Injecting uniformities into Peano arithmetic

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Abstract

We present a functional interpretation of Peano arithmetic that uses Gödel's computable functionals and which systematically injects uniformities into the statements of finite-type arithmetic. As a consequence, some uniform boundedness principles (not necessarily set-theoretically true) are interpreted while maintaining unmoved the Π^0_2-sentences of arithmetic. We explain why this interpretation is taylored to yield conservation results.

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1 Introduction

In 1958 [1], Kurt Gödel introduced an interpretation of Peano arithmetic into a quantifier-free theory of finite-type functionals. Gödel's interpretation consists of two steps. First, Peano arithmetic is interpreted into Heyting arithmetic by a negative translation. Afterwards, Heyting arithmetic is interpreted into the quantifier-free theory via what is now known as Gödel's (functional) *dialectica* interpretation. Almost ten years later, Joseph Shoenfield defined in his well-known textbook [2] a direct functional interpretation of Peano arithmetic. Shoenfield's interpretation and its variants are specially perspicuous for an undeviating study of classical theories. A case in point is the work of functional interpretations of admissible set theories (see [3]).

Both interpretations of Gödel and Shoenfield are based on a transformation of formulas whose analysis of $\forall\exists$-formulas is given in terms of witnessing functionals. (As an aside, recent work of Thomas Streicher and Ulrich Kohlenbach in

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[4] shows that Shoenfield's interpretation can be "factored" into Krivine's negative translation [5] and the *dialectica* interpretation.) By maintaining Gödel's functionals but relaxing their witnessing role to that of a mere bound (and, in the process, introducing some uniformities), a new functional interpretation, with a novel assigment of formulas – dubbed the bounded functional interpretation (with hindsight, it should have been called *uniform functional interpretation*) – was recently introduced by Paulo Oliva and the present author in [6]. In common with Gödel's, this interpretation is also two-barreled. In this paper we introduce a direct bounded functional interpretation of Peano arithmetic, in the style of Shoenfield.

The interpretation defined in the sequel is not set-theoretically faithful, in the sense that it introduces uniformities which collide with set-theoretic truth. For instance, the axiom of extensionality is refuted. Due to its simplicity, the interpretation shows very distinctly what are its characteristic principles, i.e. the principles that we can add to Peano arithmetic and still obtain a soundness theorem. One of these characteristic principles subsumes the so-called uniform boundedness principle introduced by Kohlenbach in [7] (and discussed interestingly in [8] in a wider setting), as well as a Brouwerian FAN type principle. The role of this characteristic principle is to *inject uniformities*, and a simple (but somewhat superficial) way of describing it is to say that it is a vast higher-order generalization of the bounded collection scheme in the first-order arithmetic setting.

Classically, weak König's lemma (WKL) is a consequence of the Brouwerian FAN principle alluded to above. As an illustration, we give a very straight proof of Harvey Friedman's conservation result of the second-order arithmetical theory WKL_0 over the base theory RCA_0.

2 Majorizability unvarnished

In the bounded functional interpretation, it is necessary to introduce a notion of *intensional*, i.e. rule-governed, majorizability. Except for the intensional bit, the notion of majorizability in question is Mark Bezem's notion of strong majorizability given in [9], a modification of the majorizability notion introduced by William Howard in [10]. The fact that this notion needs to be governed by a rule, instead of mere axioms, is crucial in proving the soundness theorem for the new interpretation. As we will see, without this feature our main theory below would be inconsistent. In a sense, the rule deactivates the computational capacity of the majorizability relation with respect to the functional interpretation.

The language $\mathcal{L}^{\omega}_{\trianglelefteq}$ is described in detail in sections 2 and 6 of [6]. However, since we are in a classical setting, we restrict $\mathcal{L}^{\omega}_{\trianglelefteq}$ to the logical words $\vee$, $\forall$, $\neg$ and the bounded quantifier $\forall x \trianglelefteq t$ (x does not appear in the term t). The other logical words are defined classically in the usual manner. *Mutatis mutandis* for the existential bounded quantifier. Note the presence of the (intensional) majorizability binary relation symbols $\trianglelefteq_{\tau}$ – one for each finite type τ (we usually omit the type-subscript) – in the bounded quantifiers. The majorizability relation symbols are governed by the axioms

$$\mathsf{M}_1 \ : \ x \trianglelefteq_0 y \leftrightarrow x \leq y$$
$$\mathsf{M}_2 \ : \ x \trianglelefteq_{\rho\to\sigma} y \to \forall u \trianglelefteq_{\rho} v(xu \trianglelefteq_{\sigma} yv \wedge yu \trianglelefteq_{\sigma} yv)$$

Note that we do *not* have the biconditional above (that would give Bezem's extensional notion). In its stead, we have the *rule* $\mathsf{RL}_{\trianglelefteq}$

$$\frac{A_{\mathrm{bd}} \wedge u \trianglelefteq v \to su \trianglelefteq tv \wedge tu \trianglelefteq tv}{A_{\mathrm{bd}} \to s \trianglelefteq t}$$

where s and t are terms of $\mathcal{L}^{\omega}_{\trianglelefteq}$, A_{bd} is a bounded formula and u and v are variables which do not occur free in the conclusion. The only quantifiers in a *bounded formula* are the bounded quantifiers, and these are regulated by the axiom scheme

$$\mathsf{B}_{\forall} \ : \ \forall x \trianglelefteq t A(x) \leftrightarrow \forall x(x \trianglelefteq t \to A(x)).$$

Concerning equality, we adopt the minimal treatment described in Anne Troelstra's commentary [11] to Gödel's seminal *dialectica* paper, whereby there is only an equality sign '$=_0$' infixing between terms of type 0. The question of equality must always be dealt with some care in functional interpretations. In point of fact, the main theory introduced in the next section *refutes* the axiom of extensionality.

