Two General Results on Intuitionistic Bounded Theories

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Abstract. We study, within the framework of intuitionistic logic, two well-known general results of (classical logic) bounded arithmetic. Firstly, Parikh's theorem on the existence of bounding terms for the provably total functions. Secondly, the result which states that adding the scheme of bounded collection to (suitable) bounded theories does not yield new Π_2 consequences.

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1 Introduction

The theory $I\Delta_0$ is that subsystem of Peano Arithmetic in which the scheme of induction is restricted to bounded formulas. These are the formulas generated from atomic formulas via the successive applications of Boolean connectives and bounded quantifications, i.e., quantifications of the form $\forall y \leq t(\overline{x})\,(\ldots)$ or of the form $\exists y \leq t(\overline{x})\,(\ldots)$, where $t(\overline{x})$ is any term of the language in which the variable y does not occur (we are abbreviating a sequence of variables $x_1, \ldots, x_k$ by $\overline{x}$). In 1971, Rohit Parikh proved in [6] the following theorem:

Theorem. *If $I\Delta_0 \vdash \forall \overline{x} \exists y A(\overline{x}, y)$, where A is a bounded formula, then there is a term $t(\overline{x})$ such that $I\Delta_0 \vdash \forall \overline{x} \exists y \leq t(\overline{x})\, A(\overline{x}, y)$.*

The conclusion is false if A is not restricted to bounded formulas. For instance, $I\Delta_0$ proves the sentence $\forall x \exists y \forall z (Exp(x, z) \rightarrow y = z)$, where $Exp(x, z)$ is a bounded formula that suitable formalizes the relation $2^x = z$. Observe that there can be no bounding term for y. However, *classical* reasoning (in the form of a use of the excluded middle) seems to be needed to argue for the above sentence. As a matter of fact, it follows from Theorem I below that the above sentence is not provable intuitionistically in $I\Delta_0$ (nor even in $I\Delta_0$ supplemented with the law of excluded middle for bounded formulas).

In 1980, Jeff Paris in [7] showed that the theory $I\Delta_0$ *plus* the collection scheme $B\Sigma_0$ for bounded formulas is a Π_2-conservative extension of $I\Delta_0$. In other words:

Theorem. *If $I\Delta_0 + B\Sigma_0 \vdash \forall \overline{x} \exists y A(\overline{x}, y)$, where A is a bounded formula, then $I\Delta_0 \vdash \forall \overline{x} \exists y A(\overline{x}, y)$.*

Here, $B\Sigma_0$ is the scheme

$$\forall u \leq x \exists y A(u, y) \rightarrow \exists w \forall u \leq x \exists y \leq w A(u, y)$$

where A is any bounded formula that may contain additional free variables as parameters.

A compactness argument yields a very simple proof of the earlier theorem. This argument readily generalizes to other bounded theories of arithmetic, such as those introduced by Samuel

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Buss in his doctoral dissertation [2]. The original proof of the latter theorem does not extend to more general situations, although the theorem still holds for a wide class of bounded theories of arithmetic. This was stated and shown by Samuel Buss in [3].

In this paper, we envisage studying the above two theorems within the intuitionistic setting. We will place ourselves in a rather general framework, since our results (as well as the classical results) need not be framed in theories of arithmetic. Results such as the above two theorems – and their intuitionistic versions – depend solely on some general features of the binary relation $\leq$ and of the provable structure of the terms (however, see the final section).

We shall be concerned with intuitionistic theories formulated in first-order recursively presented languages (this condition is here for convenience only, and can be discarded – we will comment on this in the final section) which include a distinguished binary relation symbol $\lhd$. A quantification is *bounded* if it is of the form $\forall x(x \lhd t \to \ldots)$ or of the form $\exists x(x \lhd t \wedge \ldots)$, where t is a term of the language in which x does not occur. These quantifications are abbreviated by $\forall x \lhd t\,(\ldots)$ and $\exists x \lhd t\,(\ldots)$, respectively. A formula is called $\exists^{\Omega}$*-free* if all its existential quantifications are bounded. Note that bounded formulas are $\exists^{\Omega}$-free.

