Notre Dame Journal of Formal Logic

A simple proof of Parsons' theorem

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Abstract Let $\mathsf{I}\Sigma_1$ be the fragment of elementary Peano Arithmetic in which induction is restricted to Σ_1-formulas. More than three decades ago, Charles Parsons showed that the provably total functions of $\mathsf{I}\Sigma_1$ are exactly the primitive recursive functions. In this paper, we observe that Parsons' result is a consequence of Herbrand's theorem concerning the $\exists\forall\exists$-consequences of universal theories. We give a self-contained proof requiring only basic knowledge of mathematical logic.

1 Introduction

Primitive recursive arithmetic, or Skolem arithmetic, was invented in 1923 by the Norwegian mathematician Thoralf Skolem. It presents a way of developing arithmetic in a quantifier-free calculus in which theorems are stated by free-variable formulas (asserting, in effect, Π_1-sentences of arithmetic). The work of Skolem [Sko23] was given ample attention by David Hilbert and Paul Bernays in [HB34], where they took up the task of formalizing it in a propositional calculus of equations. A few year later, independently of each other, Haskell Curry [Cur41] and Reuben Goodstein [Goo45] carried the work of Skolem a step further, showing how to develop primitive recursive arithmetic in a "logic-free" calculus based solely on equations.

The interest of Hilbert and Bernays in primitive recursive arithmetic stemmed from their conviction that the arguments carried in it correspond to the point of view of the "evident, finitistic theory of numbers" (*anschaulichen, finiten Zahlentheorie*, p. 286 of [HB34] – in italics in the original).[1] Hilbert's foundational program aimed at reducing infinitistic, set-theoretic mathematics, to finitism. As explained by Hilbert (e.g., [Hil25]), the reduction was to be accomplished by means of finitistic proofs of conservation results for Π_1-sentences or, equivalently, by means of finitistic consistency proofs.[2] It is well-known that Gödel's second incompleteness theorem refuted Hilbert's original foundational program.

Printed September 11, 2003
2001 Mathematics Subject Classification: Primary, 03F30

Hilbert's programmatic ideas didn't die with Gödel's theorem. Rather, they were reformulated in the light of Gödel's results. *Beweistheorie*, the mathematical discipline that Hilbert invented to carry out finitistic consistency proofs, eventually redirected its aims and broadened its methods (the reader can find a clear and accessible description of this change of direction, as well as more specialized references to this topic, in Feferman's lecture [Fef00]). Somewhat surprisingly, Hilbert's *original* program resurfaced after a torpor of more than fifty years. It has been studied in detail in the form of the following question: What parts of mathematics can be reduced to finitism in Hilbert's original sense? In other words: Gödel showed that Hilbert's original program is not feasible in its entirety, but it remains a matter of investigation what *partial* realizations of Hilbert's program may still be vindicated. This line of research was forcefully articulated by Stephen Simpson in [Sim88], and can be viewed as a sub-program of Simpson and Harvey Friedman's wider program of *Reverse Mathematics* (see [Sim99]).

It is against this background that it is important to study formal systems of arithmetic that are (finitistically) conservative over primitive recursive arithmetic. Plainly, the parts of mathematics able to be carried out in these systems constitute partial realizations of Hilbert's original program. Charles Parsons' conservation theorem – independently proved by Grigori Mints and Gaisi Takeuti – is an important and central result of this sort.

A modern exposition of primitive recursive arithmetic can be found in section 2.1 of the textbook of Anne Troelstra and Dirk van Dalen [vDT88]. Their presentation is of a piece with the original presentations of Skolem and Hilbert/Bernays, in that it is framed in a quantifier-free calculus. Nevertheless, we opt for a framework based on a first-order language with equality, as expounded in section IX.3 of Simpson's book [Sim99]. In the sequel, PRA is such system: It is a first-order universal theory (i.e., axiomatized by purely universal formulas), with a function symbol for each (description of a) primitive recursive function, and in which the principle of induction for quantifier-free formulas holds. By Herbrand's theorem, PRA is conservative over quantifier-free Skolem arithmetic (a result which, by itself, constitutes a conservation result in the sense described above – see the next section). The theory $\mathsf{I\Sigma_1}$ is the fragment of elementary Peano arithmetic in which induction is restricted to Σ_1-formulas. It is well known that the primitive recursive functions can be suitably introduced in this theory. Thus, by a harmless abuse of language, PRA is a subtheory of $\mathsf{I\Sigma_1}$. Charles Parsons' result of [Par70], [Par71] and [Par72] can be formulated as follows (note that it even applies to Π_2-consequences):