Our theory has the usual arithmetical axioms, including the scheme of induction for all formulas of the language (parameters are permitted). At this point, we finish our brief presentation of the classical theory $\mathsf{PA}^{\omega}_{\trianglelefteq}$. In the sequel, we shall use some simple results provable in it, viz concerning the majorizability relations. All these results are stated and proved in [6].

A term is t is *monotone* if $t \trianglelefteq t$. A *monotone quantification* is a quantification of the form $\forall b(b \trianglelefteq b \to \ldots)$, abbreviated by $\tilde{\forall} b\,(\ldots)$. Note that monotone quantifications are *not* bounded quantifications (nor are they vacuous for nonzero types). In the following, the underlined letters are meant to represent (possibly empty) tuples of variables. The mixed use of these abbreviations is self-explanatory.

Definition 1 *To each formula A of the language $\mathcal{L}^{\omega}_{\trianglelefteq}$ we assign formulas* $(A)^{\mathrm{U}}$

and A_{U} so that $(A)^{\mathrm{U}}$ is of the form $\tilde{\forall}\underline{b}\tilde{\exists}\underline{c}A_{\mathrm{U}}(\underline{b},\underline{c})$, with $A_{\mathrm{U}}(\underline{b},\underline{c})$ a bounded formula, according to the following clauses:

1. $(A)^{\mathrm{U}}$ and A_{U} are simply A, for bounded formulas A.

For the remaining cases, if we have already interpretations for A and B given by $\tilde{\forall}\underline{b}\tilde{\exists}\underline{c}A_{\mathrm{U}}(\underline{b},\underline{c})$ and $\tilde{\forall}\underline{d}\tilde{\exists}\underline{e}B_{\mathrm{U}}(\underline{d},\underline{e})$ (respectively) then we define:

2. $(A \vee B)^{\mathrm{U}}$ is $\tilde{\forall}\underline{b},\underline{d}\tilde{\exists}\underline{c},\underline{e}(A_{\mathrm{U}}(\underline{b},\underline{c}) \vee B_{\mathrm{U}}(\underline{d},\underline{e}))$,
3. $(\forall x A(x))^{\mathrm{U}}$ is $\tilde{\forall}a\tilde{\forall}\underline{b}\tilde{\exists}\underline{c}\forall x \trianglelefteq a A_{\mathrm{U}}(\underline{b},\underline{c},x)$,
4. $(\neg A)^{\mathrm{U}}$ is $\tilde{\forall}\underline{f}\tilde{\exists}\underline{b}\,\tilde{\exists}\underline{b}' \trianglelefteq \underline{b}\neg A_{\mathrm{U}}(\underline{b}',\underline{f}\underline{b}')$,
5. $(\forall x \trianglelefteq t A(x))^{\mathrm{U}}$ is $\tilde{\forall}\underline{b}\tilde{\exists}\underline{c}\forall x \trianglelefteq t\, A_{\mathrm{U}}(\underline{b},\underline{c},x)$.

The matrix $(\neg A)_{\mathrm{U}}$ includes the bounded quantification '$\tilde{\exists}\underline{b}' \trianglelefteq \underline{b}$' in order to make the following *crucial* monotonicity condition hold true:

Lemma 1 *For each formula A of the language $\mathcal{L}^{\omega}_{\trianglelefteq}$, we have:*

$$\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash \forall\underline{b}\forall\underline{c}\forall\underline{c}' \trianglelefteq \underline{c}(A_{\mathrm{U}}(\underline{b},\underline{c}') \to A_{\mathrm{U}}(\underline{b},\underline{c})).$$

Implications $A \to B$ are defined by $\neg A \vee B$. A simple computation shows that $(A \to B)^{\mathrm{U}}$ is

$$\tilde{\forall}\underline{f},\underline{d}\tilde{\exists}\underline{b},\underline{e}(\tilde{\forall}\underline{b}' \trianglelefteq \underline{b}A_{\mathrm{U}}(\underline{b}',\underline{f}\underline{b}') \to B_{\mathrm{U}}(\underline{d},\underline{e})).$$

3 Characteristic principles

There are three principles which play an important role in the interpretation defined in the previous section. The proper formulation of the first two principles should be with tuples of variables. To ease readability, we formulate them with single variables. However, the reader should keep in mind that arguments pertaining to these principles should comprehend the tuple case. This can be achieved either by introducing product types in the language or, better still, by arguing directly (at the cost of slight complications *vis a vis* the single variable case).

I. *Monotone Bounded Choice* $\mathsf{mAC}^{\omega}_{\mathrm{bd}}$:

$$\tilde{\forall}b\tilde{\exists}cA_{\mathrm{bd}}(b,c) \to \tilde{\exists}f\tilde{\forall}b\tilde{\exists}c \trianglelefteq fb\, A_{\mathrm{bd}}(b,c),$$

where A_{bd} is a bounded formula of $\mathcal{L}^{\omega}_{\trianglelefteq}$.

II. *Bounded Collection Principle* bC^{ω}:

$$\forall z \trianglelefteq c \exists y A_{\mathrm{bd}}(y,z) \to \tilde{\exists} b \forall z \trianglelefteq c \exists y \trianglelefteq b\, A_{\mathrm{bd}}(y,z),$$

where A_{bd} is a bounded formula of $\mathcal{L}^{\omega}_{\trianglelefteq}$.

III. *Majorizability Axioms* MAJ^{ω}: $\forall x \exists y (x \trianglelefteq y)$.

It is worth making some brief comments on the second principle. Its contrapositive permits the conclusion of the existence of an element z (with $z \trianglelefteq c$) such that $\forall y \neg A_{\mathrm{bd}}(y,z)$ from the weaker statement that such z's only exist *locally*, in the sense that for each (monotone) b there exists z (with $z \trianglelefteq c$) such that $\forall y \trianglelefteq b\, \neg A_{\mathrm{bd}}(y,z)$. We may regard such a z as an *ideal* element that works *uniformly* for each b and whose postulation does not affect (as we will see) *real* consequences. (We thank Reinhard Kahle for suggesting this Hilbertian reading of the soundness theorem below).