The following definition is an adaptation to our setting of a definition of Samuel Buss in [3]:

Definition. *Let Γ be a theory in a first-order language as above. We say that Γ is a* term-sufficient theory *if there is a binary term maj, a unary term $trans$, and for each n-ary functional symbol f there is a n-ary term σ_f, such that:*

(1) $\Gamma \vdash_i \forall x\,(x \lhd x)$;

(2) $\Gamma \vdash_i \forall x\,\forall y\,(x \lhd maj(x,y) \wedge y \lhd maj(x,y))$;

(3) $\Gamma \vdash_i \forall x\,\forall y \lhd x\,\forall z \lhd y\,(z \lhd trans(x))$;

(4) for any function symbol $f(\overline{x})$ of the language, $\Gamma \vdash_i \forall \overline{x}\,\forall \overline{y} \lhd \overline{x}\,(f(\overline{y}) \lhd \sigma_f(\overline{x}))$,

where $\vdash_i$ denotes intuitionistic provability. For convenience, we also demand that there is at least a constant c in the language. We say that Γ is a $\exists^{\Omega}$-free term-sufficient theory *if it is a term sufficient theory and if it is recursively axiomatized by a set of $\exists^{\Omega}$-free formulas.*

We are using some obvious abbreviations; for instance, $\forall \overline{y} \lhd \overline{x}$ abbreviates $\forall y_1 \lhd x_1 \ldots \forall y_k \lhd x_k$. Condition (3) is complicated because we are striving for generality. In many cases $\lhd$ will be a transitive relation and (3) comes automatically by taking $trans(x) := x$. Also, in many theories of bounded arithmetic, the function symbols of the language are (provably) monotonous and, thus, σ_f can be taken to be f itself. This is the case with the function symbols of the theories $I\Delta_0$, S_2^i, T_2^i, etc.

In our discussion, we will need two simple facts concerning term-sufficient theories. Their proofs are simple and we omit them:

Lemma. *In a term-sufficient theory Γ, for each positive natural number k there is an k-ary term $maj_k(x_1,\ldots,x_k)$ of the language such that:*

$$\Gamma \vdash_i \forall x_1 \ldots \forall x_k \forall y_1 \lhd x_1 \ldots \forall y_k \lhd x_k \bigwedge_{i=1}^{k} (y_i \lhd maj_k(x_1,\ldots,x_k)).$$

Lemma. *Let Γ be a term-sufficient theory. For each term $t(\overline{x})$ of the language there is a term $\rho_t(\overline{x})$ such that,* $\Gamma \vdash_i \forall \overline{x}\,\forall \overline{y} \lhd \overline{x}\,\forall w \lhd t(\overline{y})\,(w \lhd \rho_t(\overline{x}))$.

The main aim of this paper is to prove the following two theorems:

Theorem I. *If Γ is a $\exists^\Omega$-free term-sufficient theory and if $\Gamma \vdash_i \forall\overline{x}\exists y A(\overline{x}, y)$, where A is* any *(first-order) formula, then there is a term $t(\overline{x})$ of the language such that $\Gamma \vdash_i \forall\overline{x}\exists y \lhd t(\overline{x})\, A(\overline{x}, y)$.*

The scheme of $\exists^\Omega$*-free collection*, denoted by $B\overline{\exists}^\Omega$, consists of all formulas

$$\forall u \lhd x \exists y\, A(u, y) \to \exists w \forall u \lhd x \exists y \lhd w\, A(u, y)$$

where A is a $\exists^\Omega$-free formula, possibly with parameters.

Theorem II. *If Γ is a $\exists^\Omega$-free term-sufficient theory and if $\Gamma + B\overline{\exists}^\Omega \vdash_i \forall\overline{x}\exists y A(\overline{x}, y)$, where A is $\exists^\Omega$-free, then $\Gamma \vdash_i \forall\overline{x}\exists y\, A(\overline{x}, y)$.*

The strategy for proving these theorems follows a technique introduced in [5]. This technique hinges on realizability arguments that take place within theories whose languages permit countable infinite disjunctions. For the first theorem, the theory is the same as the original theory. For the second theorem, we need an extra infinitary axiom schema. In the next section, we briefly describe these infinitary theories and mention the pertinent results concerning them. The third section introduces infinitary realizability notions convenient to proving the above theorems. The two theorems themselves are consequences of soundness theorems pertaining to the realizability notions.

