ANALYSIS IN WEAK SYSTEMS

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Abstract. The authors survey and comment their work on weak analysis. They describe the basic set-up of analysis in a feasible second-order theory and consider the impact of adding to it various forms of weak König's lemma. A brief discussion of the Baire categoricity theorem follows. It is then considered a strengthening of feasibility obtained (fundamentally) by the addition of a counting axiom and showed how it is possible to develop Riemann integration in the stronger system. The paper finishes with three questions in weak analysis.

1. Introduction

This paper is the third and last part of a triptych whose two other parts consist of "Techniques in weak analysis for conservation results" [13] and "Interpretability in Robinson's Q" [20]. This triptych of papers mostly surveys the work of the authors in weak analysis. The present paper is representative of our work in this field and it is the way that the authors have found to jointly render homage to Amílcar Sernadas on the occasion of his 64th birthday. The number sixty-four is a round number in binary notation and appears in the title of a famous song of the *Beatles* but, more prosaically, Amílcar's 64th birthday was the occasion of a deserved *Festschrift* in his honour. The influence and leadership of Amílcar on *his side* of logic has been enormous at Instituto Superior Técnico and, indeed, in Portugal. One should also mention his international collaborations, specially with Brazilian logicians. Amílcar's influence is also felt in Portugal on the *other side*, namely in important matters of academic support and institutional collaboration. The authors of this paper work on the *other side* of logic and are grateful for the chance to present here the final paper of their triptych. They accepted with pleasure the kind invitation of Francisco Dionísio and the other organizers of the *Festschrift* to participate in the conference (via a presentation of Fernando Ferreira) and to make a contribution to this volume. We must also thank them for giving us the opportunity to render a public homage to Amílcar. We warmly salute Amílcar!

Both this paper and the *Techniques* paper (also written by the three of us) survey our work on the formalization of analysis in weak sub-exponential systems of second-order arithmetic (the theme of this paper) and on the metamathematical

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properties of these systems (the theme of the *Techniques* paper). This line of research was inaugurated by F. Ferreira in his doctoral dissertation [14], written in 1988 in *The Pennsylvania State University* under the supervision of Stephen Simpson. Three years before, Samuel Buss had submitted to Princeton University a doctoral dissertation [5] studying weak systems of arithmetic connected to well-known classes of computational complexity (polytime and polyspace computability, for instance). Simpson asked F. Ferreira to look into it and see if a conservation result regarding weak König's lemma could be proved in the weak setting. Of course, the project was motivated by similar results in reverse mathematics according to which the second-order system WKL_0 is Π^1_1-conservative over the base system RCA_0 (see [26] for this result and for reverse mathematics in general). It was also connected with the following challenge of Wilfried Sieg: Find a mathematically significant subsystem of analysis whose class of provably recursive functions consists only of the computationally "feasible" ones (see the end of [25]). In his dissertation, F. Ferreira re-worked Buss's arithmetical theory S^1_2 in terms of a system of binary words (finite 0-1 strings), thereby obtaining the first-order system Σ^b_1-NIA. The recast was done not only because the work with binary strings is more congenial with computational complexity, but also because it made the formulation of weak König's lemma very transparent. The dissertation proceeded model-theoretically and it was soon realized that, as opposed to the usual setting of reverse mathematics, weak König's lemma has various interesting formulations in the weak setting. It can be formulated by saying that every infinite binary set-tree (i.e., a tree in the range of the second-order variables of the weak system) has an infinite path, but this most natural form turns out to be insufficient for basic reversals (e.g., Theorem 7). It can also be formulated thus: For each bounded formula of arithmetic which defines an infinite binary tree (not necessarily a set in the weak system) there is an infinite path through the tree. The latter formulation of weak König's lemma is denoted by Σ^b_∞-WKL. Intermediate formulations do exist and are important for weak analysis (see Section 3). Stronger forms of weak König's lemma can also be envisaged, as it is the case with the principle of strict Π^1_1-reflection discussed in Section 6. Note that all of these different formulations of weak König's lemma collapse over RCA_0. F. Ferreira was able to prove that the theory BTFA together with Σ^b_∞-WKL is Π^0_2-conservative over Σ^b_1-NIA. The corresponding first-order conservation result is false because $\mathsf{BTFA} + \Sigma^b_\infty$-$\mathsf{WKL}$ proves bounded collection, and this form of collection is independent from Σ^b_1-NIA. The matter was finally clarified in [16], where it was shown that the first-order consequences of $\mathsf{BTFA} + \Sigma^b_\infty$-$\mathsf{WKL}$ are exactly the consequences of bounded collection over Σ^b_1-NIA. It was actually shown that $\mathsf{BTFA} + \Sigma^b_\infty$-$\mathsf{WKL}$ is first-order conservative over the theory Σ^b_1-$\mathsf{NIA} + \mathsf{B}\Sigma^b_\infty$, where $\mathsf{B}\Sigma^b_\infty$ is the bounded collection scheme. Unsurprisingly, the proof of this result used Harrington's forcing argument (N.B. It is sometimes thought that Harrington's conservation result can only be proved using a forcing technique whereas, in fact, it can also be proved by pure proof-theoretic means, as it was shown in [18] via a cut-elimination argument). Most of these matters are discussed in our *Techniques* paper, but in the above we wanted to convey to the reader how the results first appeared.

In F. Ferreira's doctoral dissertation there is little of analysis (nevertheless, e.g., the dissertation considers and studies the Heine/Borel theorem for the Cantor space). The formalization of the real number system and of continuous real functions was only worked out systematically in [11], together with António Fernandes. Section 2 below outlines this systematization. The section finishes with the proof of the intermediate value theorem in BTFA (Theorem 4). This result entails that Tarski's theory of real closed ordered fields RCOF is interpretable in BTFA. The interest of this remark is that F. Ferreira showed in the last section of [11] that BTFA is interpretable in Raphael Robinson's theory of arithmetic Q. We have, therefore, the following interesting relationship between the very weak arithmetical theory Q and the geometric theory RCOF: The first is not interpretable in the second (because Q is essentially undecidable and RCOF is a decidable theory by Tarski's theorem of quantifier-elimination), whereas the second is interpretable in the first. The theories Q and RCOF could not be more different from the metamathematical point of view, and this issue was the theme of the talk of F. Ferreira in the *Festschrift* (the title of the talk was "Arithmetic and geometry from the formal point of view: some notes, some lessons"). The fact that BTFA is interpretable in Q is *prima facie* quite amazing. It rests on the work of people like Edward Nelson, Alex Wilkie, Robert Solovay, Petr Hájek and others on interpretability in Q. This work was surveyed by F. Ferreira and G. Ferreira in the triptych paper "Interpretability in Robinson's Q," where an old unpublished argument of Solovay appeared for the first time. From this body of work it transpires that the interpretability of BTFA in Q is far from being a strange and isolated phenomena. It is a practical application of the rule of thumb according to which a bounded theory of arithmetic, short of proving the totality of exponentiation (actually, for the knowledgeable, short of $\mathsf{I}\Delta_0 + \Omega_\infty$), must be interpretable in Q. (Alex Wilkie proved that $\mathsf{I}\Delta_0 + \mathsf{exp}$ is *not* interpretable in Q, a result discussed in [20], but as far as we know it is still an open question whether $\mathsf{I}\Delta_0 + \Omega_\infty$ is interpretable in Q.) For instance, the *Interpretability* paper shows that a weak theory of analysis associated with polyspace computability (which contains the counting theory of Section 5, where Riemann integration can be developed) is interpretable in Q.