The theory $\mathsf{PA}^{\omega}_{\trianglelefteq}$ with the above principles is not set-theoretically sound. For instance, it *refutes* the weakest form of extensionality. That is, it proves the *negation* of the sentence $\forall \Phi^2 \forall \alpha^1, \beta^1 \,(\forall k^0 (\alpha k = \beta k) \to \Phi\alpha = \Phi\beta)$. In effect, assume this form of extensionality. In particular, one has

$$\forall \Phi \trianglelefteq_2 1^2\, \forall \alpha, \beta \trianglelefteq_1 1^1 \exists k\, (\alpha k = \beta k \to \Phi\alpha = \Phi\beta),$$

where $1^1 := \lambda k^0.1^0$ and $1^2 := \lambda \gamma^1.1^1$. By bC^{ω}, one may infer

$$\exists n \forall \Phi \trianglelefteq_2 1 \forall \alpha, y \trianglelefteq_1 1\, (\forall k < n(\alpha k = \beta k) \to \Phi\alpha = \Phi\beta).$$

Take one such $n = n_0$. Define Φ according to:

$$\gamma^1 \rightsquigarrow_{\Phi} \begin{cases} 0 \text{ if } \forall k \leq n_0\, (\gamma k = 0) \\ 1 \text{ otherwise} \end{cases}$$

It is clear that for $\alpha := \lambda k.0$ and $\beta := \lambda k.\delta_{n_0,k}$ (Kronecker's delta) one has $\forall k < n_0\, (\alpha k = \beta k)$ but $\Phi\alpha \neq \Phi\beta$. Since it is easy to show that $\Phi \trianglelefteq 1^2$ and $\alpha, \beta \trianglelefteq 1^1$, we are faced with a contradiction.

Let us write $Ext(\Phi)$ for saying that the type 2 functional Φ is *extensional*, i.e. $\forall \alpha^1, \beta^1\, (\forall k^0 (\alpha k = \beta k) \to \Phi\alpha = \Phi\beta)$. Despite the classical setting, we can prove the following version of the Brouwerian FAN principle: Every extensional type 2 functional is uniformly continuous on the Cantor space (see, also, [8]). In symbols,

$$\forall \Phi^2 (Ext(\Phi) \to \exists n \forall \alpha, \beta \leq_1 1\, (\forall k \leq n(\alpha k = \beta k) \to \Phi\alpha = \Phi\beta)).$$

The argument is easy. Suppose that $Ext(\Phi)$. Then,

$\forall \alpha, \beta \trianglelefteq_1 1 \exists k(\alpha k = \beta k \to \Phi\alpha = \Phi\beta)$,

As in the previous argument, by bC^ω there is a natural number n such that $\forall \alpha, \beta \trianglelefteq_1 1(\forall k \leq n(\alpha k = \beta k) \to \Phi\alpha = \Phi\beta)$. Take now arbitrary $\alpha, \beta \leq_1 1$ and suppose that $\forall k \leq n(\alpha k = \beta k)$. Using the rule $\mathsf{RL}_\trianglelefteq$, it can be proved that $\min(\alpha, 1^1) \trianglelefteq_1 1$ and $\min(\beta, 1^1) \trianglelefteq_1 1$, where the minima are taken pointwise. Note that α and $\min(\alpha, 1)$ (respectively, β and $\min(\beta, 1)$) are type 1 functionals which take the same values for each natural number. By the above, we get $\Phi(\min(\alpha, 1)) = \Phi(\min(\beta, 1))$. Now, by the extensionality of Φ, we conclude that $\Phi\alpha = \Phi(\min(\alpha, 1))$ and $\Phi\beta = \Phi(\min(\beta, 1))$. We are done.

4 The soundness theorem

Prima facie, it is not even clear whether the theory $\mathsf{PA}^\omega_\trianglelefteq$ together with the three principles above is consistent. As we will see in a forthcoming section, the "flattened" version of $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{mAC}^\omega_{\mathrm{bd}} + \mathsf{bC}^\omega + \mathsf{MAJ}^\omega$ is *inconsistent*. Notwithstanding, the soundness theorem below does in fact guarantee the consistency of the original version (modulo Peano arithmetic). The following theorem is crucial for the proof of soundness :

Theorem 1 (Howard) *For each closed term t of $\mathcal{L}^\omega_\trianglelefteq$, there is a closed term q of the same type such that $\mathsf{PA}^\omega_\trianglelefteq \vdash t \trianglelefteq q$.*

Essentially, this theorem appeared in [10]. It was shown for Howard's "flattened" majorizability relation and a corresponding "flattened" theory (see Section 6), but his argument goes through in the intensional setting (cf. [6]). Note that the result only holds for *closed* terms (the theory $\mathsf{PA}^\omega_\trianglelefteq$ does *not* prove MAJ^ω).

Theorem 2 (Soundness) *Suppose that*

$$\mathsf{PA}^\omega_\trianglelefteq + \mathsf{mAC}^\omega_{\mathrm{bd}} + \mathsf{bC}^\omega + \mathsf{MAJ}^\omega \vdash A(\underline{z}),$$

where A is an arbitrary formula of $\mathcal{L}^\omega_\trianglelefteq$ (with free variables as shown). Then there are closed monotone terms $\underline{t}$ of appropriate types such that

$$\mathsf{PA}^\omega_\trianglelefteq \vdash \tilde{\forall} \underline{a} \forall \underline{z} \trianglelefteq \underline{a} \tilde{\forall} \underline{b}\, A_{\mathrm{U}}(\underline{b}, \underline{t}(\underline{a}, \underline{b}), \underline{z}).$$

Note. The reader might have been expecting

$$\mathsf{PA}^\omega_\trianglelefteq \vdash \tilde{\forall} \underline{a} \forall \underline{z} \trianglelefteq \underline{a} \tilde{\forall} \underline{b} \exists y \trianglelefteq \underline{t}(\underline{a}, \underline{b})\, A_{\mathrm{U}}(\underline{b}, y, \underline{z})$$

at this point. However, note that A_{U} is monotone in the second variable (cf. Lemma 1). We used the notation $\underline{t}(\underline{a}, \underline{b})$ instead of the official $(\underline{t}\, \underline{a})\underline{b}$. We also use $\underline{t}\,\underline{a}\,\underline{b}$ in the sequel.