2 The Technical Framework

In the next section, we shall introduce realizability notions framed in a language which permits countable infinite disjunctions. We deal with these countable infinite disjunctions semantically, by adding an extra forcing clause in the Kripkean semantics ([9] is a good reference for the intuitionistic notions used in this paper). Given a Kripke model $\mathcal{K}$ of a term-sufficient theory Γ and given a node α of $\mathcal{K}$ we let,

$$\Vdash_\alpha \bigvee_n F_n \quad := \quad \exists m \in \omega \ \Vdash_\alpha F_m$$

Now it makes sense to say that Γ forces a sentence A of the extended infinitary language. It means that the least node (hence, every node) of every Kripke model of Γ forces the sentence A, and we write $\Gamma \Vdash A$. With a view of proving Theorem II, we need to consider a special class Ω_Γ of Kripke models of Γ. This class is, by definition, the class of Kripke models of Γ which force the following infinitary sentences (henceforth called the *infinitary axioms*):

$$\forall x \left(A(x) \to \bigvee_n F_n(x) \right) \to \bigvee_n \forall x \left(A(x) \to \bigvee_{k \le n} F_k(x) \right)$$

where $A(x)$ is a first-order formula and $F_0(x), F_1(x), F_2(x), \ldots$ is a recursive enumeration of first-order formulas with only a finite number of parameters. Given a (first-order or infinitary) sentence A, we write $\Gamma_\infty \Vdash A$ if every Kripke structure of Ω_Γ forces A.

The following proposition is crucial:

Proposition (Conservativity). *Let Γ be a term-sufficient theory of arithmetic and suppose that*

$$\Gamma_\infty \Vdash \forall\overline{x} \bigvee_t \exists y \lhd t(\overline{x})\, A(\overline{x}, y) \qquad (\star)$$

where $A(\overline{x}, y)$ is a first-order formula with its free variables as shown and t ranges over all the appropriate terms. Then there is a term t of the language such that $\Gamma \vdash_i \forall\overline{x}\, \exists y \lhd t(\overline{x})\, A(\overline{x}, y)$.

In assertion $(\star)$, if Γ were instead of Γ_∞, a single compactness argument would suffice to prove the proposition. In the general case, we need a compactness argument on top of multiple compactness arguments (in fact, on top of a recursive saturation argument). The framework for making these arguments consists in looking (sic) "at Kripke models from outside, as a complicated concoction of classical structures, and hence as a classical structure itself" (Dirk van Dalen, page 264 of [8]). We will not describe the procedure of associating to each Kripke structure $\mathcal{K}$ its corresponding classical structure $\mathcal{K}^c$. We refer the reader to the above mentioned work of van Dalen. For the present purposes, it is sufficient to say that the language $\mathcal{L}^c$ supporting $\mathcal{K}^c$ is a modification of the original language $\mathcal{L}$ tailored to describe the structure of the original Kripke structure $\mathcal{K}$ and, conversely, that we restrict ourselves to classical structures $\mathcal{M}$ which validate a number of (first-order) laws so conceived as to permit the extraction from $\mathcal{M}$ of a natural Kripke structure $\mathcal{M}^i$. In actual fact, we have $(\mathcal{M}^i)^c \approx \mathcal{M}$ and $(\mathcal{K}^c)^i \approx \mathcal{K}$.

By making a detour through the classical structures as described above, we can mimic the recursive saturation argument of the proof of the main lemma of section 3 of [5] and obtain,

Lemma. *Suppose that $\mathcal{K}$ is a (countable) Kripke model of Γ for which the associated classical structure $\mathcal{K}^c$ is recursively saturated. Then $\mathcal{K} \in \Omega_\Gamma$.*

We are now ready to prove the conservativity result.

