjoin these schemes, the true sentence $\neg(\overline{n} + \overline{k} = \overline{r} \wedge \overline{n} + \overline{k} \neq \overline{r})$ is rendered by the first-order scheme,

$$[\exists_{\overline{n}} x A(x) \wedge \exists_{\overline{k}} x B(x) \wedge \neg\exists x (A(x) \wedge B(x)) \to \neg\exists_{\overline{r}} x (A(x) \vee B(x))] \vee$$
$$[\exists_{\overline{n}} x C(x) \wedge \exists_{\overline{k}} x D(x) \wedge \neg\exists x (C(x) \wedge D(x)) \to \exists_{\overline{r}} x (C(x) \vee D(x))] \,.$$

Observe that the instances of the above scheme are logically valid, no matter what are the numerals $\overline{n}$, $\overline{k}$ and $\overline{r}$. This is as it should be.

We finish this section with a catalog of the checking points of arithmetic, one for each atomic sentence of L_a. As we go along, the reader should pause and convince himself that each checking point is constituted by logically valid sentences if, and

[3] Thus, L_a has an infinite number of constants. Of course, the idea of an infinite alphabet goes beyond a purely syntactic view. However, it is well-known that in cases such as the above, the formal apparatus can be reformulated suitably by replacing each numerical constant c_n by an expression consisting of $n + 1$ consecutive strokes '|'. To facilitate reading, I use in the sequel the more congenial expression $\overline{n}$ instead of c_n

only if, the corresponding arithmetical sentence is true in the standard interpretation of L_a. We have already discussed the checking points of arithmetic associated with the sentences of the form $Add(\overline{n}, \overline{k}, \overline{r})$ and $\widetilde{Add}(\overline{n}, \overline{k}, \overline{r})$. These are the schemata (1) and (2), respectively. There remains the following cases:

(3) The checking point of $\overline{n} = \overline{k}$ is the scheme:

$$\exists_{\overline{n}} x A(x) \leftrightarrow \exists_{\overline{k}} x A(x).$$

(4) The checking point of $\overline{n} \neq \overline{k}$ is the scheme:

$$\exists_{\overline{n}} x A(x) \rightarrow \neg \exists_{\overline{k}} x A(x).$$

(5) The checking point of $Suc(\overline{n}, \overline{k})$ is the scheme:

$$\exists_{\overline{k}} x A(x) \rightarrow \exists y (A(y) \wedge \exists_{\overline{n}} x (A(x) \wedge x \neq y)).$$

(6) The checking point of $\widetilde{Suc}(\overline{n}, \overline{k})$ is the scheme:

$$\exists_{\overline{k}} x A(x) \rightarrow \forall y (A(y) \rightarrow \neg \exists_{\overline{n}} x (A(x) \wedge x \neq y)).$$

(7) The checking point of $Mul(\overline{n}, \overline{k}, \overline{r})$ is the scheme:

$$(\exists_{\overline{n}} x A(x) \wedge \forall x \forall w \forall y (A(x) \wedge A(w) \wedge C(x, y) \wedge C(w, y) \rightarrow x = w) \wedge$$

$$\forall x (A(x) \rightarrow \exists_{\overline{k}} y C(x, y))) \ \rightarrow \ \exists_{\overline{r}} y (\exists x (A(x) \wedge C(x, y))).$$

(8) Finally, the checking point of $\widetilde{Mul}(\overline{n}, \overline{k}, \overline{r})$ is the scheme:

$$(\exists_{\overline{n}} x A(x) \wedge \forall x \forall w \forall y (A(x) \wedge A(w) \wedge C(x, y) \wedge C(w, y) \rightarrow x = w) \wedge$$

$$\forall x (A(x) \rightarrow \exists_{\overline{k}} y C(x, y))) \ \rightarrow \ \neg \exists_{\overline{r}} y (\exists x (A(x) \wedge C(x, y))).$$

Let me give an application of (7). Suppose $A(x)$ means that x is a soccer team in the Portuguese upper division soccer championship league. Suppose $C(x, y)$ means that y is a soccer player of team x at the start of a given complete round of soccer matches. There are 18 soccer teams in the championship league and each team starts a match with 11 players (it goes without saying that a soccer player cannot play for two teams simultaneously). By a logically valid instance of (7), we can conclude that there are 198 soccer players at the start of the given complete round of soccer matches.