Proof. The proof proceeds by induction on the length of the derivation of $A(\underline{z})$. We rely on the complete axiomatization of classical logic described by Shoenfield in sections 2.6 and 8.3 of his textbook [2]. In the following, we discuss a few cases only. To ease readability, we use single variables instead of tuples and we usually omit the free variabes $\underline{z}$.

Let us start with the propositional axiom $\neg A \vee A$. Let $(A)^{\mathrm{U}}$ be $\tilde{\forall} b \tilde{\exists} c A_{\mathrm{U}}(b, c)$. A simple computation shows that $(\neg A \vee A)^{\mathrm{U}}$ is

$$\tilde{\forall} f, b \tilde{\exists} a, c\, (\tilde{\exists} a' \trianglelefteq a \neg A_{\mathrm{U}}(a', fa') \vee A_{\mathrm{U}}(b, c)).$$

Therefore, we need to find closed monotone terms t and q such that

$$\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall} f, b\, (\tilde{\exists} a' \trianglelefteq tbf \neg A_{\mathrm{U}}(a', fa') \vee A_{\mathrm{U}}(b, qbf)).$$

It is clear that we may put $t := \lambda b, f.b$ and $q := \lambda b, f.fb$.

We now consider the contraction rule which permits the inference A from $A \vee A$. This seemingly innocuous principle is always a delicate matter for functional interpretations (cf. the discussions in [11] and in [12]). In the present case, the interpretation of this principle is obtained by a slight of hand using the properties of the majorizabilty relation (and requiring no characteristic terms for bounded formulas). Again, assume that $(A)^{\mathrm{U}}$ is $\tilde{\forall} b \tilde{\exists} c A_{\mathrm{U}}(b, c)$. By induction hypothesis there are closed monotone terms t and q such that

$$\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall} b, d (A_{\mathrm{U}}(b, tbd) \vee A_{\mathrm{U}}(d, qbd)).$$

We must find a closed monotone term r such that $\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall} b A_{\mathrm{U}}(b, rb)$. Well, in the theory $\mathsf{PA}^{\omega}_{\trianglelefteq}$ it is possible to define, for each type τ, a monotone closed term $\max_{\tau}$ of type $\tau \to (\tau \to \tau)$ such that

$$\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash x \trianglelefteq_{\tau} x \wedge y \trianglelefteq_{\tau} y \to x \trianglelefteq \max_{\tau}(x, y) \wedge x \trianglelefteq \max_{\tau}(x, y).$$

This is explained in [6]. Therefore, using the monotonicity of A_{U} in the second variable, we readily see that the term $r := \lambda b. \max(tbb, qbb)$ does the job.

Let us now consider the Cut Rule that allows the inference of $B \vee C$ from $A \vee B$ and $\neg A \vee C$. Assume that $(A)^{\mathrm{U}}$ is $\tilde{\forall} b \tilde{\exists} c A_{\mathrm{U}}(b, c)$, $(B)^{\mathrm{U}}$ is $\tilde{\forall} d \tilde{\exists} e B_{\mathrm{U}}(d, e)$ and $(C)^{\mathrm{U}}$ is $\tilde{\forall} u \tilde{\exists} v C_{\mathrm{U}}(u, v)$. By induction hypothesis there are closed monotone terms t, q, r and s such that

(1) $\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall} b, d (A_{\mathrm{U}}(b, tbd) \vee B_{\mathrm{U}}(d, qbd))$ and

(2) $\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall} f, u (\tilde{\exists} b \trianglelefteq rfu \neg A_{\mathrm{U}}(b, fb) \vee C_{\mathrm{U}}(u, sfu))$.

We must find closed monotone terms k and l such that

$$\mathsf{PA}^\omega_\trianglelefteq \vdash \tilde{\forall} d, u(B_{\mathrm{U}}(d, kdu) \vee C_{\mathrm{U}}(u, ldu)).$$

Let us put $k := \lambda d, u.q(r(\lambda b.tbd, u), d)$ and $l := \lambda d, u.s(\lambda b.tbd, u)$. We now check that these closed monotone terms do the job. We reason inside $\mathsf{PA}^\omega_\trianglelefteq$. Fix monotone d and u. By (2) above, either $\tilde{\exists} b \trianglelefteq rf_0 u \neg A_{\mathrm{U}}(b, f_0 b)$ or $C_{\mathrm{U}}(u, sf_0 u)$, where f_0 abbreviates $\lambda b.tbd$. If the latter case holds, we are done. Otherwise, there is a monotone b with $b \trianglelefteq rf_0 u$ and $\neg A_{\mathrm{U}}(b, f_0 b)$, i.e. $\neg A_{\mathrm{U}}(b, tbd)$. By (1) above, we get $B_{\mathrm{U}}(d, qbd)$ and hence, by monotonicity, $B_{\mathrm{U}}(d, q(rf_0 u, d))$. Therefore, $B_{\mathrm{U}}(d, kdu)$ and we are done.