Theorem 1.1 *Any Π_2-consequence of $\mathsf{I\Sigma_1}$ is also a consequence of PRA.*

Parsons' proof uses a variant of Gödel's functional interpretation.[3] The proofs of Grigori Mints and Gaisi Takeuti use quite different ideas, namely the no-counterexample interpretation and a Gentzen-style assignment of ordinals to proofs, respectively.[4] A dozen or so years ago, Wilfried Sieg [Sie91] gave a very perspicuous proof of Parsons' theorem by systematically applying Herbrand's theorem for $\exists$-consequences of universal theories at the induction inferences of a suitable normalized proof.[5] Very recently, Jeremy Avigad [Avi02] provided a very elegant model-theoretic analogue of Sieg's proof. In his book [Sim99], Simpson gives a model-theoretic proof of Parsons' theorem based on the notion of 'semiregular cut', a notion due to Laurie Kirby and Jeff Paris in [KP77]. Simpson attributes to these two authors the idea of the proof.

In the present paper, we observe that Parsons' result is a simple consequence of Herbrand's theorem concerning the $\exists\forall\exists$-consequences of universal theories. Our proof can be followed with only basic knowledge of mathematical logic. It also readily applies to similar situations, e.g., to show that the polytime computable functions witness the $\forall\Sigma^b_1$-consequences of Buss' theory S^1_2 (as in [Bus85]).

2 Herbrand's theorem

Herbrand's theorem characterizes first-order validities in terms of suitable tautologies. In the sequel, we need a form of Herbrand's theorem for $\exists\forall\exists$-consequences of universal theories.[6] The particular form in question has a quite elegant statement, and can be proved by a very simple compactness argument due to Jan Krajíček, Pavel Pudlák and Gaisi Takeuti in [KPT91]. Their argument was given in the somewhat arcane setting of bounded arithmetic. It is however a general argument, and merits to be more widely known. We include their argument below (making our exposition self-contained).

Theorem 2.1 *Let* U *be a universal theory in the first-order language* $\mathcal{L}$.

(1) *Suppose* $\exists x\varphi(x,u)$ *is a consequence of* U, *where* φ *is a quantifier-free formula with its variables as shown. Then there are terms* $t_1(u)$, $t_2(u)$, $\dots$, $t_k(u)$ *of* $\mathcal{L}$ *(with at most the variable* u*) such that*

$$\mathsf{U} \models \varphi(t_1(u),u) \vee \varphi(t_2(u),u) \vee \ldots \vee \varphi(t_k(u),u).$$

(2) *Suppose* $\exists x\forall y\varphi(x,y,u)$ *is a consequence of* U, *where* φ *is an existential formula, with its free variables as shown. Then there are terms* $t_1(u)$, $t_2(u,y_1)$, $\dots$, $t_k(u,y_1,\dots,y_{k-1})$ *of* $\mathcal{L}$ *(with its variables among the ones shown) such that*

$$\mathsf{U} \models \varphi(t_1(u),y_1,u) \vee \varphi(t_2(u,y_1),y_2,u) \vee \ldots \vee \varphi(t_k(u,y_1,\dots,y_{k-1}),y_k,u).$$

Proof Note that (1) is a particular case of (2): just insert *two* dummy quantifiers and substitute the ys by the variable u in the terms. Alternatively, one can prove (1) directly by a compactness argument. We will not do this, since the same proof idea (albeit more involved) appears in the proof of (2) below.