3 Substitutional arithmetic done right (?)

Let L be a first-order referential language enhanced with numerical quantifiers of the form $\exists_{\overline{n}}x$. I extend L to a language L_{sa} in which substitutional quantification plays a prominent role. With slight differences of terminology, I essentially follow the paradigmatic treatment of substitutional quantification as expounded by Kripke in [Kri76]. Let $\mathrm{x}_1, \mathrm{x}_2, \ldots$ be an infinite list of variables not occurring in L. (Following Kripke's notation, I use italicized variables $x_1, x_2, \ldots$ for the referential variables of the given language L; the new unitalicized variables play a substitutional role in the extended language L_{sa}.) An *atomic form* is an expression obtained from a *sentence* of L by replacing zero or more numerals $\overline{n}$ occurring in the context of a numerical quantifier $\exists_{\overline{n}}x$ by an unitalicized variable. For instance, given $A(x)$ a formula of L with parameters z_1, ..., z_s, then

$$\forall z_1 \cdots \forall z_s(\exists_{\mathrm{y}} x A(x) \rightarrow \exists y(A(y) \wedge \exists_{\mathrm{x}} x(A(x) \wedge x \neq y)))$$

is an atomic form with free substitutional variables x and y (the prefix $\forall z_1 \cdots \forall z_s$ ensures that the above atomic form comes from a *sentence* of L). We are now ready to define inductively the notion of a *form* of L_{sa}. An atomic form is a form. If ϕ and ψ are forms then so are $\phi \wedge \psi$, $\phi \vee \psi$, $\Sigma\mathrm{x}\phi$ and $\Pi\mathrm{x}\phi$. (It is assumed that the truth-functions and the existential Σ and the universal Π substitutional quantifiers are *new* notations, not to be found in L.) A *sentence form* of L_{sa} is a form without free substitutional variables (note that, by definition, forms do not have free referential variables). Ever since Tarski [Tar83], we know how to define truth for sentences of L with respect to a previously given interpretation. We extend Tarski's truth conditions to account for the notion of a true sentence form of L_{sa}. The conditions are:

(a) If ϕ is an atomic form which is a sentence (thus ϕ is a sentence of L), then ϕ is true in the extended sense if, and only if, it is true in the original sense.

(b) $\phi \vee \psi$ is true if, and only if, either ϕ is true or ψ is true (or both).

(c) $\phi \wedge \psi$ is true if, and only if, both ϕ and ψ are true.

(d) $\Sigma\mathrm{x}\phi$ is true if, and only if, there is a numeral $\overline{n}$ such that ϕ' is true, where ϕ' comes from ϕ by replacing all free occurrences of x by $\overline{n}$.

(e) $\Pi\mathrm{x}\phi$ is true if, and only if, for all numerals $\overline{n}$, ϕ' is true, where ϕ' comes from ϕ by replacing all free occurrences of x by $\overline{n}$.

For instance, according to the above definition, the scheme of sentence forms

$$(**) \qquad \Pi\mathrm{x}\Sigma\mathrm{y}\forall z_1 \cdots z_s\,(\exists_{\mathrm{y}} x A(x) \rightarrow \exists y(A(y) \wedge \exists_{\mathrm{x}} x(A(x) \wedge x \neq y)))$$

is always true, no matter what is the given interpretation of the underlying referential language L (as before, z_1, ..., z_s are the parameters of $A(x)$).

Let L_a be the language of first-order arithmetic, as described in the last section. I define a correspondence between formulas S of L_a and schemata of forms $\mathcal{S}$ of L_{sa}. The correspondence is such that if x, y, z (say) are the free variables of S, then x, y, z are the free substitutional variables of the forms occurring in $\mathcal{S}$. The correspondence is defined inductively, according to the following clauses:

(a) If S is an atomic *sentence* of L_a, then $\mathcal{S}$ consists of the first-order referential universal closures of the formulas of the corresponding checking points (i.e., the checking points are prefixed by a row of universal objectual quantifiers $\forall z_1 \cdots \forall z_s$ to ensure that all referential variables are bound). The case of atomic *formulas* is obtained from the previous case by suitably replacing numerals in the numerical quantifiers by substitutional variables. E.g., the scheme that corresponds to $Add(\overline{3}, z, x)$, consists of the universal referential closures of

$$\exists_{\overline{3}} x A(x) \wedge \exists_{\mathrm{z}} x B(x) \wedge \neg \exists x (A(x) \wedge B(x)) \rightarrow \exists_{\mathrm{x}} x (A(x) \vee B(x)).$$

(b) The scheme corresponding to $Q_1 \vee Q_2$ consists of the forms $\phi \vee \psi$, where ϕ is in the scheme corresponding to Q_1 and ψ is in the scheme corresponding to Q_2.