Let us now consider the substitution axioms $\forall x A(x) \to A(t)$, where t is a term free for x in A. Assume that $(A(x))^{\mathrm{U}}$ is $\tilde{\forall} b \tilde{\exists} c A_{\mathrm{U}}(b, c, x)$. A simple computation shows that we must find closed monotone terms r, s and l such that

$$\mathsf{PA}^\omega_\trianglelefteq \vdash \tilde{\forall} f, b(\tilde{\forall} a \trianglelefteq rfb \tilde{\forall} b' \trianglelefteq sfb \tilde{\forall} x \trianglelefteq a A_{\mathrm{U}}(b', fab', x) \to A_{\mathrm{U}}(b, lfb, t)).$$

By Howard's majorizability Theorem 1, take a closed monotone term q such that $\mathsf{PA}^\omega_\trianglelefteq \vdash t \trianglelefteq q$ (see the comment ahead). Now, put $r := \lambda f, b.q$, $s := \lambda f, b.b$ and $l := \lambda f, b.fqb$. It is clear that these terms do the job. Let us comment briefly on the case in which parameters (free variables) z occur, e.g. t is of the form $t[z]$. In this case, we apply Howard's theorem to the *closed* term $\lambda z.t[z]$. Since in the statement of the soundness theorem we only have to consider those z below a certain given monotone element, everything goes fine.

We finish the study of the *logical* reasoning by considering the $\forall$-introduction rule that infers $\forall x A \vee B$ from $A \vee B$, provided that x is not free in B. Assume that $(A(x))^{\mathrm{U}}$ is as in the previous case and $(B)^{\mathrm{U}}$ is $\tilde{\forall} d \tilde{\exists} e B_{\mathrm{U}}(d, e)$. By induction hypothesis, there are closed monotone terms t and q such that

$$\mathsf{PA}^\omega_\trianglelefteq \vdash \tilde{\forall} a, b, d \forall x \trianglelefteq a(A_{\mathrm{U}}(b, tabd, x) \vee B_{\mathrm{U}}(d, qabd)).$$

We obviously get $\mathsf{PA}^\omega_\trianglelefteq \vdash \tilde{\forall} a, b, d(\forall x \trianglelefteq a A_{\mathrm{U}}(b, tabd, x) \vee B_{\mathrm{U}}(d, qabd))$. But this is what we want.

The axioms regarding combinators, the axioms M_1 and M_2 and the equality axioms for $=_0$ have trivial interpretations, since they are universal. The rule $\mathsf{RL}_\trianglelefteq$ also poses no difficulty (see [6]). At this juncture, let us observe that the soundness theorem wouldn't go through if instead of the *rule* one would have the *axioms* $\forall v \forall u \trianglelefteq_\rho v(xu \trianglelefteq_\sigma yv \wedge yu \trianglelefteq_\sigma yv) \to x \trianglelefteq_{\rho\to\sigma} y$. The bounded functional interpretation would ask for closed monotone terms t such that the theory $\mathsf{PA}^\omega_\trianglelefteq$ proves

$$\tilde{\forall} a, b, c \forall x \trianglelefteq a \forall y \trianglelefteq b \forall v \trianglelefteq c\, (\forall v \trianglelefteq tabc \forall u \trianglelefteq_\rho v\, (xu \trianglelefteq_\sigma yv \wedge yu \trianglelefteq_\sigma yv) \to x \trianglelefteq_{\rho\to\sigma} y),$$

and such terms are simply not available (in order to conclude $x \trianglelefteq y$, the value v cannot be bounded). Such an impossibility can be argued directly, but it is also a consequence of the inconsistency of the "flattened" theory discussed in Section 6 below.

The axioms $\mathsf{B}_\forall$ are easily dealt with, specially if we see them as abbreviations of two corresponding conditionals. The scheme of induction is better analysed via the induction rule. Of course, the recursors are needed here (this is the only place where they are needed) and the analysis poses no difficulty (even though one has to be careful with ensuring the monotonicity of terms).

Finally, the characteristic principles trivialize under the interpretation, and the witness terms are readily forthcoming. For instance, after some computations, the interpretation of the bounded collection principle bC^ω asks for a closed monotone term t such that

$$\tilde{\forall} a, b \forall c \trianglelefteq a (\forall z \trianglelefteq c \exists y \trianglelefteq b A_{\mathrm{bd}}(y, z) \to \forall z \trianglelefteq c \exists y \trianglelefteq tba A_{\mathrm{bd}}(y, z)),$$

and $t := \lambda a, b.b$ obviously works. Similarly, the majorizability axioms MAJ^ω ask for a closed term t such that $\mathsf{PA}^\omega_\trianglelefteq \vdash \tilde{\forall} a \forall x \trianglelefteq a (x \trianglelefteq ta)$. The analysis of monotone bounded choice $\mathsf{mAC}^\omega_{\mathrm{bd}}$ is also straightforward. $\Box$

A particular case of the above theorem section is the important:

Corollary 1 *If $\mathsf{PA}^\omega_\trianglelefteq + \mathsf{mAC}^\omega_{\mathsf{bd}} + \mathsf{bC}^\omega_{\mathsf{bd}} + \mathsf{MAJ}^\omega \vdash \forall x \exists y A_{\mathrm{bd}}(x, y)$, where A_{bd} is a bounded formula with its free variables as shown, then one already has $\mathsf{PA}^\omega_\trianglelefteq \vdash \forall a \forall x \trianglelefteq a \exists y A_{\mathrm{bd}}(x, y)$.*

This corollary can be refined in an interesting way. As it is well-known, Georg Kreisel has often remarked that the use of true universal lemmata in the proof of $\forall\exists$-sentences does not affect the extraction of bounds. Within the framework of Gödel's *dialectica* interpretation, this can be readily seen by observing that the *dialectica* interpretation of a universal sentence is the universal sentence itself. We can even disregard whether the lemmata are *true* provided that we accept the very same lemmata in the *verification* of the bounds for the $\forall\exists$-consequences. Kohlenbach generalized Kreisel's observation by considering a wider class of sentences. In his framework of the *monotone functional interpretation* (see [13]), the verification of the bounds of the $\forall\exists$-consequences takes place using slightly *stronger* lemmata than the original ones (nevertheless, the stronger lemmata are true if the original lemmata are: this is of importance for the applications of the monotone functional interpretation).