We have already disclosed a bit of the structure of this paper. In the next section, we show how to develop the basic notions of analysis in a weak base second-order theory. Section 3 considers various formulations of weak König's lemma and their relationships with basic theorems of analysis. In Section 4 we briefly discuss the Baire category theorem in the feasible setting. We answer a question of [9] and remark that Cohen's forcing does not preserve bounded collection in the absence of the totality of exponentiation. The following section strengthens the base theory to a theory in which counting is possible. We sketch how Riemann integration can be developed in this strengthened theory. The material of this section comes mainly from the doctoral dissertation of Gilda Ferreira [21] and the paper [19]. It is perhaps worth remarking at this point that recently Stephen Cook and Akitoshi Kawamura considered a polytime version of Weihrauch reducibility with the view of classifying the computational complexity of problems in analysis (cf. [7]). Even more recently, in the executive summary [3] of a conference on Weihrauch reducibility, the following passage can be read: "one could expect relations between weak complexity theoretic versions of arithmetic as studied by Fernando Ferreira *et al.*, on the one hand, and the polynomial-analogue of Weihrauch reducibility studied by Cook, Kawamura *et al.*, on the other hand." The relationships are certainly there, but they are perhaps more subtle than the known relationships between plain Weihrauch reducibility and ordinary reverse mathematics. We close the paper with a section where we pose three questions in weak analysis.

2. Groundwork for weak analysis

The theory BTFA (an acronym for 'Base Theory for Feasible Analysis') is a second-order theory of 0-1 strings. Its language directly describes finite binary words. The intended *standard* model has first-order domain ${}^{<\omega}2$ (the set of finite sequences of zeros and ones). Let $\mathcal{L}$ be the first-order language with three constant symbols ϵ (for the empty word), 0 and 1, two binary function symbols $\frown$ (for concatenation, usually omitted) and $\times$ (the intended interpretation of $x \times y$ is that of the word x concatenated with itself length of y times) and two binary relation symbols $=$ and $\subseteq$ (for *equality* and *initial subwordness*, respectively).

$\mathcal{L}_2$, the second-order language of BTFA, extends the language $\mathcal{L}$ with:

- second-order set variables X, Y, Z, $\ldots$ (we reserve lower-case roman variables x, y, z, $\ldots$ for first-order variables).
- a binary relation symbol $\in$ which infixes between a term t of $\mathcal{L}$ and a second-order variable.

A structure for $\mathcal{L}_2$ has domain (M, S) with M the domain of a structure for $\mathcal{L}$ and $S \subseteq \mathcal{P}(M)$. The first-order variables range over M and the second-order variables range over S. Note that we are allowing Henkin models, i.e., S need not be $\mathcal{P}(M)$. As it is well-known, second-order logic with Henkin semantics is essentially first-order. The *full* standard model for $\mathcal{L}_2$ has domain $({}^{<\omega}2, \mathcal{P}({}^{<\omega}2))$.

We denote by $x \preceq y$ (respectively, $x \equiv y$) the formula $1 \times x \subseteq 1 \times y$ (respectively, $1 \times x = 1 \times y$). In the standard model $x \preceq y$ (respectively, $x \equiv y$) expresses that the length of x is less than or equal (respectively, equal) to the length of y. By $l(x)$ we denote $1 \times x$ (the tally length of x). Quantifications of the form $\forall x\, (x \preceq t \to \ldots)$ and $\exists x\, (x \preceq t \wedge \ldots)$, usually abbreviated by $\forall x \preceq t\, (\ldots)$ and $\exists x \preceq t\, (\ldots)$, are called *bounded quantifications*. A *subword quantification formula* is a formula where all quantifications appear in the form $\forall x\, (x \subseteq^* t \to A)$ or in the form $\exists x\, (x \subseteq^* t \wedge A)$, where $\subseteq^*$ stands for *subwordness*, i.e., $x \subseteq^* t$ abbreviates $\exists z (z \frown x \subseteq t)$. A subword quantification can be seen as a very particular type of bounded quantification. Note also that a bounded quantification, over elements x such that $x \preceq t$, ranges over exponential many elements in the length of t, whereas a subword quantification over elements x such that $x \subseteq^* t$, ranges over polynomial many elements in the length of t.

Let us introduce some important classes of formulas. The Σ^b_1*-formulas* (respectively Π^b_1*-formulas*) are the formulas of the form $\exists y \preceq t\, A$, (respectively $\forall y \preceq t\, A$) where A is a subword quantification formula, possibly with parameters, and t is a term in which y does not occur. The *bounded formulas* (also known as the class of Σ^b_∞*-formulas*) are the formulas where all the quantifications are bounded (i.e., there are no second-order quantifications and all first-order quantifications are bounded). It is well-known that in the first-order language, the Σ^b_1-formulas define exactly the NP-predicates in the standard model of domain ${}^{<\omega}2$ (a detailed proof of this fact can be found in [14]); that, dually, the Π^b_1-formulas define the co-NP predicates; and that the Σ^b_∞-formulas define the predicates in the polytime hierarchy.

Definition 1. BTFA *is the second-order theory, in the language $\mathcal{L}_2$, which has the following axioms:*

- Basic axioms
  $x\epsilon = x$, $x(y0) = (xy)0$ and $x(y1) = (xy)1$;
  $x \times \epsilon = \epsilon$, $x \times y0 = (x \times y)x$ and $x \times y1 = (x \times y)x$;
  $x \subseteq \epsilon \leftrightarrow x = \epsilon$, $x \subseteq y0 \leftrightarrow x \subseteq y \vee x = y0$ and $x \subseteq y1 \leftrightarrow x \subseteq y \vee x = y1$;
  $x0 = y0 \to x = y$ and $x1 = y1 \to x = y$;
  $x0 \neq y1$, $x0 \neq \epsilon$ and $x1 \neq \epsilon$;
- Induction on notation scheme
  $$A(\epsilon) \wedge \forall x\,(A(x) \to A(x0) \wedge A(x1)) \to \forall x A(x),$$
  where A is a Σ^b_1-formula (possibly with first and second-order parameters);
- Bounded collection scheme ($\mathsf{B}\Sigma^b_\infty$)
  $$\forall x \preceq w \exists y A(x,y) \to \exists z \forall x \preceq w \exists y \preceq z A(x,y),$$
  where A is a bounded formula (possibly with first and second-order parameters) and z is a new variable;
- Comprehension scheme
  $$\forall x\,(\exists y A(x,y) \leftrightarrow \forall z B(x,z)) \to \exists X \forall x\,(x \in X \leftrightarrow \exists y A(x,y))$$
  where A is a Σ^b_1-formula and B is a Π^b_1-formula (possibly with first and second-order parameters) and X does not occur in A nor in B.

If, instead of $\mathcal{L}_2$, we take the language $\mathcal{L}$, the basic axioms together with the above scheme of induction on notation form the first-order theory Σ^b_1-NIA. It is well-known that Σ^b_1-NIA is equivalent to Buss's theory S^1_2 defined in his doctoral dissertation (see [23] for a formal interpretation of S^1_2 in Σ^b_1-NIA). Thus, by the celebrated witnessing theorem of Buss in his dissertation, the provably total functions of Σ^b_1-NIA (with Σ^b_1-graphs) are the polytime computable functions. The theory BTFA is Π^0_2-conservative over Σ^b_1-NIA and, therefore, these two theories are of the same proof-theoretic strength. It also follows that whenever $\mathsf{BTFA} \vdash \forall x \exists y A(x,y)$, where A is a Σ^b_1-formula, then there exists a polytime computable function $f : {}^{<\omega}2 \to {}^{<\omega}2$ such that $A(x, f(x))$, for all $x \in {}^{<\omega}2$. The reader can consult our *Techniques* paper for these and related results and for pointers to the original papers.