Proof of the Proposition : Suppose that the conclusion is false. Let $t(x)$ be a term of the language (to simplify notation, we work with a single variable x). Then, $\Gamma \nvdash_i \forall x \exists y \lhd t(x)\, A(x, y)$. By the Kripke completeness theorem, there is a (rooted) Kripke model $\mathcal{K}^t$ such that,

$$\nVdash_0 \forall x \exists y \lhd t(x)\, A(x, y)$$

where 0 is the root of $\mathcal{K}^t$. This means that there is a node α of $\mathcal{K}^t$ and an element d in the domain $D(\alpha)$ of the node α such that

$$\forall d' \in D(\alpha)\, (\Vdash_\alpha d' \lhd t(d) \;\rightarrow \nVdash_\alpha A(d, d'))$$

Consider the theory Σ consisting of Γ *plus* the following sentences (one for each appropriate term t):

$$k \in D(\kappa) \wedge \forall y \in D(\kappa)\, (\Vdash_\kappa y \lhd t(k) \;\rightarrow \nVdash_\kappa A(k, y))$$

where k and κ are new constant symbols. (We are abusing language in the above definition of Σ: in rigour, the sentences of Σ should have been stated in the language $\mathcal{L}^c$ discussed earlier.) Using the first lemma after the definition of a term-sufficient theory, it is easy to argue that every finite subset of Σ has a (classical) model. Hence, by compactness, Σ has a countable (classical) model $\mathcal{M}$. Let $\mathcal{M}_\infty$ be a recursively saturated structure elementarily equivalent to $\mathcal{M}$. From $\mathcal{M}_\infty$ we can read-off a Kripke structure $\mathcal{K}_\infty = \mathcal{M}^i_\infty$. By the previous lemma, this structure is in the class Ω_Γ. Thus, by hypothesis, the following (infinitary) sentence is forced in $\mathcal{K}_\infty$:

$$\forall x \bigvee_t \exists y \lhd t(x)\, A(x, y)$$

However, by construction, $\mathcal{K}_\infty$ has a node α and an element $d \in D(\alpha)$ such that, for every term t:

$$\forall d' \in D(\alpha)\, (\Vdash_\alpha d' \lhd t(d) \;\rightarrow \nVdash_\alpha A(d, d'))$$

This yields a contradiction. □

3 Notions of Infinitary Realizabiliy

With the help of the conservativity result of the previous section, Theorems I and II are corollaries of soundness theorems for suitable notions of realizability. In the following definition, we introduce the two notions of realizibility that we need.

Definition. *Let $\mathcal{L}$ be the (first-order) language of a term-sufficient theory Γ. To each (first-order) formula A of $\mathcal{L}$ we associate a new formula $z\mathbf{r}A$ of the extended infinitary language according to the following clauses (z is a new variable):*

1. *$z\mathbf{r}A$ is A, if A is an atomic formula;*
2. *$z\mathbf{r}(A \wedge B)$ is $\exists z_0 \lhd z\, \exists z_1 \lhd z\, (z_0\mathbf{r}A \wedge z_1\mathbf{r}B)$;*
3. *$z\mathbf{r}(A \vee B)$ is $\exists w \lhd z\, (w\mathbf{r}A \vee w\mathbf{r}B)$;*
4. *$z\mathbf{r}(A \to B)$ is $\forall x(x\mathbf{r}A \to \bigvee_t \exists w \lhd t(z,x)\, w\mathbf{r}B)$;*
5. *$z\mathbf{r}\forall x A(x)$ is $\forall x \bigvee_t \exists w \lhd t(z,x)\, w\mathbf{r}A(x)$;*
6. *$z\mathbf{r}\exists x A(x)$ is $\exists x_0 \lhd z\, \exists x_1 \lhd z\, (x_0\mathbf{r}A(x_1))$;*

where t ranges over appropriate terms. The notion of **q***-realizability has the same clauses as* **r***-realizability, except for an extra requirement in the case of implication:*

4*. *$z\mathbf{q}(A \to B)$ is $(A \to B) \wedge \forall x(x\mathbf{q}A \to \bigvee_t \exists w \lhd t(z,x)\, w\mathbf{q}B)$.*

In the above definition we omitted talk of parameters. However, it should be clear from the context what parameters are permitted in each term. For instance, in clause 5, if $\overline{u}$ are the parameters of the matrix A then t ranges over all terms of the form $t(z,x,\overline{u})$, where all the variables are as displayed.