(c) The scheme corresponding to $Q_1 \wedge Q_2$ consists of the forms $\phi \wedge \psi$, where ϕ is in the scheme corresponding to Q_1 and ψ is in the scheme corresponding to Q_2.

(d) The scheme corresponding to $\exists x Q$ consists of the forms $\Sigma \mathrm{x} \phi$, where ϕ is in the scheme corresponding to Q.

(e) The scheme corresponding to $\forall x Q$ consists of the forms $\Pi \mathrm{x} \phi$, where ϕ is in the scheme corresponding to Q.

We are now ready to state and prove the following theorem:

Theorem. *If S is a true sentence of L_a, then the corresponding scheme $\mathcal{S}$ consists of true sentence forms of L_{sa}.*

Proof : This is a simple proof by induction on the complexity of the sentence S. If S is a true atomic sentence, a direct inspection of the checking points (1) - (8) of the last section shows that the corresponding scheme indeed consists of true sentence forms. Suppose S is the disjunction $S_1 \vee S_2$. Then either S_1 is true or S_2 is true. Assume, without loss of generality, the earlier case. By definition, the scheme corresponding to S consists of the sentence forms $\phi \vee \psi$, where ϕ is in the scheme corresponding to S_1 and ψ is in the scheme corresponding to S_2. By induction hypothesis, the above ϕs are true. Therefore, so are the disjunctions $\phi \vee \psi$. Now, suppose S is $\exists x Q(x)$. Then there is a numeral $\overline{n}$ such that $Q(\overline{n})$ is true. By definition, the scheme $\mathcal{S}$ corresponding to S consists of the forms $\Sigma \mathrm{x} \phi(\mathrm{x})$, where $\phi(\mathrm{x})$ is in the scheme corresponding to the formula $Q(x)$. By induction hypothesis, $\phi(\overline{n})$ is a true sentence form for every such form ϕ. Hence, $\Sigma \mathrm{x} \phi(\mathrm{x})$ is true.

The cases of conjunction and universal quantifications are also easy to handle. Q. E. D.

In the above theorem, it is implicit a given interpretation of the original referential language L (from which L_{sa} is built). This interpretation can be *any* interpretation. Therefore, in a definite sense, if S is a true sentence of number theory, then $\mathcal{S}$ consists of *logically valid* sentence forms.[4] In fact, more is true. A scheme of logically valid sentence forms of the type $\Sigma \mathrm{x} \phi(\mathrm{x})$ – where $\phi(\mathrm{x})$ is a form in a previously given scheme $\mathcal{S}'$ – is called *robust* if there is a numeral $\overline{n}$ such that all sentence forms $\phi(\overline{n})$ – where $\phi(\mathrm{x})$ is in $\mathcal{S}'$ – are logically valid. It is a consequence of the proof of the above theorem that if $\exists x Q(x)$ is a true sentence of arithmetic, then the corresponding logically valid scheme is robust.[5] This property of robustness is crucial for a correct rendering of arithmetical truth. It says, in effect, that the substitutions that make each instance of a scheme – corresponding to a true arithmetical sentence – a true sentence form are *independent* from the particular instance in question. If we were working with slightly more logical resources, we could have expressed the above mentioned independence in a direct linguistic (as opposed to a meta-linguistic) way. Indeed, if we allow quantification for the predicates of the underlying referential language, then we could associate with each sentence S of arithmetic a *sentence* $\mathcal{S}$ of the expanded language L_{sa} by replacing the checking points of arithmetic with a corresponding universal predicate closure. For instance, the arithmetical statement $\forall x \exists y \, Suc(x,y)$ would be rendered by

$$\Pi \mathrm{x} \Sigma \mathrm{y} \forall F \, (\exists_{\mathrm{y}} x Fx \rightarrow \exists y (Fy \wedge \exists_{\mathrm{x}} x (Fx \wedge x \neq y))) \,.^{6,7}$$

[4] To put it explicitly, I am using the following *semantic* notion of logical validity: a sentence form is logically valid if, and only if, it is true in every interpretation (with non-empty domain). I caution the reader that this notion of logical validity differs from Kripke's notion as expounded in pages 335-337 of [Kri76]. Kripke's notion is too tied up with the first-order objectual case.