The structure $({}^{<\omega}2, \textit{Recursive Sets})$ is the smallest model of BTFA with the first-order part ${}^{<\omega}2$, and this fact may give the impression that the comprehension scheme is stronger than in fact is. The comprehension scheme states that recursive sets exist *under the condition* that their recursiveness is shown. In a weak theory like BTFA, this condition may be impossible to show. In any case, the *unbounded* quantifications in the antecedent of the comprehension scheme are extremely convenient for the development of analysis within BTFA. Since in this weak base system functions are formalized as *sets* of codes of ordered pairs, to show the existence of the composition of two functions $f : X \to Y$ and $g : Y \to Z$ we just need to write $(g \circ f)(x) = z$ in the following two $\exists/\forall$ forms: $(g \circ f)(x) = z$ iff $x \in X \wedge \exists y\,((x,y) \in f \wedge (y,z) \in g)$ iff $x \in X \wedge \forall y\,((x,y) \in f \to (y,z) \in g)$. A similar observation ensures the existence of the inverse image: $f(x) \in Z$ can be stated by $x \in X \wedge \exists y\,((x,y) \in f \wedge y \in Z)$ or by $x \in X \wedge \forall y\,((x,y) \in f \to y \in Z)$.

In the remainder of this section we formalize the basics of real analysis (e.g. the real number system, the notion of continuous function on the reals, etc.) in BTFA. This material is taken from [11], where more details can be found. We start by considering two sorts of natural numbers in BTFA: the tally numbers and the dyadic natural numbers.

The *tally numbers*, denoted by $\mathbb{N}_1$, are the binary words satisfying $x = 1 \times x$. These numbers are concatenations of 1s (tallies). We can define $0_{\mathbb{N}_1}$, $\leq_{\mathbb{N}_1}$, $+_{\mathbb{N}_1}$ and $\cdot_{\mathbb{N}_1}$ as ϵ, $\subseteq$, $\hat{}$ and $\times$ respectively. BTFA proves that $\mathbb{N}_1$ is an ordered semi-ring.

The *dyadic natural numbers*, denoted by $\mathbb{N}_2$, are the binary words satisfying $x = \epsilon \vee x = 1\hat{}y$ (with y a word). The idea is that the dyadic number $1x_1x_2 \cdots x_{n-1}$, where each x_i is 0 or 1, represents the number $2^{n-1} + \sum_{i=1}^{n-1} x_i 2^{n-i-1}$. The empty string ϵ represents the number zero. We can define $0_{\mathbb{N}_2}$, $\leq_{\mathbb{N}_2}$, $+_{\mathbb{N}_2}$ and $\cdot_{\mathbb{N}_2}$ in order to reproduce the usual operations of the natural numbers and show in BTFA that $\mathbb{N}_2$ is an ordered semi-ring.

The *dyadic rational numbers*, denoted by $\mathbb{D}$, are the triples (i, x, y) (coded as strings in a smooth way), with $i = 0$ or $i = 1$, $x \in \mathbb{N}_2$ and $y = \epsilon \vee y = z\hat{}1$ (with z a word). The idea is that the triple $(s, x_0 \ldots x_{n-1}, y_0 \ldots y_{m-1})$ represents the rational number $(-1)^s(\sum_{i=0}^{n-1} x_i 2^{n-i-1} + \sum_{j=0}^{m-1} \frac{y_j}{2^{j+1}})$. We denote such dyadic rational number by $\pm x_0x_1 \ldots x_{n-1} \cdot y_0 \ldots y_{m-1}$. By x^* we denote the binary word x with its rightmost zeros chopped off. Thus $\cdot x^*$ is a (positive) dyadic rational number. We define $0_{\mathbb{D}}$, $\leq_{\mathbb{D}}$, $+_{\mathbb{D}}$ and $\cdot_{\mathbb{D}}$ extending, to the dyadic rational numbers, the operations already mentioned in the dyadic natural numbers. Such operations reproduce the usual operations in the rational numbers and turn $\mathbb{D}$ in an ordered ring. The dyadic rational numbers do not form a field, but divisions by 2 are possible. We can also introduce in $\mathbb{D}$ the operations $-_{\mathbb{D}}$ and $|.|_{\mathbb{D}}$ with the expected meaning of subtraction and absolute value function, respectively.

We use the notations 2^n and 2^{-n}, with n a tally number, to stand for the dyadic rational numbers

$$+1\underbrace{00\ldots0}_{n\ zeros}\cdot\epsilon \quad \text{and} \quad +\epsilon\cdot\underbrace{00\ldots0}_{n-1\ zeros}1$$

respectively. Note that this exponential notation makes sense, even though BTFA does not prove the totality of exponentiation.

Definition 2. *A function* $\alpha : \mathbb{N}_1 \to \mathbb{D}$ *is a* real number if $|\alpha(n) - \alpha(m)| \leq 2^{-n}$ for all tallies n and m with $n \leq m$. Two real numbers α and β are said to be *equal* (written $\alpha = \beta$) if $\forall n \in \mathbb{N}_1\, |\alpha(n) - \beta(n)| \leq 2^{-n+1}$.

The above definition resembles the definition of real number in RCA_0. The point in need of attention is that in the feasible context the domain of the function α is $\mathbb{N}_1$. In systems in which the totality of exponentiation is provable, there is no difference between $\mathbb{N}_1$ and $\mathbb{N}_2$. More precisely, in BTFA it is possible to define (in a natural way) an embedding of $\mathbb{N}_1$ into $\mathbb{N}_2$, and the totality of exponentiation may be taken to affirm that this embedding is surjective. In short, in a system where the totality of exponentiation is not provable (like BTFA), $\mathbb{N}_1$ and $\mathbb{N}_2$ are essentially different entities.

A particular real numbers is a *set* (of ordered pairs) whose existence needs to be shown by the comprehension available in BTFA. The relation of real equality is defined by a formula of the form $\forall x A$, with A a Π^b_1-formula (it is a $\forall\Pi^b_1$-formula). A *dyadic real number* is a triple (i, x, X) with $i = 0$ or $i = 1$, $x \in \mathbb{N}_2$ and X an infinite binary sequence. Informally, the idea is that the triple $(s, x_0 \ldots x_{n-1}, X)$ gives the real number $(-1)^s(\sum_{i=0}^{n-1} x_i 2^{n-i-1} + \sum_{i=0}^{\infty} \frac{X(i)}{2^{i+1}})$, where $X(i)$ is the (i+1)-th bit of X ($i \in \mathbb{N}_1$). We denote dyadic real numbers by $\pm x_0x_1 \ldots x_{n-1} \cdot X$. Dyadic real numbers are a natural generalization of the dyadic rational numbers. It is shown in [12] that, over BTFA, the two alternative definitions of real numbers give the same numbers (this is far from straightforward, and in [11] it is deduced as a consequence of the definition of the value of a continuous function at a point of its domain: see a later comment on this issue). More precisely, to a dyadic real number $\pm x \cdot X$ we can associate the real number $\alpha_X : \mathbb{N}_1 \to \mathbb{D}$ given by $\alpha_X(n) := \pm x \cdot X[n]^*$ ($X[n]$ denotes the first n bits of X), and BTFA is able to prove that every real number α (according to Definition 2) is equal, as a real number, to a dyadic real number α_X (this is the hard part).