Assume that no disjunction as in (2) is a consequence of the theory U. Let $v_0, v_1, \dots$ be the list of the formal variables of $\mathcal{L}$, and fix $t_1, t_2, t_3, \dots$ a enumeration of all the terms of the language such that the variables of $t_j(v_0, v_1, \dots, v_{j-1})$ occur among $v_0, v_1, \dots, v_{j-1}$.

Consider the set of sentences U together with

$$\{\neg\varphi(t_1(c),d_1,c), \neg\varphi(t_2(c,d_1),d_2,c), \dots, \neg\varphi(t_j(c,d_1,d_2,\dots,d_{j-1}),d_j,c), \dots\},$$

where $c, d_1, d_2, \dots, d_j, d_{j+1}, \dots$ are new constants. It follows from our assumption that this set is finitely satisfiable. By compactness, it has a model $\mathcal{M}$. Let us consider the following subset of the domain of $\mathcal{M}$,

$$\{t_1(c), t_2(c,d_1), \dots, t_j(c,d_1,d_2,\dots,d_{j-1}), \dots\},$$

where we are identifying the terms with their interpretations in $\mathcal{M}$. Note that all elements $c, d_1, d_2, \ldots$ are members of the above subset because the variables v_j appear in the enumeration of terms. It is also clear that the above subset defines a *substructure* $\mathcal{M}^*$ of $\mathcal{M}$. Using the fact that U is a universal theory, $\mathcal{M}^*$ is a model of U. But:

$$\mathcal{M}^* \models \forall x \exists y \neg \varphi(x, y, c).$$

In fact, for $x = t_j(c, d_1, \ldots, d_{j-1})$ take $y = d_j$ and use the fact that $\neg\varphi$ is a universal formula and, therefore, downward absolute between $\mathcal{M}$ and $\mathcal{M}^*$. $\square$

We have restricted the statement of the theorem to single variables u, x and y in order to make the proof more readable. It is clear, however, that the theorem holds for several variables $\overline{u} := u^1, \ldots, u^i$, $\overline{x} := x^1, \ldots, x^j$ and $\overline{y} := y^1, \ldots, y^r$. In this case, we must consider appropriate terms $\bar{t}_1 := t_1^1, \ldots, t_1^j; \ldots; \bar{t}_k = t_k^1, \ldots, t_k^j$. One should also point out that part (1) of the theorem simplifies if the universal theory U admits *definition by cases*,[7] as it is the case with PRA. In this case, we may take $k = 1$. Note, however, that no such simplification is forthcoming for part (2) of the theorem!

The above theorem (in general, Herbrand's theorem for prenex formulas) can also proved through the analysis of a suitable complete proof system. The theorem is a simple consequence of Gentzen's "verschärfter Hauptsatz," known in English as Gentzen's midsequent theorem (see [TS00] for this route). It can also be proved using Gentzen's plain Hauptsatz, as Buss does in [Bus95]. Herbrand's own method appears in his doctoral dissertation [Her30]. The reader can find a partial translation into English of Herbrand's thesis in the volume [vHe67], together with commentaries and corrections of Herbrand's proof. Both analyses (*à la* Herbrand or *à la* Gentzen) automatically entail that a quantifier-free first-order consequence of a universal theory is a quasi-tautological consequence[8] of a finite number of substitution instances of its axioms. When applied to the theory PRA, this additional feature explains why PRA is conservative over quantifier-free Skolem arithmetic, as observed in the previous section.