We may formulate a similar observation in our setting. In the above soundness theorem, it is clear that we can substitute (both in the hypothesis and the thesis) the theory $\mathsf{PA}^\omega_\trianglelefteq$ by the stronger $\mathsf{PA}^\omega_\trianglelefteq + \Delta$, where Δ is constituted by universal closures of bounded formulas. In particular, the use of lemmata of this form does not affect the extraction of bounds. This can be generalized to a wider class of sentences, with the added twist that the verification takes place using slightly *weaker* sentences (see also Section 7.1 of [6]):

Corollary 2 *Let* Δ *be a set of sentences of the form* $\tilde{\forall} b \exists u \trianglelefteq rb \forall v B_{\mathrm{bd}}(v,u,b)$*, with* r *a (closed) term and* B_{bd} *a bounded formula. If*

$$\mathsf{PA}^{\omega}_{\trianglelefteq} + \mathsf{mAC}^{\omega}_{\mathsf{bd}} + \mathsf{bC}^{\omega}_{\mathsf{bd}} + \mathsf{MAJ}^{\omega} + \Delta \vdash \forall x \exists y A_{\mathrm{bd}}(x,y),$$

where A_{bd} *is a bounded formula with its free variables as shown, then one already has* $\mathsf{PA}^{\omega}_{\trianglelefteq} + \Delta_w \vdash \forall a \forall x \trianglelefteq a \exists y A_{\mathrm{bd}}(x,y)$*, where* Δ_w *is the weakening of* Δ *consisting of the sentences of the form* $\tilde{\forall} b, c \exists u \trianglelefteq rb \forall v \trianglelefteq c B_{\mathrm{bd}}(v,u,b)$*, each one corresponding to a sentence of* Δ*.*

Proof. Note that the sentences in Δ are consequences of Δ_w together with the bounded collection principle. Observe now that the sentences in Δ_w are universal sentences with bounded matrices. □

Let us advance a speculative note regarding the above issue. Mathematicians are very liberal (in the sense of *not caring*) in their use of induction (and comprehension). They are oblivious to the complexity of the statements they are inducting over. Logicians, on the other hand, are very sensitive to issues of definability and know that induction (together with comprehension) is the main reason for the advent of fast growing bounds. Nevertheless, as a matter of common mathematical experience, *really* fast growing functions almost never show up in ordinary mathematics. This is a puzzling phenomenon. I want to point that certain forms of induction are *tame* in this respect, namely induction for bounded formulas. In these cases, induction takes the form $A_{\mathrm{bd}}(0) \wedge \forall n < m(A_{\mathrm{bd}}(n) \to A_{\mathrm{bd}}(n+1)) \to A_{\mathrm{bd}}(m)$, with A_{bd} a bounded formula. As we saw above, statements like this are dealt by our interpretation effortlessly, with no need of recursors. NB after "flattening" (see Section 6), bounded formulas may have very high logical complexity. To what extent can inductions in ordinary mathematics be put in this form? The use of tame forms of induction is a particular case of using lemmata which have trivial interpretations (and which are true after "flattening"). Can lemmata of this kind formulate statements which have mathematically interesting consequences? (In a sense, the answer to this question is a trivial 'yes' because corresponding statements considered by Kohlenbach in his work can be dealt by lemmata of this kind. The question is really meant for mathematical statements beyond those.)

For other comments concerning our interpretation, including its relation to the Gödel-Shoenfield interpretation and Kohlenbach's monotone functional interpretation, see our recent [14].

5 The characterization theorem

The characterization theorem shows that a formula A and its *uniformization* A^U are equivalent provided that we are allowed to use certain principles. A conspicuous difference between Shoenfield's interpretation and the present interpretation is that the principles allowed for the former – namely, the axiom of choice for quantifier-free matrices (cf. [15]) – have an unproblematic interpretation and are set-theoretically correct.

Theorem 3 (Characterization) *Let A be an arbitrary formula of $\mathcal{L}^{\omega}_{\trianglelefteq}$. Then,*

$$\mathsf{PA}^{\omega}_{\trianglelefteq} + \mathsf{mAC}^{\omega}_{\mathrm{bd}} + \mathsf{bC}^{\omega} + \mathsf{MAJ}^{\omega} \vdash A \leftrightarrow (A)^{\mathrm{U}}.$$

Proof. The proof is by induction on the complexity of A. Let us discuss the case of negation. Suppose $(A)^{\mathrm{U}}$ is $\tilde{\forall} b \tilde{\exists} c A_{\mathrm{U}}(b,c)$. By $\mathsf{mAC}^{\omega}_{\mathrm{bd}}$, the latter formula is equivalent to $\tilde{\exists} f \tilde{\forall} b \tilde{\exists} c \trianglelefteq fb\, A_{\mathrm{U}}(b,c)$. By monotonicity, this is equivalent to $\tilde{\exists} f \tilde{\forall} b A_{\mathrm{U}}(b, fb)$ and, hence, to $\tilde{\exists} f \tilde{\forall} b \forall b' \trianglelefteq b A_{\mathrm{U}}(b', fb')$. Therefore, $\neg (A)^{\mathrm{U}}$ is equivalent to $(\neg A)^{\mathrm{U}}$. By induction hypothesis, the former is equivalent to $\neg A$, and we are done for this case. The equivalence concerning the case of the bounded universal quantifier uses the principle of bounded collection bC^{ω}. This principle, as well as MAJ^{ω}, is needed for the equivalence concerning the plain universal quantifier. The case of the disjunction sign is straightforward. □

It is not apparent what is accomplished by showing the equivalence between a formula A and its uniformization $(A)^{\mathrm{U}}$ within the intensional theory $\mathsf{PA}^{\omega}_{\trianglelefteq} + \mathsf{mAC}^{\omega}_{\mathrm{bd}} + \mathsf{bC}^{\omega} + \mathsf{MAJ}^{\omega}$. The problem lies with the status and interpretation of this theory (see the next section). Nevertheless, the Characterization Theorem has the following theoretical consequence: It ensures that we are not missing any principles besides $\mathsf{mAC}^{\omega}_{\mathrm{bd}}$, bC^{ω} and MAJ^{ω} in our statement of the soundness theorem. To see this, suppose that we could state the soundness theorem with a further principle P. Since P is a consequence of itself, soundness would give the existence of a closed monotone term t such that $\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall} b\, \mathsf{P}_{\mathrm{U}}(b, tb)$ and, therefore, $\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash (\mathsf{P})^{\mathrm{U}}$. By the Characterization Theorem above, we get $\mathsf{PA}^{\omega}_{\trianglelefteq} + \mathsf{mAC}^{\omega}_{\mathrm{bd}} + \mathsf{bC}^{\omega} + \mathsf{MAJ}^{\omega} \vdash \mathsf{P}$. In conclusion: P would be already superfluous.