The arithmetical operations on the real numbers can be defined as follows:

- $\alpha + \beta$ is the real number $n \rightsquigarrow \alpha(n+1) + \beta(n+1)$
- $\alpha - \beta$ is the real number $n \rightsquigarrow \alpha(n+1) - \beta(n+1)$
- $\alpha \cdot \beta$ is the real number $n \rightsquigarrow \alpha(n+k) \cdot \beta(n+k)$, where k is the least *tally* such that $|\alpha(0)| + |\beta(0)| + 2 \leq 2^k$ (the symbol $\cdot$ is usually omitted)
- $\alpha \leq \beta$ is defined by the $\forall\Pi_1^b$-formula $\forall n(\alpha(n) \leq \beta(n) + 2^{-n+1})$
- $\alpha < \beta$ is defined by the formula $\alpha \leq \beta \wedge \alpha \neq \beta$ (which is equivalent to a $\exists\Sigma_1^b$-formula)
- $|\alpha|$ is the real number $n \rightsquigarrow |\alpha(n)|$,

and it is possible to prove (in BTFA) that the real numbers form an ordered field.

By $\forall \alpha \in \mathbb{R}\,(\ldots)$ or $\alpha \in [\beta, \gamma]$ we abbreviate $\forall \alpha\,($*if* α *is a real number then* $\ldots)$ or α *is a real number and* $\beta \leq \alpha \leq \gamma$, respectively. Note that the language of BTFA does not allow for the formation of sets of sets, and so $\mathbb{R}$ does not make literal sense in BTFA. With this proviso, there plainly exists a natural embedding of $\mathbb{D}$ into $\mathbb{R}$, by identifying each dyadic rational number x with the real number α_x defined by the constant function $\alpha_x(n) = x$, for all $n \in \mathbb{N}_1$.

In the following definition $(x, n)\Phi(y, k)$ can informally be seen as stating that the elements in the interval $]x - 2^{-n}, x + 2^{-n}[$ are applied under Φ into elements of the interval $[y - 2^{-k}, y + 2^{-k}]$.

Definition 3. *A* continuous partial function *from* $\mathbb{R}$ *to* $\mathbb{R}$ *is a set* Φ *of codes of quintuples* (*denoted by* $\langle w, x, n, y, k\rangle$) *satisfying:*

- *if* $\langle w, x, n, y, k\rangle \in \Phi$ *then* w *is a first-order element,* $x, y \in \mathbb{D}, n, k \in \mathbb{N}_1$
- *if* $(x, n)\Phi(y, k)$ *and* $(x, n)\Phi(y', k')$ *then* $|y - y'| \leq 2^{-k} + 2^{-k'}$
- *if* $(x, n)\Phi(y, k)$ *and* $(x', n') < (x, n)$ *then* $(x', n')\Phi(y, k)$
- *if* $(x, n)\Phi(y, k)$ *and* $(y, k) < (y', k')$ *then* $(x, n)\Phi(y', k')$,

where $(x, n)\Phi(y, k)$ *stands for* $\exists w\, \langle w, x, n, y, k\rangle \in \Phi$ *and* $(x', n') < (x, n)$ *abbreviates* $|x - x'| + 2^{-n'} < 2^{-n}$.

The above definition follows closely Simpson's definition of continuous function in the context of reverse mathematics. Simple examples of continuous functions (as sets of quintuples) can be given, like the constant functions, the identity function, the modulus of a function, and the sum and product of two functions (cf. [11] for details). Now we present some standard definitions:

- Let Φ be a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$. A real number α *is in the domain of* Φ, denoted by $\alpha \in \mathrm{dom}(\Phi)$, if
$$\forall k \in \mathbb{N}_1 \exists n \in \mathbb{N}_1 \exists x, y \in \mathbb{D}\,(|\alpha - x| < 2^{-n} \wedge (x, n)\Phi(y, k)).$$
- Let Φ be a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$, and let α be a real number in the domain of Φ. We say that a real number β *is the value of* α *under the function* Φ, denoted by $\Phi(\alpha) = \beta$, if
$$\forall x, y \in \mathbb{D}\, \forall n, k \in \mathbb{N}_1((x, n)\Phi(y, k) \wedge |\alpha - x| < \tfrac{1}{2^n} \to |\beta - y| \leq \tfrac{1}{2^k}).$$

In BTFA it is possible to prove that if Φ is a continuous partial function from $\mathbb{R}$ into $\mathbb{R}$ and $\alpha \in \mathrm{dom}(\Phi)$, then there is a unique real number β satisfying $\Phi(\alpha) = \beta$ (the uniqueness condition is easy to check and, of course, it refers to uniqueness with respect to equality of reals). The proof of this fact (see [11] pages 569-572) is very detailed and sensitive, making a strong use of classical logic. It is worth remarking that the real-number theoretic relations $\Phi(\alpha) = \beta$ and $\Phi(\alpha) \leq \beta$ are given by $\forall\Pi_1^b$-formulas, while the relation $\Phi(\alpha) < \beta$ is given by a $\exists\Sigma_1^b$-formula.

Theorem 4. *Let Φ be a continuous function total in the interval $[0,1]$ such that $\Phi(0) < 0 < \Phi(1)$. Then there is a real number $\alpha \in [0,1]$ such that $\Phi(\alpha) = 0$.*

Proof. If there is a dyadic rational number $x \in [0,1]$ such that $\Phi(\alpha_x) = 0$, the proof is done. Suppose this is not the case. Thus, the value of a dyadic rational number in the interval $[0,1]$ under Φ is strictly positive or strictly negative. With the comprehension available in BTFA it is possible to form the sets

$$X_1 = \{x : x \in \mathbb{D} \cap [0,1] \wedge \Phi(\alpha_x) < 0\} \text{ and } X_2 = \{x : x \in \mathbb{D} \cap [0,1] \wedge \Phi(\alpha_x) > 0\}$$

Note that the formulas defining the sets above, say $A_1(x)$ and $A_2(x)$ respectively, are both $\exists\Sigma_1^b$-formulas and are such that $\neg A_1(x)$ is equivalent to $A_2(x) \vee x \notin \mathbb{D} \cap [0,1]$ and $\neg A_2(x)$ is equivalent to $A_1(x) \vee x \notin \mathbb{D} \cap [0,1]$.