The realizability clauses for bounded quantifications simplify somewhat:

Lemma. *In a term-sufficient theory Γ the following holds (intuitionistically):*

5'. *$z\mathbf{r}\ \forall x \lhd s(\overline{u})\, A(x,\overline{u}) \ \longleftrightarrow\ \forall x \lhd s(\overline{u}) \bigvee_t \exists w \lhd t(z,\overline{u})\, w\mathbf{r}A(x,\overline{u})$;*

6'. *$z\mathbf{r}\ \exists x \lhd s(\overline{u})\, A(x,\overline{u}) \ \longleftrightarrow\ \exists x_0 \lhd z\, \exists x_0' \lhd x_0\, \exists x_1 \lhd z\, (x_1 \lhd t(\overline{u}) \wedge x_0'\mathbf{r}A(\overline{u},x_1))$.*

Proof : We will prove the left-to-right implication of 5'. The other three implications are easy. Suppose $z\mathbf{r}\forall x(x \lhd s(\overline{u}) \to A(x,\overline{u}))$. Take x with $x \lhd s(\overline{u})$. According to the definition of **r**-realizability, there is a term q and an element $w' \lhd q(z,x,\overline{u})$ such that

$$\forall v(v\mathbf{r}(x \lhd s(\overline{u})) \ \to\ \bigvee_r \exists w \lhd r(w',v,x,\overline{u})\ w\mathbf{r}A(x,\overline{u})).$$

Taking $v := c$, we get a term r' and an element $w \lhd r'(w',x,\overline{u})$ such that $w\mathbf{r}A(x,\overline{u})$. It is not difficult to check that the term $t(z,\overline{u}) := \rho_{r'}(\rho_q(z,s(\overline{u}),\overline{u}),s(\overline{u}),\overline{u})$ does the job.

□

The folowing proposition is handy:

Proposition. *Let Γ be a term-sufficient theory. For each $\exists^\Omega$-free formula $A(\overline{x})$ there is a term $t_A(\overline{x})$ such that,*

(i) $\Gamma \vdash_i \exists y\, (y\mathbf{r}A(\overline{x})) \ \to\ A(\overline{x})$;

(ii) $\Gamma \vdash_i A(\overline{x}) \ \to\ \exists y \lhd t_A(\overline{x})\, y\mathbf{r}A(\overline{x})$.

The same holds for the notion of **q***-realizability.*

Proof : t_A is constructed by induction on the complexity of A:

(a) $t_A(\overline{x}) := c$, if A is an atomic formula;

(b) $t_{A\wedge B}(\overline{x}) := maj_2(t_A(\overline{x}), t_B(\overline{x}))$;

(c) $t_{A\vee B}(\overline{x}) := maj_2(t_A(\overline{x}), t_B(\overline{x}))$;

(d) $t_{A\to B}(\overline{x}) := c$;

(e) $t_{\forall u A(u)}(\overline{x}) := c$;

(f) $t_{\exists u \lhd s\, A(u)}(\overline{x}) := maj_2(\rho_{t_A}(s(\overline{x}), \overline{x}), s(\overline{x}))$.

We establish (i) and (ii) simultaneously, by induction on the complexity of A. We will only check cases (d) and (f). The statements (i) corresponding to these cases are straightforward. Let us argue for (ii) of case (d). Suppose that $A(\overline{x}) \to B(\overline{x})$ and assume that $z\mathbf{r}A(\overline{x})$. By induction hypothesis and by Modus Ponens, we get $B(\overline{x})$. Again by induction hypothesis, we conclude that $\exists w \lhd t_B(\overline{x})\, w\mathbf{r}B(\overline{x})$. Thus, we have argued that

$$z\mathbf{r}A(\overline{x}) \;\to\; \bigvee_t \exists w \lhd t(\overline{x})\; w\mathbf{r}B(\overline{x}).$$

This shows that $c\mathbf{r}(A(\overline{x}) \to B(\overline{x}))$. Let us now check (ii) of case (f). Suppose that there is $u \lhd s(\overline{x})$ such that $A(u, \overline{x})$. By induction hypothesis, there is $y \lhd t_A(u, \overline{x})$ such that $y\mathbf{r}A(u, \overline{x})$. Clearly, both u and y are in the relation $\lhd$ with $maj_2(\rho_{t_A}(s(\overline{x}), \overline{x}), s(\overline{x}))$. Thus, our conclusion follows.

Similar (and sometimes more immediate) arguments work for the notion of **q**-realizability.

□

The notion of **q**-realizability is, as in usual realizability notions, **r**-realizability *plus* truth. The following mimics a standard result of the literature:

Lemma. *Let Γ be a term-sufficient theory. If z does not occur in the formula A then $\Gamma \Vdash (z\mathbf{q}A) \to A$.*

We have the following soundness theorem.