Both in the Shoenfield interpretation and in the present interpretation, the treatment of negation is responsible for the raising of types. Negations, and specially iterated negations, have the effect of raising the types and making the translation somewhat opaque. As Georg Kreisel commented in a related context: "those iterated [negations] make my head spin" (cf. p. 147 of [16]; Kreisel actually wrote 'implications'). This is in general unavoidable, but not so in the case of conjuntion. If we translate a conjunction $A \wedge B$ in terms of our primitive logical words, we get $\neg(\neg A \vee \neg B)$: three negations appear, two of which nested. However, in the presence of $\mathsf{PA}^{\omega}_{\trianglelefteq} + \mathsf{mAC}^{\omega}_{\mathrm{bd}} + \mathsf{bC}^{\omega} + \mathsf{MAJ}^{\omega}$, $(A \wedge B)^{\mathrm{U}}$ is equivalent to $A \wedge B$ and, therefore, to $(A)^{\mathrm{U}} \wedge (B)^{\mathrm{U}}$. The latter is classically equivalent to $\tilde{\forall} b, d \tilde{\exists} c, e\, (A_{\mathrm{U}}(b,c) \wedge B_{\mathrm{U}}(d,e))$, given that $(A)^{\mathrm{U}}$ is $\tilde{\forall} b \tilde{\exists} c A_{\mathrm{U}}(b,c)$ and $(B)^{\mathrm{U}}$ is $\tilde{\forall} d \tilde{\exists} e B_{\mathrm{U}}(d,e)$. Indeed, we could have started with the conjunction sign as primitive and *define* $(A \wedge B)^{\mathrm{U}}$ by $\tilde{\forall} b, d \tilde{\exists} c, e\, (A_{\mathrm{U}}(b,c) \wedge B_{\mathrm{U}}(d,e))$: the soundness theorem would still hold.

6 Flattening

The theory $\mathsf{PA}^{\omega}_{\leq^*}$ is Peano arithmetic in finite-types (with the minimal treatment of equality referred to above) formulated in the usual language of arithmetic extended by *primitive* majorizability symbols $\leq^*_\tau$ (one for each type τ) and the associated bounded quantifiers. We call this language $\mathcal{L}^{\omega}_{\leq^*}$. The following axioms (Bezem's majorizability relations) govern these symbols:

$$x \leq^*_0 y \leftrightarrow x \leq y$$
$$x \leq^*_{\rho\to\sigma} y \leftrightarrow \forall u \leq^*_\rho v (xu \leq^*_\sigma yv \wedge yu \leq^*_\sigma yv)$$

(note the biconditional above) and the axioms of the form $\forall x \leq^* t\, A(x) \leftrightarrow \forall x (x \leq^* t \to A(x))$. Of course, $\mathsf{PA}^{\omega}_{\leq^*}$ is a straightforward extension by definitions of plain PA^{ω}. The former formulation is considered only for convenience.

The next result is clear:

Lemma 2 (Flattening) *Suppose that* $\mathsf{PA}^{\omega}_{\trianglelefteq} \vdash A$*, where* A *is a sentence of the language* $\mathcal{L}^{\omega}_{\trianglelefteq}$*. Then* $\mathsf{PA}^{\omega}_{\leq^*} \vdash A^*$*, where* A^* *is the sentence of* $\mathcal{L}^{\omega}_{\leq^*}$ *obtained from* A *by replacing throughout the binary symbols* $\trianglelefteq_\tau$ *by* $\leq^*_\tau$*.*

This simple lemma provides the passageway from the intensional theories to $\mathsf{PA}^{\omega}_{\leq^*}$ and, therefore, to the set-theoretical world. As a typical illustration, suppose that the intensional theory $\mathsf{PA}^{\omega}_{\trianglelefteq} + \mathsf{mAC}^{\omega}_{\mathrm{bd}} + \mathsf{bC}^{\omega} + \mathsf{MAJ}^{\omega}$ proves a Π^0_2-sentence of first-order arithmetic. By the corollary of the soundness theorem, this sentence is provable in $\mathsf{PA}^{\omega}_{\trianglelefteq}$ and, by flattening, in $\mathsf{PA}^{\omega}_{\leq^*}$. As a matter of fact, the sentence is even provable in first-order Peano arithmetic because $\mathsf{PA}^{\omega}_{\leq^*}$ has a suitable interpretation in it (there is an internal coding of the finite-type functionals within first-order Peano arithmetic: the hereditarily recursive operations – cf. [17]).