[5] A similar property of robustness holds for schemata corresponding to disjunctive true sentences of arithmetic: if the schema $\mathcal{S}$ consists of sentence forms of the type $\phi_1 \vee \phi_2$, where ϕ_1 and ϕ_2 come from previously given schemata $\mathcal{S}_1$ and $\mathcal{S}_2$ (respectively), then either $\mathcal{S}_1$ consists only of logically valid sentence forms or $\mathcal{S}_2$ does.

[6] The reader should compare this rendering with (∗∗).

[7] There are similarities between this alternative approach (which, nevertheless, assents to the correctness constraint) and Gottlieb's account in [Got80]. However, the latter's account makes essential use of existential predicate quantifications. This is not the case for the alternative account.

Although the alternative account uses a new sort of quantification, it nevertheless seems committed to the same ontology as the original account. In fact, the new predicate quantifiers are always universal and never appear in subordinate clauses. Predicative expressions (or predicates, or classes, or whatever) are required for making the (alternative) rendering of true arithmetical sentences logically valid schemata, as much as they are required to account for the schematic validities of first-order logic.[8] And nothing is required for the latter account, as Quine forcefully argues in [Qui53]. The reason why I did not pursue the alternative account is that I wanted to press to the limits a minimal use of logical resources. But have I pressed too much? Is my account at all faithful to the fabric of arithmetic? To put it more exactly: are the truths of arithmetic determined by the checking points of arithmetic *plus* logical validity *plus* my minimal substitutional apparatus? The answer to this question is affirmative. That is the content of the next theorem, which applies to sufficiently rich underlying referential languages:

Theorem. *If S is a false sentence of L_a, then the corresponding scheme $\mathcal{S}$ is not constituted only by logically valid sentence forms.*

A proof of this theorem is given in the appendix.

4 Final remarks

We can read the sentences of the language L_a of first-order arithmetic as encapsulations of certain schemata derived from first-order referential languages. The common ground between these schemata and referential first-order languages are the checking points of arithmetic. These checking points make claims about multiplicity attributions and it is upon these claims that arithmetic is ultimately founded and that its applications are accounted for. As we saw, these schemata are logically valid if, and only if, the (assumed) underlying arithmetical sentence is true. Insofar as arithmetical truth is subsumed under a notion of logical validity, we may classify the above rendering as a brand of logicism, albeit of a *non-Fregean* type since it is not framed on second-order logic nor does it regard numbers as objects *via* Hume's principle. The deductive calculus of first-order arithmetic can be seen as a calculus for producing (certain) logically valid schemata. Even though this calculus is formally consonant with a direct referential reading, its semantics is in no way directly referential. For instance, the semantic interpretation of the negation sign cannot, in any sense, be considered *negation*. A similar phenomenon concerns the behaviour of the equality sign: formally, it is like equality in a referential language. In actual fact, it does not even stand for a relation between objects.

[8] Of course, I am here disposing of the issue concerning the ontological commitments to *numerical* expressions.

My rendering of arithmetic and Gottlieb's account in [Got80] are founded on the same two fundamental ideas: application to multiplicity attributions and substitutional quantification. There is a definite sense in which my proposal is both an amendment and an amelioration of Gottlieb's arithmetic. On the one hand, it is an amendment because Gottlieb's arithmetic does not satisfy the correctness constraint. On Gottlieb's account, there are true sentences of arithmetic (even quantifier-free) which are rendered false in certain circunstances. I have already given an example of this phenomenon in section 2, but it is worthwhile to point out that Gottlieb's problem already shows up at the really fundamental level of the interpretation of inequality. According to Gottlieb, given a certain referential language and a certain interpretation thereof, the sentence $\overline{n} \neq \overline{k}$ is true if, and only if, it is *not* the case that $\forall F(\exists_{\overline{n}} xFx \leftrightarrow \exists_{\overline{k}} xFx)$. On this account, if the domain of the interpretation has less than k elements, then $\overline{n} \neq \overline{k}$ is rendered false for all $n > k$. Gottlieb's way out is to revise arithmetic, thus dissenting from the correctness constraint. On the other hand, my proposal ameliorates Gottlieb's account in that it is more parsimonious in the use of logical and semantic resources. As I have already remarked, Gottlieb makes use of substitutional quantification not only for the numerals occurring in expressions of the form $\exists_{\overline{n}} x$, but also for the predicates of the given referential language. Moreover, Gottlieb introduces the truth conditions for multiplication *via* a recursive definition that reduces these conditions to the truth conditions of addition. Only a somewhat indirect reading of these truth conditions for multiplication would understand them in terms of multiplicity attributions. My approach to multiplication is not mediated by addition, does not use the apparatus of a recursive definition for the truth-conditions, and has a direct reading in terms of multiplicity attributions.