Soundness Theorem I. *If Γ is a $\exists^\Omega$-free term-sufficient theory and if $\Gamma \vdash_i A(\overline{x})$, where $A(\overline{x})$ is a (first-order) formula, then*

$$\Gamma \Vdash \bigvee_t \exists w \lhd t(\overline{x})\, w\mathbf{r}A(\overline{x})$$

where t ranges over all terms whose variables are among the free variables $\overline{x}$ of $A(\overline{x})$. The same is true for the notion of **q***-realizability.*

Proof : The proof is by induction on the lengths of derivations in a Hilbert-type deduction system. For determinateness, we work with the deductive system described in page 68 of [9]. By the above proposition, the mathematical axioms of Γ pose no trouble since, by hypothesis, they are $\exists^\Omega$-free. The equality axioms pose no trouble either. It would be tedious to go throught all the thirteen logical axioms of our deduction system. All these axioms are – in fact – realized by any element (by c, in particular), and we will see this for three typical axioms (to make the discussion more clear, we will usually avoid the consideration of parameters).

The logical axiom $A \to (B \to A)$ is **r**-realized by c if, and only if,

$$\forall z \left(z\mathbf{r}A \;\to\; \bigvee_t \exists z_0 \lhd t(z)\, \forall w \;\left(w\mathbf{r}B \;\to\; \bigvee_q \exists w_0 \lhd q(z_0, w)\; w_0\mathbf{r}B) \right) \right).$$

Given z with $z\mathbf{r}A$, take $t(z) = z$ and let $z_0 = z$. Then, given w with $w\mathbf{r}B$, take $q(z_0, w) = z_0$ and let $w_0 = q(z_0, w)$.

Now let us consider the axiom $(A \to C) \to ((B \to C) \to (A \vee B \to C))$. We must see that $z\mathbf{r}(A \to C)$ implies

$$\bigvee_r \exists z' \lhd r(z)\, \forall w \left(w\mathbf{r}(B \to C) \;\to \bigvee_s \exists z'' \lhd s(z', w)\, \forall y \left(y\mathbf{r}(A \vee B) \;\to\; \bigvee_p \exists z''' \lhd p(z'', y)\; z'''\mathbf{r}c \right)\right).$$

Suppose that $z\mathbf{r}(A \to C)$, $w\mathbf{r}(B \to C)$ and $y\mathbf{r}(A \vee B)$. Take $y_0 \lhd y$ such that $y_0\mathbf{r}A \vee y_0\mathbf{r}B$. Suppose that the first case holds: $y_0\mathbf{r}A$. Then there is a term t and there is $z_0 \lhd t(z, y_0)$ such that $z_0\mathbf{r}C$. On the other hand, if $y_0\mathbf{r}B$ there is a term q and there is $w_0 \lhd q(w, y_0)$ such that $w_0\mathbf{r}C$. It is now a matter of simple checking to see that we can take $r(z) = z$, $z' = z$, $s(z', w) = maj_2(z', w)$, $z'' = s(z', w)$ and $p(z'', y) = maj_2(\rho_t(z'', y), \rho_q(z'', y))$.

The third axiom that we consider is $\forall x(A(x) \to B) \to (\exists y A(y) \to B)$, where x is not free in B. We must see that,

$$z\mathbf{r}\forall x(A(x) \to B) \;\to \bigvee_r \exists z' \lhd r(z)\, \forall w \left(w\mathbf{r}\exists y A(y) \to \bigvee_p \exists w' \lhd p(z', w)\, w'\mathbf{r}B \right).$$

Suppose that $z\mathbf{r}\forall x(A(x) \to B)$ and $w\mathbf{r}\exists y A(y)$. Then, for some $w_0 \lhd w$ and for some $w_1 \lhd w$, we have $w_0\mathbf{r}A(w_1)$. Hence, there is a term $t(z, w_1)$ and an element $z_0 \lhd t(z, w_1)$ such that $z_0\mathbf{r}(A(w_1) \to B)$. Thus, there exists a term s and an element $y_0 \lhd s(z_0, w_0)$ such that $y_0\mathbf{r}B$. It is not difficult to check that we can take $r(z) = z$, $z' = z$ and $p(z', w) = \rho_s(\rho_t(z', w), w)$.