In particular, the above argument shows that the theory $\mathsf{PA}^{\omega}_{\trianglelefteq} + \mathsf{mAC}^{\omega}_{\mathrm{bd}} + \mathsf{bC}^{\omega} + \mathsf{MAJ}^{\omega}$ is consistent (relative to first-order Peano arithmetic). This is a syntactic consistency argument, and one wonders whether one can find a veridical interpretation of the theory $\mathsf{PA}^{\omega}_{\trianglelefteq} + \mathsf{mAC}^{\omega}_{\mathrm{bd}} + \mathsf{bC}^{\omega} + \mathsf{MAJ}^{\omega}$. This seems to be a delicate task because the "flattened" version of this theory is *inconsistent*. Let us show that the theory $\mathsf{PA}^{\omega}_{\leq^*}$ together with the following "flattening" of a particular case of bC^{ω}:

$$\forall\gamma \leq_1 1\exists n^0 A_{\mathrm{bd}}(n,z) \to \exists m\forall\gamma \leq_1 1\exists n \leq_0 m A_{\mathrm{bd}}(n,z),$$

is already inconsistent (in the above, A_{bd} is a bounded formula in the flattened sense, i.e. it is of the form A^* for a bounded formula A of the language $\mathcal{L}^\omega_\trianglelefteq$). By classical logic, $\forall\gamma \leq_1 1\exists n(\neg\gamma \leq_1 0^1 \to \gamma n \neq 0)$, where $0^1 := \lambda k^0.0^0$. Since $\neg\gamma \leq_1 0^1$ is a bounded formula in the flattened sense, we may infer by the above form of collection that there is a natural number m such that

$$\forall\gamma \leq_1 1(\exists n(\gamma n \neq 0) \to \exists n \leq m(\gamma n \neq 0)).$$

Obviously, this can be refuted.

7 The conservativity of weak König's lemma

The Shoenfield-like bounded functional interpretation provides a very perspicuos proof of Friedman's Π^0_2-conservation result of the theory WKL_0 over RCA_0 (see [18] for the terminology and the result). The proper setting for discussing this result is not the theory $\mathsf{PA}^\omega_\trianglelefteq$ but rather its subtheory $\mathsf{PRA}^\omega_\trianglelefteq$. The latter differs from the former in that it only allows the recursor R_0 of type 0 – resulting in the finite-type functionals in the sense of Kleene (cf. section 5.1 and footnote 10 of [12]) – and restricting the scheme of induction

$$A_{\mathrm{qf}}(0) \wedge \forall n^0(A_{\mathrm{qf}}(n) \to A_{\mathrm{qf}}(\mathrm{S}n)) \to \forall n A_{\mathrm{qf}}(n)$$

to quantifier-free formulas A_{qf} (possibly with parameters of arbitrary type) in which the new predicate symbols $\trianglelefteq$ do not occur. It is clear that one can formulate and prove a soundness theorem for $\mathsf{PRA}^\omega_\trianglelefteq$ as in Section 4. *Mutatis mutandis* for the ensuing corollaries.

The second-order language of arithmetic can be embedded in $\mathcal{L}^\omega_\trianglelefteq$ by letting the first-order variables run over type 0 arguments, letting the second-order variables run over type 1 variables α such that $\alpha \trianglelefteq_1 1$, and by interpreting $n \in \alpha$ by $\alpha n = 0$. Under this embedding, we claim that WKL_0 is a subtheory of $\mathsf{PRA}^\omega_\trianglelefteq + \mathsf{mAC}^\omega_{\mathrm{bd}} + \mathsf{bC}^\omega + \mathsf{MAJ}^\omega$. We need to check that the latter theory proves induction for Σ^0_1-formulas, recursive comprehension and weak König's lemma. It is folklore (in a slightly different setting) that the first two principles follow from $\mathsf{PRA}^\omega_\trianglelefteq$ together with the numerical axiom of choice for quantifier-free matrices, i.e.

$$\forall n^0\exists m^0 A_{\mathrm{qf}}(n,m) \to \exists\alpha^1\forall n A_{\mathrm{qf}}(n,\alpha n),$$

where A_{qf} is quantifier-free. Note that this form of choice is an immediate consequence of $\mathsf{mAC}^\omega_{\mathrm{bd}}$.

It remains to prove weak König's lemma, i.e.

$$\forall T \trianglelefteq_1 1(Tree_\infty(T) \to \exists \alpha \trianglelefteq_1 1 \forall k^0\, T(\overline{\alpha}k) = 0),$$

where $\overline{\alpha}k$ denotes the number-code of the binary sequence $\langle \alpha 0, \ldots, \alpha(k-1)\rangle$ and $Tree_\infty(T)$ abbreviates the conjunction of

$$\forall s^0(T(s) = 0 \to Seq_2(s)) \wedge \forall s, r(T(r) = 0 \wedge s \preceq r \to T(s) = 0)$$

with the infinity clause $\forall n^0 \exists s^0(T(s) = 0 \wedge |s| = n)$. We are using standard notation: $Seq_2(s)$ expresses that s is the number-code of a binary sequence, $s \preceq r$ says that the binary sequence given by s is an initial segment of the binary sequence given by r, and $|s|$ is the length of the binary sequence given by s.

Assume $Tree_\infty(T)$. By the infinity clause and the fact that T is a tree, we may conclude that $\forall n^0 \exists \alpha \trianglelefteq_1 1 \forall k \leq n\, T(\overline{\alpha}k) = 0$. The reason is simple: given (the code for) a binary sequence s of length $n+1$, the type 1 function α which prolongs s by zeros satisfies $\alpha \trianglelefteq_1 1$. Now, using the contrapositive of bC^ω, we get $\exists \alpha \trianglelefteq_1 1 \forall k^0\, T(\overline{\alpha}k) = 0$. Q.E.D.

We just need the punch line to prove Friedman's conservation result. Suppose that WKL_0 proves a Π^0_2-sentence ϕ. Then, $\mathsf{PRA}^\omega_\trianglelefteq + \mathsf{mAC}^\omega_{\mathrm{bd}} + \mathsf{bC}^\omega + \mathsf{MAJ}^\omega \vdash \phi$. By the corollary to the soundness theorem, $\mathsf{PRA}^\omega_\trianglelefteq \vdash \phi$. Therefore, by flattening, $\mathsf{PRA}^\omega_{\leq^*} \vdash \phi$. In fact, $\mathsf{RCA}_0 \vdash \phi$ because there is a suitable internal coding of the finite-type functionals in the sense of Kleene in first-order Peano arithmetic with induction restricted to Σ^0_1-formulas (this is a watered down version of the hereditarily recursive operations). Of course, this restricted version of Peano arithmetic is a subtheory of RCA_0.

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