However, one need not lay down and analyze a complete proof system in order to obtain the extra information above. Plain semantic considerations suffice. Here is why. First, we may work with *pure* first-order logic (no equality present) and, in tandem, with tautological (*vs.* quasi-tautological) consequences, since the equality axioms may be taken to be universal sentences. Secondly, it is easy to argue semantically that a pure quantifier-free first-order validity must be a tautology (where the propositional letters are the atomic formulas). After these preliminaries, suppose that U is a (pure) universal theory, and that $\mathsf{U} \models \varphi(\overline{u})$, where $\varphi(\overline{u})$ is a quantifier-free formula with its variables as shown. By compactness, $\varphi(\overline{u})$ is a consequence of finitely many axioms of U. Without loss of generality, we may suppose that $\forall \overline{x} \psi(\overline{x}) \models \varphi(\overline{u})$, for a single axiom '$\forall \overline{x} \psi(\overline{x})$' of U. Therefore, the sentence $\forall \overline{u} \exists \overline{x} (\psi(\overline{x}) \to \varphi(\overline{u}))$ is a first-order validity. By Herbrand's theorem (1), applied to the empty theory, there are terms $\bar{t}_1(\overline{u}), \ldots, \bar{t}_k(\overline{u})$ such that the implication

$$\psi(\bar{t}_1(\overline{u})) \wedge \ldots \wedge \psi(\bar{t}_k(\overline{u})) \to \varphi(\overline{u})$$

is a first-order validity and, hence, a tautology. In short, $\varphi(\overline{u})$ is a tautological consequence of finitely many substitutions instances of axioms of U.[9]

3 A proof of Parsons' result

We are now ready to prove Parsons' theorem. Suppose that the Π_2-sentence $\forall u \exists v \theta(u,v)$ is a consequence of $\mathsf{I\Sigma}_1$, where θ is an open formula (in the language of PRA). By compactness, the given Π_2-sentence is a consequence of finitely many instances of the Σ_1-induction scheme. It is not difficult to see that these finitely many instances can be subsumed by a single instance. Therefore:

$$\mathsf{PRA} \models Ind_\varphi \to \forall u \exists v \theta(u,v),$$

where Ind_φ abbreviates the sentence

$$\forall c \forall z (\varphi(c,0) \wedge \forall x(\varphi(c,x) \to \varphi(c,x+1)) \to \varphi(c,z))$$

for a certain Σ_1-formula $\varphi(c,x) := \exists y\, \psi(c,x,y)$, ψ quantifier-free (it is all right to consider only a single parameter c because PRA has a pairing function).

We now put the sentence $Ind_\varphi \to \forall u \exists v \theta(u,v)$ in prenex form and obtain,

$$(*) \qquad \mathsf{PRA} \models \exists v, c, z, y_0 \forall x, y, w \exists y' (\theta(u,v) \vee \chi(c,z,y_0,x,y,w,y')),$$

where $\chi(c,z,y_0,x,y,w,y')$ is the quantifier-free formula

$$\psi(c,0,y_0) \wedge (\psi(c,x,y) \to \psi(c,x+1,y')) \wedge \neg\psi(c,z,w).$$

Lemma 3.1 *Let $t(\overline{p})$, $s(\overline{p})$, $r(\overline{p})$ and $q(\overline{p},x,y,w)$ be terms of the language of* PRA*, with the variables as shown. Then,*

$$\mathsf{PRA} \models \forall \overline{p} \exists x, y, w\, \neg\chi(t(\overline{p}), s(\overline{p}), r(\overline{p}), x, y, w, q(\overline{p},x,y,w)).$$

Proof We reason inside PRA. In order to get a contradiction, suppose that there is $\overline{p}$ such that $\forall x, y, w\, \chi(t(\overline{p}), s(\overline{p}), r(\overline{p}), x, y, w, q(\overline{p},x,y,w))$. We get

(1) $\psi(t(\overline{p}), 0, r(\overline{p}))$;
(2) $\forall x, y, w\, (\psi(t(\overline{p}), x, y) \to \psi(t(\overline{p}), x+1, q(\overline{p},x,y,w)))$; and
(3) $\forall w\, \neg\psi(t(\overline{p}), s(\overline{p}), w)$.