Quine's criterium of ontological commitment for an objectual first-order language states that some given object is required in a theory if that object is required, for the truth of the theory, to be among the values over which the bound variables range (see [Qui69]). Orthodox wisdom complements this criterium with the following principle: when the language of a theory is not objectual, then the ontological commitments of the theory cannot be assessed directly – the language must be first translated into an objectual first-order language and only then can the commitment be assessed. In the case of substitutional quantification, Quine proposes a translation which is mathematically indistinguishable from the truth theory given in the first paragraphs of the previous section. It follows that a theory framed in a substitutional language is, in general, committed to an ontology of expressions and of (finite) sequences of objects. Substitutionalism, on the other hand, is the doctrine according to which substitutional quantification carries no ontological commitments. A substitutionalist rejects orthodoxy and, accordingly, feels entitled to reject *abstracta* like expressions. A third view, proposed by Parsons in [Par83], maintains that substitutional quantifications 'could express a genuine concept of existence, different from that of the objectual quantifier.'[9] The question of adjudicating between orthodoxy, substitutionalism or Parsons's *middle way* is an intricate ontological question whose discussion is beyond the scope of this paper.

All the same, I would like to draw attention to a feature of the substitutional framework for arithmetical validity presented in this article. According to this interpretation, sentences of first-order arithmetic are read as encapsulations of certain schemata derived from first-order languages. I showed in the previous section that if a sentence of first-order arithmetic is true in the standard interpretation of number theory then that sentence encapsulates (robust) logically valid schemata. As a consequence, the correctness constraint is uphold. There is, of course, an alternative substitutional interpretation of first-order arithmetic which assents to the correctness constraint. Such an interpretation is based on algorithms that discriminate the truths from the falsehoods for atomic sentences of first-order arithmetic. Quantifications are then interpreted substitutionally, i.e., a sentence of the form $\exists x Q(x)$ is true if, and only if, there is some closed term t of the language of arithmetic such that $Q(t)$ is true. In my view, this interpretation has a limited philosophical interest unless it is complemented by an account regarding the applications of arithmetic (Parsons sketches such an account in pp. 416-417 of [Par82]). However, this "combinatorial" interpretation and my interpretation are altogether different, even with respect to the ontological commitments to numerals under an orthodox analysis. Let us see why.

On the orthodox interpretation, the "combinatorial" view is ontologically committed to *all* numerals. However, this need not be the case for the substitutional interpretation presented in this article. In fact, if the domain of the underlying language L is finite – say, of cardinality n – then, on the orthodox interpretation, my framework is only ontologically committed to the first $n + 2$ numerals.[10] This can be seen as follows. Given any form $\phi(\mathrm{x}, \mathrm{y})$ associated with a given atomic formula $Q(x, y)$ (say) of L_a, and given natural numbers k and r, $\phi(\overline{k}, \overline{r})$ is a true sentence form if, and only if, $\phi(\overline{\min\{k, n+1\}}, \overline{\min\{r, n+1\}})$ is.[11] We can see this by direct inspection. Plainly, this phenomenon propagates to all first-order formulas of arithmetic.

On the surface, there is something bizarre in the above result. If we start with a singleton universe, then every sentence of referential arithmetic is translated into a schema that can be decided with only three numerals. This *seems* to fly in the face of results of Church, Gödel, Tarski *et al.* to the effect that the set of truths of number theory is not decidable. In fact, the contradiction is apparent. My set-up does *not* entail that a *falsity* of arithmetic is translated into a scheme constituted by

[9]See p. 410 of [Par82].

[10]Thus, in a sense, it is committed to no numerals at all! Just replace the existential (resp., universal) substitutional quantifiers by suitable finite disjunctions (resp., conjunctions).