We now use a divide and conquer argument to construct a real number α such that $\Phi(\alpha) = 0$. Let

$$\begin{array}{rccc} f: & \mathbb{N}_1 & \to & \mathbb{D} \times \mathbb{D} \\ & n & \mapsto & \langle f_0(n), f_1(n) \rangle \end{array}$$

be the function, defined by bounded recursion along the tally part, by $f(0) = \langle 0, 1 \rangle$ and:

$$f(n+1) = \begin{cases} \langle (f_0(n) + f_1(n))/2, f_1(n) \rangle, & \text{if } (f_0(n) + f_1(n))/2 \in X_1; \\ \langle f_0(n), (f_0(n) + f_1(n))/2 \rangle, & \text{otherwise.} \end{cases}$$

By the induction on notation (for Σ_1^b-formulas) available in BTFA, it is possible to prove that, for all tally n, $f_0(n) \in X_1$, $f_1(n) \in X_2$, $f_0(n) \leq f_0(n+1)$, $f_1(n) \geq f_1(n+1)$, $f_0(n) < f_1(n)$ and $f_1(n) - f_0(n) = 2^{-n}$. But, then, f_0 and f_1 are real numbers such that $f_0 = f_1$. With $\alpha = f_0 = f_1$, we have that $\Phi(\alpha) = 0$. $\square$

Within BTFA, we can speak of polynomials of *tally* degree. Given $d \in \mathbb{N}_1$, a sequence $(\gamma)_{i \leq d}$ of real numbers of length $d+1$ is a function

$$F : \{i \in \mathbb{N}_1 : i \leq d\} \times \mathbb{N}_1 \to \mathbb{D}$$

such that, for every $i \leq d$, the function γ_i defined by $\gamma_i(n) = F(i,n)$ is a real number. A real polynomial $P(X)$ of (tally) degree d is just such a sequence with the proviso that $\gamma_d \neq 0$. As usual, we write $P(X) = \gamma_d X^d + \cdots + \gamma_1 X + \gamma_0$. It is not difficult to define smoothly $P(\alpha)$, for each real number α. (Note that if d was not tally then a^d, for $a \in \mathbb{N}_2$, would not make sense in general because BTFA does not prove that exponentiation is total. On the other hand a^d, with $a \in \mathbb{N}_2$ and $d \in \mathbb{N}_1$, is always a well-defined dyadic number.) It is even possible to show in BTFA that to each tally polynomial P as described, we can associate a total continuous function Φ_P such that, for each real number α, $\Phi_P(\alpha) = P(\alpha)$. As a *very particular* case of this discussion, it makes sense to speak in BTFA of polynomials of *standard* degree and to see that they are given by continuous functions in the sense of Definition 3. It should now be clear, using the intermediate value theorem, that – as observed in the introduction – Tarski's theory of real closed ordered fields RCOF is interpretable in BTFA.

3. The role of weak König's lemma

Given A a formula of $\mathcal{L}_2$ and x a first-order variable, we denote by $Tree(A_x)$ the formula:

$$\forall x \forall y\,(A(x) \wedge y \subseteq x \to A(y)) \wedge \forall n \in \mathbb{N}_1 \exists x\,(l(x) = n \wedge A(x)).$$

The first condition expresses that initial subwords of words satisfying A still satisfy A, and the second condition says that there are binary sequences that satisfy A of arbitrarily large tally length (the tree given by the formula $A(x)$ is infinite). Given X a second-order variable, we denote by $Path(X)$ the formula:

$$Tree((x \in X)_x) \wedge \forall x \forall y\,(x \in X \wedge y \in X \to x \subseteq y \vee y \subseteq x).$$

Weak König's lemma for trees defined by bounded formulas, denoted by Σ^b_∞-WKL, is the following scheme:

$$Tree(A_x) \to \exists X\,(Path(X) \wedge \forall x\,(x \in X \to A(x))),$$

where A is a bounded formula and X is a fresh variable. Informally it says that every infinite binary tree (defined by a bounded formula) has an infinite path. Note that, while the tree needs not to be a set in the system, the path is a set.

Theorem 5. BTFA $+ \Sigma^b_\infty$-WKL *is* Π^1_1*-conservative over* BTFA*.*

A consequence of this theorem is that the class of provably total functions (with Σ^b_1-graphs) of BTFA $+ \Sigma^b_\infty$-WKL is still the class of polytime computable functions. Even though BTFA $+ \Sigma^b_\infty$-WKL has the same proof-theoretic strength as BTFA, weak König's lemma is a useful form of a compactness principle that increases the demonstrative power of BTFA. In what follows, we denote by Π^b_1-WKL the Σ^b_∞-WKL scheme restricted to Π^b_1-formulas. By WKL we denote the Σ^b_∞-WKL scheme above restricted to sets i.e., to formulas $A(x)$ of the form $x \in X$.

Definition 6 (BTFA)**.** *An* open set *of* $\mathbb{R}$ *is a set* U *of codes of triples of the form* $\langle w, z, n\rangle$ *such that* w *is a first-order element,* $z \in \mathbb{D}$ *and* $n \in \mathbb{N}_1$*. We say that a real number* α *is an element of* U*, and we write* $\alpha \in U$*, if*

$$\exists z \in \mathbb{D}\, \exists n \in \mathbb{N}_1\,(|\alpha - z| < \frac{1}{2^n} \wedge \exists w\, \langle w, z, n\rangle \in U).$$

The formulation of open set may appear unfamiliar at first sight, but it merely says that the open sets of $\mathbb{R}$ are given by countable unions of the form

$$\bigcup_{w} \bigcup_{\substack{z,n \\ \langle w,z,n\rangle \in U}}]z - \frac{1}{2^n}, z + \frac{1}{2^n}[.$$

The Heine/Borel theorem for the closed unit interval says that if U is an open set such that $[0,1] \subseteq U$ (i.e., every real number in the closed unit interval is an element of U), then there is $k \in \mathbb{N}_1$ such that: For all $\alpha \in [0,1]$, there are w, $z \in \mathbb{D}$, $n \in \mathbb{N}_1$, all of length less than k, such that $|\alpha - z| < \frac{1}{2^n}$ and $\langle w, z, n\rangle \in U$.

Theorem 7 (BTFA)**.** *The Heine/Borel theorem for* $[0,1]$ *is equivalent to* Π^b_1-WKL*.*

We are not going to prove this result here. The proof adapts well-known arguments of reverse mathematics and the details can be found in [12]. Instead, we will try to convey to the reader the need of Π^b_1-WKL instead of just plain WKL. How does the argument from the right side to the left side goes? It hinges on considering a tree given by a formula $T(x)$ of the form $\forall y\,(\langle x, y\rangle \in X)$, for a certain *set* X. The tree is proven to be bounded by contradiction. If it were infinite, it would have an infinite path and this gives rise to absurdity. Now, and this is the critical point, how can we conclude from the infinitude of the tree that it has an infinite path? It is well-known that we can associate to the above tree $T(x)$ the tree $T'(x) :\equiv \forall w \subseteq x \forall y \preceq x\,(\langle w, y\rangle \in X)$. Moreover, T' is infinite if T is, and any path through T' is also a path through T. We apply weak König's lemma to T'. This tree is defined by a Π^b_1-formula, and this is why plain WKL is not enough in the feasible setting. Over RCA_0 the problem does not arise because the formulas in the comprehension scheme of RCA_0 are closed under bounded quantifications.

The above result shows the need for fine-tuned versions of weak König's lemma when doing reverse mathematics over a feasible base theory.

Definition 8. *Let $\Phi : [0, 1] \to \mathbb{R}$ be a continuous total function. We say that Φ is* uniformly continuous *if*

$$\forall k \in \mathbb{N}_1 \exists m \in \mathbb{N}_1 \forall \alpha, \beta \in [0, 1](|\alpha - \beta| \leq \tfrac{1}{2^m} \to |\Phi(\alpha) - \Phi(\beta)| < \tfrac{1}{2^k}).$$

Proposition 9. (BTFA) *Let $\Phi : [0, 1] \mapsto \mathbb{R}$ be a uniformly continuous function. Then there is $n \in \mathbb{N}_1$ such that, for all $\alpha \in [0, 1]$, $|\Phi(\alpha)| \leq 2^n$.*

Proof. Take $m \in \mathbb{N}_1$ such that if $|\alpha - \beta| \leq 2^{-m}$ then $|\Phi(\alpha) - \Phi(\beta)| < 1$, for $\alpha, \beta \in [0, 1]$. It is easy to see, by bounded collection, that there is $r \in \mathbb{N}_1$ such that $\forall x\,(\ell(x) = m \to |\Phi(.x^*)| < 2^r)$. Since every real in the closed unit interval is within 2^{-m} of a certain $.x^*$ for x of length m, it is clear that $n = r + 1$ does the job. $\square$

In the next result there is a gap that we are unable to fill.