Let us now consider the two rules of our deduction system: *Modus Ponens* and Generalization. Suppose that both,

$$\Gamma \Vdash \bigvee_t \exists w \lhd t(x)\; w\mathbf{r}A(x)$$

and

$$\Gamma \Vdash \bigvee_r \exists y \lhd r(x)\, \forall z \left(z\mathbf{r}A(x) \to \bigvee_q \exists z_0 \lhd q(z, y, x)\; z_0\mathbf{r}B(x) \right).$$

We reason (intuitionistically) inside Γ. Take t and $w \lhd t(x)$ with $w\mathbf{r}A(x)$. We know that there is a term r and an element $y \lhd r(x)$ such that for some term q and element $z_0 \lhd q(w, y, x)$ we have $z_0\mathbf{r}B(x)$. Let $s(x) := \rho_q(t(x), r(x), x)$. It is clear that $\exists z_0 \lhd s(x)\; z_0\mathbf{r}B(x)$.

Finally, we consider the rule of Generalization:

$$\frac{A(x, \overline{u})}{\forall x A(x, \overline{u})}$$

By induction hypothesis,

$$\Gamma \Vdash \bigvee_t \exists w \lhd t(x, \overline{u})\; w\mathbf{r}A(x, \overline{u}).$$

Inside Γ, we must see that,

$$\bigvee_q \exists y \lhd q(\overline{u})\, \forall x \bigvee_s \exists z \lhd s(y, x)\; z\mathbf{r}A(x, \overline{u}).$$

Just take $q(\overline{u}) = maj_k(u_1, \ldots, u_k)$, $y = q(\overline{u})$ and $s(y, x) = \rho_t(x, y, \ldots y)$ (k many y's), where $u_1, \ldots, u_k$ is the sequence of variables $\overline{u}$.

Although we gave an argument for the result pertaining to the notion of **r**-realizability, it should be clear that a similar argument yields the result for the notion of **q**-realizability.

$\square$

We are now ready to prove Theorem I:

Proof of Theorem I: Suppose that Γ is a $\exists^{\Omega}$-free term-sufficient theory and that $\Gamma \vdash_i \exists y A(\overline{x}, y)$, where A is a (first-order) formula. By the above soundness theorem for the notion of **q**-realizability,

$$\Gamma \Vdash \bigvee_t \exists w \lhd t(\overline{x})\, \exists w_0 \lhd w\, \exists y \lhd w\ w_0 \mathbf{q} A(\overline{x}, y).$$

Since **q**-realizability implies truth, we get:

$$\Gamma \Vdash \bigvee_t \exists y \lhd trans(t(\overline{x}))\ A(\overline{x}, y).$$

The conclusion of the theorem follows from the conservativity result. □

Soundness Theorem II. *If Γ is a $\exists^{\Omega}$-free term-sufficient theory and if $\Gamma + B\overline{\exists}^{\Omega} \vdash_i A(\overline{x})$, where $A(\overline{x})$ is a first-order formula, then*

$$\Gamma_{\infty} \Vdash \bigvee_t \exists w \lhd t(\overline{x})\ w \mathbf{r} A(\overline{x})$$

where t ranges over all terms whose variables are among the free variables $\overline{x}$ of $A(\overline{x})$.

Proof : We just have to supplement the argument given in proving Soundness Theorem I with the proof that each instance of the $B\overline{\exists}^{\Omega}$-scheme is **r**-realized by c within Γ_{∞}. Let

$$\forall u \lhd x\, \exists y A(u, y) \ \rightarrow\ \exists w \forall u \lhd x\, \exists y \lhd w\, A(u, y)$$

where A is a $\exists^{\Omega}$-free formula, be an instance of the $B\overline{\exists}^{\Omega}$-scheme. We must see that

$$\Gamma_{\infty} \Vdash \forall z \left(z \mathbf{r} \forall u \lhd x\, \exists y A(u, y) \ \rightarrow \bigvee_t \exists v \lhd t(z, x)\ v \mathbf{r} \exists w \forall u \lhd x\, \exists y \lhd w\, A(u, y) \right).$$

We reason (intuitionistically) inside Γ_{∞}. Suppose that $z \mathbf{r} \forall u \lhd x\, \exists y A(u, y)$. Thus,

$$\forall u \lhd x \bigvee_q \exists y \lhd q(z, x)\, \exists y_0 \lhd y\, \exists y_1 \lhd y\ y_0 \mathbf{r} A(u, y_1).$$