[11]$\min\{a, b\}$ is the least number among a and b.

false sentences. It just entails that a *truth* of arithmetic is translated into a scheme constituted by true sentences. The injunction is: use true sentences of arithmetic in your study of the world, not false ones.[12]

5 Appendix

Let L be a first-order language with equality with at least two binary relation symbols $A(x,u)$ and $B(x,v)$ and a ternary relation symbol $C(x,y,w)$. We associate with each formula S of the language L_a of arithmetic a *special* form F_S of the scheme $\mathcal{S}$ of forms corresponding to S. This is done as follows:

(a) If S is an atomic sentence of L_a, e.g. $\overline{n}+\overline{k}=\overline{r}$, then F_S is the sentence form

$$\forall u \forall v (\exists_{\overline{n}} x A(x,u) \wedge \exists_{\overline{k}} x B(x,v) \wedge \neg \exists x (A(x,u) \wedge B(x,v)) \rightarrow \exists_{\overline{r}} x (A(x,u) \vee B(x,v))).$$

Similarly for the other atomic sentences of arithmetic (the notation is set-up in such a way as to make obvious what are the special forms associated with each atomic sentence). The case of atomic *formulas* is obtained from the previous case by suitably replacing numerals in the numerical quantifiers by substitutional variables.

(b) $F_{Q_1 \vee Q_2}$ is $F_{Q_1} \vee F_{Q_2}$.

(c) $F_{Q_1 \wedge Q_2}$ is $F_{Q_1} \wedge F_{Q_2}$.

(d) $F_{\exists x Q}$ is $\Sigma \mathrm{x} F_Q$.

(e) $F_{\forall x Q}$ is $\Pi \mathrm{x} F_Q$.

The last theorem of section 3 is a consequence of the following lemma:

Lemma. *There is an interpretation $\mathcal{M}$ of L such that for all sentences S of the language L_a of arithmetic, S is true if, and only if, F_S is true in $\mathcal{M}$.*

Proof : The domain of $\mathcal{M}$ is, by definition, the set of the natural numbers. The interpretations $A^{\mathcal{M}}$, $B^{\mathcal{M}}$ and $C^{\mathcal{M}}$ of A, B and C (respectively) are defined by:

(i) $(x,u) \in A^{\mathcal{M}}$ if, and only if, x is even and $x < 2u$;

(ii) $(x,v) \in B^{\mathcal{M}}$ if, and only if, x is odd and $x < 2v+1$;

[12] I am grateful to Manuel Lourenço who commented this article in his characteristically thoughtful manner. I would also like to thank António Zilhão for the kind invitation to contribute with a paper for this issue of *Grazer Philosophische Studien*.

(iii) $(x, y, w) \in C^{\mathcal{M}}$ if, and only if, $xw \leq y < (x+1)w$.

Given an atomic sentence of arithmetic S, it is clear that S is true if, and only if, F_S is true in $\mathcal{M}$. It is straightforward to argue that this equivalence propagates to all first-order sentences of arithmetic. Q. E. D.

References

[Got80] Dale Gottlieb. *Ontological Economy. Substitutional Quantification and Mathematics.* Oxford: Clarendon Press, 1980.

[Kri76] Saul Kripke. Is there a problem about substitutional quantification? In G. Evans and J. McDowell, editors, *Truth and Meaning*, pages 325–419. Oxford: Clarendon Press, 1976.

[Par82] Charles Parsons. Substitutional quantification and mathematics. *British Journal for the Philosophy of Science*, 33:409–421, 1982. Review of Gottlieb, *Ontological Economy.*

[Par83] Charles Parsons. A plea for substitutional quantification. In *Mathematics in Philosophy*, pages 63–70. Cornell University Press, 1983. First published in 1971.

[Qui53] Willard Quine. Logic and the reification of universals. In *From a Logical Point of View*, pages 102–129. Harvard University Press, 1953.

[Qui69] Willard Quine. Existence and quantification. In *Ontological Relativity and Other Essays*, pages 91–113. Columbia University Press, 1969.

[Tai68] William Tait. Normal derivability in classical logic. In Jon Barwise, editor, *The Syntax and Semantics of Infinitary Languages*, pages 204–236. Springer-Verlag, 1968.

[Tar83] Alfred Tarski. The concept of truth in formalized languages. In *Logic, Semantics and Metamathematics (2nd edition)*, pages 152–278. Hackett Publishing Company, 1983. First published in Poland in 1933.

Fernando Ferreira
Departamento de Matemática
Faculdade de Ciências
Universidade de Lisboa
Rua Ernesto de Vasconcelos, Bloco C1, 3
1700 Lisboa, PORTUGAL
e-mail: ferferr@ptmat.lmc.fc.ul.pt