Theorem 10 (BTFA)**.** *The principle that every total real valued continuous function defined on $[0, 1]$ is uniformly continuous implies WKL and is implied by Π^b_1-WKL.*

Proof. To prove the first assertion, suppose that WKL is not valid. Take T an infinite binary (set) tree which has no infinite paths. In BTFA it is possible to prove that there is a total continuous function Φ, defined on $[0, 1]$ such that, for all end nodes x of T, we have $\Phi(\cdot x^*) = 2^{l(x)}$ (see [12], pages 5-6 for a proof of a more general result). Since T has nodes of arbitrarily large length, Φ is unbounded. This contradicts Proposition 9.

To prove the second assertion we reason within $\mathsf{BTFA} + \Pi^b_1$-WKL. Let Φ be a (total) real valued continuous function on [0,1] and fix $k \in \mathbb{N}_1$. Let U be the open set defined by $\{\langle\langle w, y\rangle, x, n+1\rangle : \langle w, x, n, y, k+2\rangle \in \Phi\}$. Since Φ is a total function, it can be proved that $[0, 1] \subseteq U$. By Theorem 7, we have the Heine/Borel theorem. Thus, there is $m \in \mathbb{N}_1$ such that: For all $\alpha \in [0, 1]$, there are $x, y \in \mathbb{D}$, $n \in \mathbb{N}_1$, all of length less than m, such that $|\alpha - x| < \frac{1}{2^{n+1}}$ and $(x, n)\Phi(y, k + 2)$. We claim that, for all $\alpha, \beta \in [0, 1]$, if we have $|\alpha - \beta| \leq \frac{1}{2^m}$ then $|\Phi(\alpha) - \Phi(\beta)| < \frac{1}{2^k}$. Take α, β as in the claim. Take $x, y \in \mathbb{D}$ and $n \in \mathbb{N}_1$ with $n <_{\mathbb{N}_1} m$, $|\alpha - x| < \frac{1}{2^{n+1}}$ and $(x, n)\Phi(y, k + 2)$. By definition of continuous function, we have $|\Phi(\alpha) - y| \leq \frac{1}{2^{k+2}}$. Now, $|\beta - x| \leq |\beta - \alpha| + |\alpha - x| < \frac{1}{2^m} + \frac{1}{2^{n+1}} \leq \frac{1}{2^n}$. Hence, by definition of continuous function again, $|\Phi(\beta) - y| \leq \frac{1}{2^{k+2}}$. We conclude that $|\Phi(\alpha) - \Phi(\beta)| \leq \frac{1}{2^{k+1}} < \frac{1}{2^k}$. $\square$

Before we finish this section with a last result, we need to make some brief considerations concerning induction. The induction available in BTFA is induction *on notation* for Σ^b_1-formulas. In BTFA we can introduce, in a natural way, a *successor function* S defined by: $S(\epsilon) = 0$, $S(x0) = x1$ and $S(x1) = S(x)0$, i.e., we order the binary words according to length and, within the same length, lexicographically, and S is the successor function induced by this order (usually denoted by $\leq_l$, l for lexicographic order).

The scheme of "slow" induction is $A(\epsilon) \wedge \forall x\,(A(x) \to A(S(x))) \to \forall x A(x)$, where A is a Σ^b_1-formula. It corresponds to the usual $+1$ scheme of induction in ordinary theories of arithmetic (it gives Buss's theory T^1_2). This scheme does not seem to be available in BTFA (in [24] it is shown that if this is the case – actually if Buss's theories S^1_2 and T^1_2 are the same – then the polytime computable hierarchy collapses).

Theorem 11. *Over* $\mathsf{BTFA} + \Sigma^b_\infty$-$\mathsf{WKL}$*, the following are equivalent:*

(a) *Every continuous real valued function defined on* $[0, 1]$ *has a maximum.*
(b) *Every continuous real valued function defined on* $[0, 1]$ *has a supremum.*
(c) *Slow induction for* Σ^b_1*-formulas.*

See [12] for a detailed proof. Here we just sketch the strategy to prove that (b) implies (c). We consider (c) in the following (BTFA equivalent) formulation: every non-empty set of binary words of equal length has a lexicographically greatest element. It is easy to construct a piecewise linear continuous real valued function Φ defined on [0,1] such that, if $x \in X$ then $\Phi(\cdot x^*) = \cdot x^*$, and if $x \notin X$ then $\Phi(\cdot x^*) = 0$. By hypothesis (b), Φ has a supremum, from which we can extract the desired maximum. The proof that (c) implies (a) is much more involved and requires (first) the formation of an infinite tree defined by a Π^0_1-formula so that an infinite path Y through it satisfies $\forall \alpha \in [0, 1](\Phi(\alpha) \leq \cdot Y)$ and (second) to prove, using Σ^b_∞-WKL again, that there is $\alpha \in [0, 1]$ such that $\Phi(\alpha) = \cdot Y$.

4. A brief digression on the Baire category theorem

Let us consider in BTFA the following formulation of the Baire category theorem for the Cantor space:

$$\forall z \forall a \exists x (a \subseteq x \wedge A(x, z)) \to \exists X (Path(X) \wedge \forall z \exists x\,(x \in X \wedge A(x, z)),$$

where A is an arithmetical formula (possibly with first and second-order parameters). The above scheme says that if an arithmetical formula $A(x, z)$ defines a countable family $D^A_z := \{x : A(x, z)\}$ of dense open sets in the Cantor space, then there is a path X intersecting all these sets. Let us call Π^0_∞-BCT this form of the Baire category theorem. It is well-known that $\mathsf{RCA}_0 + \Pi^0_\infty$-$\mathsf{BCT}$ is Π^1_1-conservative over RCA_0. This was shown by Simpson and Douglas Brown in [4] using Cohen forcing. Fernandes asked in [9] whether the theory $\mathsf{BTFA} + \Pi^0_\infty$-$\mathsf{BCT}$ is Π^1_1-conservative over BTFA. The answer is negative by a wide margin:

Theorem 12. *The theory* $\mathsf{BTFA} + \Pi^0_\infty$-$\mathsf{BCT}$ *proves the totality of exponentiation.*

Remark 1. *We will use the fact that in* BTFA *the totality of exponentiation can be formulated in the following manner:* $\forall a \exists c \forall z \preceq a\,(z \subseteq^* c)$*. See* [15].