Since A is an $\exists^{\Omega}$-free formula, the (possibly infinitary) formula "$y_0 \mathbf{r} A(u, y_1)$" can be replaced in the above by the first-order formula "$A(u, y_1)$". Using the infinitary axioms (and the first lemma after the definition of a term-sufficient theory), we may then conclude that there exists a term q such that,

$$\forall u \lhd x\, \exists y \lhd q(z, x)\, \exists y_1 \lhd y\, A(u, y_1).$$

Using, again, the fact that A is a $\exists^{\Omega}$-free formula, we get,

$$\forall u \lhd x\, \exists y \lhd t(z, x)\, \exists y_0 \lhd y\, \exists y_1 \lhd y\, y_0 \mathbf{r} A(u, y_1),$$

where $t(z, x)$ is the term $maj_2(q(z, x), \rho_{t_A}(x, trans(q(z, x))))$. A small amount of detailed checking shows that this implies:

$$t(z, x) \mathbf{r} \exists w \forall u \lhd x\, \exists y \lhd w\ A(u, y).$$

This yields our result. □

Proof of Theorem II: Suppose that Γ is a $\exists^{\Omega}$-free term-sufficient theory and that $\Gamma + B\overline{\exists}^{\Omega} \vdash_i \exists y A(\overline{x}, y)$, where A is a $\exists^{\Omega}$-free formula. By the above soundness theorem,

$$\Gamma_{\infty} \Vdash \bigvee_{t} \exists w \lhd t(\overline{x})\, \exists w_0 \lhd w\, \exists y \lhd w\ w_0 \mathbf{r} A(\overline{x}, y).$$

Since A is a $\exists^{\Omega}$-free formula, $\mathbf{r}$-realizability implies truth. Thus,

$$\Gamma_{\infty} \Vdash \bigvee_{t} \exists y \lhd trans(t(\overline{x}))\ A(\overline{x}, y).$$

The conclusion of the theorem follows from the conservativity result. □

4 Remarks and Questions

The restriction of the above theorems to recursively presented languages and to recursively axiomatized theories is not necessary. The same arguments hold for any countable language by relativizing everything to the Turing degree of the theory. Even the restriction to countable languages is not necessary because the arguments can be localized to the pertinent countable fragment of the language.

The proofs of this paper are infinitary in nature since they use the semantic apparatus of Kripke models *cum* languages which permit infinitary disjunctions. However, if we weaken the statement of Theorem I to *bounded* term-sufficient theories and to *bounded* matrices A, then the proof-theoretic argument in [2] at the end of chapter 4 readily adapts to the intuitionistic case. The question is: can we also give a finitistic argument for the general case?

A similar sort of question can be asked for Theorem II. Moreover, Ferreira in [4] presents a very simple model-theoretic proof of the classical version of Theorem II for theories in which the structure of terms does not play a prominent role (contrary to Theorem I, the statement of Theorem II does *not* mention terms anywhere). Ferreira replaces the second and third clauses of the definition of a term-sufficient theory by:

(2') $\Gamma \vdash \forall x \forall y \exists w\, (x \lhd w \wedge y \lhd w)$;

(3') $\Gamma \vdash \forall x \exists w \forall y \lhd x \forall z \lhd y\, (z \lhd w)$,

and also permits axioms of the form "$\forall \overline{x} \exists \overline{w} \forall \overline{u} \lhd \overline{x} \exists \overline{z} \lhd \overline{w}\ A(\overline{u}, \overline{z})$" where $A(\overline{u}, \overline{z})$ comes from the bounded formula $A(\overline{x}, \overline{w})$ by substituting the variables $\overline{x}$ and $\overline{w}$ by the new variables $\overline{u}$ and $\overline{z}$ (respectively). In this generality, is an intuitionistic counterpart of the theorem in [4] also true? If so, can a finitistic proof be provided?

Recently and independently, Wolfgang Burr produced some work related to the issues of this paper using (an adaptation of) Gödel's functional interpretation. As of now, it is not clear to us how Burr's work compares logically to our work. That notwithstanding, we want to draw the attention of the reader to the last section of [1].

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