Define h by primitive recursion according to the following clauses:

$$h(0,\overline{p}) = r(\overline{p})$$
$$h(x+1,\overline{p}) = q(\overline{p}, x, h(x,\overline{p}), 0)$$

By (1), (2) and quantifier-free induction, it follows that $\forall x\, \psi(t(\overline{p}), x, h(x,\overline{p}))$. In particular, $\exists w\, \psi(t(\overline{p}), s(\overline{p}), w)$. This goes against (3). $\square$

Herbrand's theorem applies to PRA. Therefore, from $(*)$ and part (2) of the theorem of the previous section, there are terms $r_1(u)$, $\bar{t}_1(u)$, $r_2(u,\overline{z}_1)$, $\bar{t}_2(u,\overline{z}_1)$,..., $r_k(u,\overline{z}_1,\ldots,\overline{z}_{k-1})$, $\bar{t}_k(u,\overline{z}_1,\ldots,\overline{z}_{k-1})$ such that the disjunction of the following formulas is a consequence of PRA:

$$\theta(u, r_1(u)) \vee \exists y' \chi(\bar{t}_1(u), \overline{z}_1, y')$$
$$\theta(u, r_2(u,\overline{z}_1)) \vee \exists y' \chi(\bar{t}_2(u,\overline{z}_1), \overline{z}_2, y')$$
$$\cdots$$
$$\theta(u, r_k(u,\overline{z}_1,\ldots,\overline{z}_{k-1})) \vee \exists y' \chi(\bar{t}_k(u,\overline{z}_1,\ldots,\overline{z}_{k-1}), \overline{z}_k, y'),$$

where each $\overline{z}_j$ abbreviates a triple of variables and each $\bar{t}_j$ abbreviates a triple of terms (with its variables as shown). Hence, the disjunction of the formula $\exists v\theta(u,v)$ together with the disjunction of the k formulas,

$$\begin{gathered}\exists y'\chi(\bar{t}_1(u),\overline{z}_1,y')\\ \exists y'\chi(\bar{t}_2(u,\overline{z}_1),\overline{z}_2,y')\\ \cdots\\ \exists y'\chi(\bar{t}_k(u,\overline{z}_1,\ldots,\overline{z}_{k-1}),\overline{z}_k,y'),\end{gathered}$$

is a consequence of PRA. By Herbrand's theorem (in the form of part (1) of the previous section), there is a term $q(u,\overline{z}_1,\ldots,\overline{z}_{k-1},\overline{z}_k)$ of the language such that the last formula of the previous list may be substituted by

$$\chi(\bar{t}_k(u,\overline{z}_1,\ldots,\overline{z}_{k-1}),\overline{z}_k,q(u,\overline{z}_1,\ldots,\overline{z}_{k-1},\overline{z}_k)).$$

By the above lemma,

$$\exists\overline{z}_k\,\neg\chi(\bar{t}_k(u,\overline{z}_1,\ldots,\overline{z}_{k-1}),\overline{z}_k,q(u,\overline{z}_1,\ldots,\overline{z}_{k-1},\overline{z}_k))$$

is a consequence of PRA. Therefore, the disjunction of $\exists v\theta(u,v)$ together with the disjunction of the $k-1$ formulas

$$\begin{gathered}\exists y'\chi(\bar{t}_1(u),\overline{z}_1,y')\\ \exists y'\chi(\bar{t}_2(u,\overline{z}_1),\overline{z}_2,y')\\ \cdots\\ \exists y'\chi(\bar{t}_{k-1}(u,\overline{z}_1,\ldots,\overline{z}_{k-2}),\overline{z}_{k-1},y'),\end{gathered}$$

is also a consequence of PRA.

If we repeat the previous argument $(k-1)$ times we eventually conclude that PRA $\models \exists v\theta(u,v)$.

Q.E.D.[10]

Notes

1. *Finitistic theory of numbers* was never made precise by Hilbert. It remained informal, presumably because an actual finitistic consistency proof would be recognized as such without disputation. The remarks of Hilbert and Bernays in the *Grundlagen* clearly endorse the thesis that primitive recursive arithmetic is part of finitistic mathematics. The substantive thesis that primitive recursive arithmetic is all there is to finitistic mathematics (modulo the arithmetization of syntax) is defended by William Tait in [Tai81].