Proof. We reason within $\mathsf{BTFA} + \Pi^0_\infty$-$\mathsf{BCT}$. Clearly, $\forall z \forall a \exists x\,(a \subseteq x \wedge z \subseteq^* x)$. By Π^0_∞-BCT, there is an infinite path X such that $\forall z \exists x\,(x \in X \wedge z \subseteq^* x)$. Let us now show that exponentiation is total. Consider an arbitrary element a. Since we have $\forall z \preceq a\, \exists x\, (x \in X \wedge z \subseteq^* x)$, by bounded collection we know that there is b such that $\forall z \preceq a\, \exists x \preceq b\, (x \in X \wedge z \subseteq^* x)$. Now, take c such that $c \equiv b$ (c has the same length as b) and $c \in X$. It is clear that $\forall z \preceq a\, (z \subseteq^* c)$. We are done. □

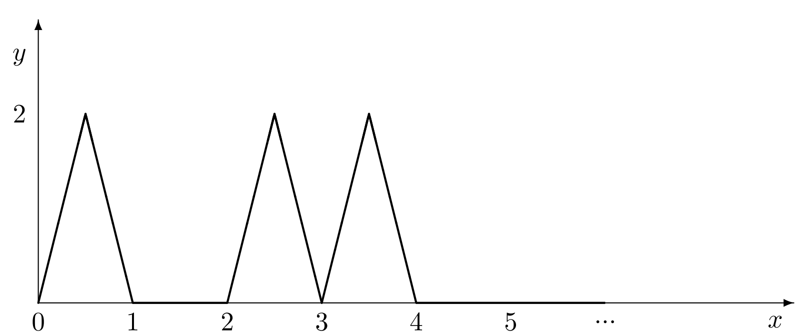
The above proof is implicit in the final section of [9], but it is brought here into the open for the first time. The use of bounded collection in the above argument is crucial. The situation seems to be different if bounded collection is not included. In fact, Takeshi Yamazaki in [28] and Fernandes in [9] studied Π^0_∞-BCT over a feasible theory weaker than BTFA. The weaker second-order theory that they considered is Σ^b_1-$\mathsf{NIA} + \nabla^b_1$-$\mathsf{CA}$, where ∇^b_1-CA is the following comprehension scheme:

$$\forall x\, (A(x) \leftrightarrow B(x)) \rightarrow \exists X \forall x\, (x \in X \leftrightarrow A(x)),$$

where A is a Σ^b_1-formula and B is a Π^b_1-formula (possibly with first and second-order parameters) and X is a fresh variable. This theory allows only the formation of NP $\cap$ co-NP sets and does *not* include bounded collection. Yamazaki showed that Σ^b_1-$\mathsf{NIA} + \nabla^b_1$-$\mathsf{CA} + \Pi^0_\infty$-$\mathsf{BCT}$ is Π^1_1-conservative over Σ^b_1-$\mathsf{NIA} + \nabla^b_1$-$\mathsf{CA}$ and Fernandes improved this conservation result to include uniqueness statements of the form $\forall X \exists! Y A(X, Y)$, for A arithmetical. Their proofs also use Cohen forcing. Brown and Simpson showed that Cohen forcing preserves Σ^0_1-induction but, as follows from Theorem 12, Cohen forcing does not preserve bounded collection (at least in the absence of the totality of exponentiation). On the other hand, Harrington's forcing preserves both bounded collection and Σ^0_1-induction.

5. Riemann integration and the theory TCA^2

How far can we go in the formalization of analysis in feasible systems? In [17] it is shown that if we are able to do a minimum of integration in BTFA, then it follows that we can count. This looks unsurprising. Given $X \subseteq \mathbb{N}_2$, we can associate (within BTFA) a continuous total function $\Phi_X : [0, \infty[\rightarrow \mathbb{R}$, with a modulus of uniform continuity (see Definition 20), in such a way that, given $w \in \mathbb{N}_2$, $\int_0^w \Phi_X(x)dx$ is the number of elements of X up to w. For instance, if $X = \{0, 2, 3\}$ then Φ_X is the function:



There are, however, some technical difficulties in pulling the above argument through in BTFA. The difficulties can, nevertheless, be met: see [17].

As mentioned in the introduction, in her doctoral dissertation (in Portuguese) G. Ferreira showed in detail how Riemann integration can be developed in a theory whose class of provably total functions is the *hierarchy of counting functions* (a computational complexity class between PTIME and PSPACE). The theme is also the subject of the paper [19]. In the rest of this section we are going to sketch why/how such formalization is possible, focusing on the logical problems of the formalization.

By $\exists X \preceq t\, A$ we abbreviate the formula $\exists X(X \preceq t \wedge A)$, where $X \preceq t$ means that every word in X has length less than or equal to the length of t. The abbreviation $\forall X \preceq t\, A$ has the corresponding dual meaning. These are the *bounded second-order quantifiers.* The $\Sigma_1^{1,b}$*-formulas* (respectively $\Pi_1^{1,b}$*-formulas*) are the formulas of the form $\exists X \preceq t\, A$, (respectively $\forall X \preceq t\, A$) where A is bounded formula. A $\Sigma_\infty^{1,b}$*-formulas* is a formula of $\mathcal{L}_2$ where all the first and second-order quantifications are bounded. This class of formulas constitutes a natural generalization of the bounded formulas.

Before presenting the *counting axiom*, which is crucial in our theory for integration, let us motivate it. Informally, given $X \preceq w$ and $y \preceq w$ we want to be able to count the number of elements $\leq_l y$ which are in X. Let f be such that $f(\epsilon) = \epsilon$, if $\epsilon \notin X$, $f(\epsilon) = 0$, if $\epsilon \in X$, and

$$f(S(x)) = \begin{cases} f(x) & \text{if } S(x) \notin X \\ S(f(x)) & \text{if } S(x) \in X \end{cases}$$

It is clear that $f(y)$ gives the result (in the $\leq_\ell$-order) of the above counting. Formally, the *counting axiom* has the form:

$$\forall X \preceq w \exists C \preceq q(w)\ Count(C, X),$$

where $q(w)$ is a certain term which depends on the variable w (for the record, the term $q(w)$ can be taken to be $wwww1111$, cf. [21]), and $Count(C, X)$ is a bounded formula (we omit it) which says the following: given $y \preceq w$, $\langle y, j\rangle \in C$ if and only if $f(y) = j$. For the formulation of the axiom, with the exact expression for $Count$, see [19, 21]. It is easy to see that this counting axiom also permits to do counting with the result given in $\mathbb{N}_2$. This is the usual counting and it is this counting that we will use from now on.

Definition 13. TCA^2 (*acronym for* Theory for Counting Arithmetic) *is the second-order theory in the language* $\mathcal{L}_2$ *which has the following axioms:*

- Basic axioms (*the previous 14 basic axioms of Definition 1);*
- Induction on notation for bounded formulas

  $$A(\epsilon) \wedge \forall x\, (A(x) \to A(x0) \wedge A(x1)) \to \forall x A(x),$$

  with A *a bounded formula* (*first and second-order parameters are permitted*)*;*
- Substitution for bounded formulas (*This is a technical axiom scheme that permits a kind of "permutation" between a bounded first-order universal quantification and a bounded second-order existential quantification. See* [19] *for its formulation. This axiom is instrumental in showing that* $\Sigma_1^{1,b}$*-formulas are provably closed under bounded first-order quantifications. A dual property holds for* $\Pi_1^{1,b}$*-formulas. We use these properties without much ado.*)*;*
- Counting axiom

  $$\forall X \preceq w \exists C \preceq q(w)\ Count(C, X)$$

- Extended bounded collection scheme ($\mathsf{B}^1\Sigma^{1,b}_\infty$)

$$\forall X \preceq w \exists y A(y, X) \to \exists z \forall X \preceq w \exists y \preceq z A(y, X),$$

where A is a $\Sigma^{1,b}_\infty$-formula (possibly with first and second-order parameters) and z is a new variable;

- Recursive comprehension scheme

$$\forall x\, (\exists y A(x, y) \leftrightarrow \forall z B(x, z)) \to \exists X \forall x\, (x \in X \leftrightarrow \exists y A(x, y))$$

where A is a $\Sigma^{1,b}_1$-formula and B is a $\Pi^{1,b}_1$-formula (possibly with first and second-order parameters) and X is a fresh variable.