2. More precisely: If S is a theory that purports to formalize infinitistic mathematics, then the consistency of S is equivalent to the reflection principle for Π_1-sentences (see [Smo77]).

3. Parsons' result appears in the last theorem of [Par70]. In its proof, Parsons refers to the abstract [Par71], where it is stated that the theory $\mathsf{I\Sigma_1}$ (actually, a seemingly stronger but equivalent theory) has a functional interpretation in $\mathsf{T_0}$, a fragment of Gödel's T. The proof of this statement is carried out in [Par72] (via a preliminary Gödel-Gentzen double negation interpretation). As a consequence, if $\exists v\theta(u,v)$, θ quantifier-free, is provable in $\mathsf{I\Sigma_1}$, then there is a closed term t of $\mathsf{T_0}$ such that $\mathsf{T_0}$ proves $\theta(u,tu)$. In order to get his conservation result, Parsons associates to t a unary term t' of the language of PRA such that the latter theory proves $\theta(u,t'(u))$. He studies this association in the initially cited paper [Par70].

4. In [Min72], Mints works directly with the sequent calculus *already restricted* to a language with one-quantifier formulas only (i.e., there are no alternations of the quantifiers $\forall$ and $\exists$ in the formulas that appear in the sequents). Clearly, these restricted systems are complete in the obvious sense. As noted, Mints' argument uses the no-counterexample interpretation which, being restricted here to one-quantifier formulas, reminds one of Samuel Buss' technique of witness functions [Bus85]. For a witness function account of Parsons' theorem, see [Bus98]. Takeuti's proof appears in [Tak75].

5. Sieg has an earlier, rather convoluted, proof of Parsons' theorem in [Sie85]. The proof technique used in [Sie91] was foreshadowed by an argument in [Fer90].

6. More precisely, we need a version of the "Propriété A" of first-order validities (of the form $\exists\forall\exists$), introduced by Jacques Herbrand in chapter V of his thesis [Her30]. This is the version of Herbrand's theorem *without* the introduction of (so-called) index functions.

7. A theory U admits definition by cases if, for any terms $t_1(\overline{u}),\ldots,t_{k+1}(\overline{u})$ and quantifier-free formulas $\theta_1(\overline{u}),\ldots,\theta_k(\overline{u})$, there is a term $t(\overline{u})$ such that

$$[\theta_1(\overline{u})\to t(\overline{u})=t_1(\overline{u})]\wedge[\theta_2(\overline{u})\wedge\neg\theta_1(\overline{u})\to t(\overline{u})=t_2(\overline{u})]\wedge\ldots$$
$$\ldots\wedge[\neg\theta_k(\overline{u})\wedge\ldots\wedge\neg\theta_1(\overline{u})\to t(\overline{u})=t_{k+1}(\overline{u})]$$

is a consequence of U.

8. I.e., a tautological (a.k.a. propositional) consequence of instances of the equality axioms.

9. This three-part semantic argument is *folklore*. The last piece is due to Mints and Nikolay Shanin for the theory PRA (see [Min72]).

10. We strove for simplicity in the above proof and, accordingly, we formulated Parsons' theorem in semantic terms and proved it in a semantic, non-finitistic, manner. The argument of this section may, nevertheless, be given a finitistic form. One must, of course, work with provability instead of semantic consequence, and rely on proof-theoretic accounts of Herbrand's theorem. The induction on k in the final step of the proof (a Σ_1-induction) can be avoided if we use the following fact: From the proof-theoretic proofs of Herbrand's theorem, one can obtain primitive recursively a PRA-term t and a PRA-proof of $\varphi(u,t(u))$ from a PRA-proof

of $\exists x\varphi(u, x)$. Applying this fact to the induction part of the proof *as well as* to the lemma, we may replace Σ_1-induction by an explicit primitive recursive construction/verification.

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Acknowledgments

Partially supported by CMAF, POCTI/FCT and FEDER.

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