The class of provably total functions of TCA^2 with $\Sigma^{1,b}_1$-graphs is exactly the computational complexity class FCH. For a proof of this result see [21, 22]. For a general blueprint of how to construct theories for weak analysis related to concrete computational complexity classes see [13]. (The FCH class consists of the *hierarchy of counting functions*. It is a computational complexity class which lies between PTIME and PSPACE introduced by Klaus Wagner [27] in 1986. More precisely, FCH is $\bigcup_{i\geq 0} i\#\mathsf{P}$, where $0\#\mathsf{P} = \mathsf{P}$ and $(i+1)\#\mathsf{P} = \#\mathsf{P}^{i\#\mathsf{P}}$, for $i \geq 0$, i.e. $(i+1)\#\mathsf{P}$ is the class of functions that "count" the number of accepting computations in a polynomial time nondeterministic Turing Machine, permitting a function in $i\#\mathsf{P}$ as an oracle.)

Before we start developing Riemann integration, we need to lay out a proposition indicating the forms of induction and minimization that are available in TCA^2. We will use them at will.

Proposition 14. *The following is provable in TCA^2 (cf. [21]):*

- *induction on notation for $\Delta^{1,b}_1$-formulas,*
- *plain $(+1)$ induction on $\mathbb{N}_2$ for $\Delta^{1,b}_1$-formulas,*
- *the minimization scheme $\exists x\, A(x) \to \exists x\, (A(x) \wedge \forall y <_l x \neg A(y))$, for $\Delta^{1,b}_1$-formulas A.*

A crucial step towards integration is the ability to sum. It is not difficult to see that the ability of counting is sufficient to perform sums. Informally, we want to show that given $f : X \times \mathbb{N}_2 \to \mathbb{D}$ there is (in TCA^2) a function $\sum_f : X \times \mathbb{N}_2 \to \mathbb{D}$ such that $\sum_f(x, n) = f(x, 0) + \ldots + f(x, n)$. $\sum_f(x, n)$ will be denoted by $\sum_{i=0}^n f(x, i)$. We start with a preliminary lemma.

Lemma 15. *Let f be a function from $X \times \mathbb{N}_2$ to $\mathbb{N}_2$. Then there is a function $g : X \times \mathbb{N}_2 \to \mathbb{N}_2$ such that $\forall x \in X \forall n \in \mathbb{N}_2 \forall i \leq n\, (f(x, i) \leq g(x, n))$.*

Proof. Let us fix $x \in X$ and $n \in \mathbb{N}_2$. By bounded collection it is easy to see that $\exists r \forall i \leq n\, (f(x, i) \leq r)$. Let ϕ be the bounded formula $\forall i \leq n\, (f(x, i) \leq r)$. Since $\forall x \in X \forall n \in \mathbb{N}_2 \exists r \in \mathbb{N}_2\, \phi(x, n, r)$, we can apply minimization and have

$$\forall x \in X \forall n \in \mathbb{N}_2 \exists r\, (\phi(x, n, r) \wedge \forall r' < r \neg\phi(x, n, r')).$$

Thus $g := \{\langle\langle x, n\rangle, r\rangle : x \in X \wedge n \in \mathbb{N}_2 \wedge r \in \mathbb{N}_2 \wedge \phi(x, n, r) \wedge \forall r' < r \neg\phi(x, n, r')\}$ is a function from $X \times \mathbb{N}_2$ to $\mathbb{N}_2$ satisfying the desired condition. $\square$

Theorem 16. *Given $f : X \times \mathbb{N}_2 \to \mathbb{N}_2$, there is a function $\Sigma_f : X \times \mathbb{N}_2 \to \mathbb{N}_2$ such that $\forall x \in X \forall n \in \mathbb{N}_2\, [\Sigma_f(x, 0) = f(x, 0) \wedge \Sigma_f(x, n+1) = \Sigma_f(x, n) + f(x, n+1)]$.*

Proof. Given $x \in X$ and $n \in \mathbb{N}_2$, let Z be the set

$$\{u : \exists i, r \preceq u\, (u = \langle r, i\rangle \wedge i \leq n \wedge r < f(x, i))\},$$

where we suppose that the pairing function (in $\mathbb{N}_2$) is monotone (in the sense of $\preceq$) in both arguments and is such that $r, i \preceq \langle r, i\rangle$. Informally, the idea is that $\Sigma_f(x, n)$ is the number of elements in Z. Note that from Lemma 15, there is a function g such that $u \in Z \to u \preceq \langle g(x, n), n\rangle$, i.e., Z is a bounded set. Let $w := \langle g(x, n), n\rangle$. By the counting axiom in $\mathbb{N}_2$, for a concrete term $q(w)$ there is $C \preceq q(w)$ such that $\langle u, j\rangle \in C$ iff there are j elements in $\mathbb{N}_2$ less than or equal to u in Z. Let $P(Z)$ abbreviate: $\forall u \preceq w\, (u \in Z \leftrightarrow \exists i, r \preceq u\, (u = \langle r, i\rangle \wedge i \leq n \wedge r < f(x, i)))$. Take Σ_f as

$$\{\langle\langle x, n\rangle, s\rangle : x \in X \wedge n \in \mathbb{N}_2 \wedge \exists Z \preceq w \exists C \preceq q(w)\, (P(Z) \wedge Count(C, Z) \wedge \langle w, s\rangle \in C)\}.$$

It can be seen that the set Σ_f exists in TCA^2. This uses the counting axiom and the recursion comprehension scheme. Note that the latter condition above is equivalent to $\forall Z \preceq w \forall C \preceq q(w)\, (P(Z) \wedge Count(C, Z) \to \langle w, s\rangle \in C)$. Clearly, Σ_f defines a function satisfying the desired conditions. □

It is not difficult to see that the above result can be extended and the following proposition proved:

Proposition 17. *Given $f : X \times \mathbb{N}_2 \to \mathbb{D}$, there is a function $\Sigma_f : X \times \mathbb{N}_2 \to \mathbb{D}$ s.t. $\Sigma_f(x, 0) = f(x, 0)$ and $\Sigma_f(x, n+1) = \Sigma_f(x, n) + f(x, n+1)$, $\forall x \in X, \forall n \in \mathbb{N}_2$.*

Note that the relation $z = \sum_{i=0}^{n} f(x, i)$ is $\Delta_1^{1,b}$-definable. The usual properties of summations can be proved by plain induction on $n \in \mathbb{N}_2$. We will not make a list of them and assume that the new notations that will occur in the computations below can be given a straightforward sense in TCA^2. All the details can be found in G. Ferreira's doctoral dissertation [21].

Given α a real number and $n \in \mathbb{N}_1$, the dyadic rational number $\alpha(n)$ is well determined. Note that the formula $\alpha(n) =_{\mathbb{D}} d$ is a bounded formula, since it abbreviates $\langle n, d\rangle \in \alpha$. However, given Φ a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$ and $\alpha \in \mathrm{dom}(\Phi)$, although the expression $\Phi(\alpha)$ is well-defined, the expression $\Phi(\alpha)(n)$ is ambiguous. This is because $\Phi(\alpha)$ is only defined modulo equality of the reals. In order to control the complexity of the formula which defines the integral, it is necessary to introduce the expression $\Phi(\alpha, n)$ that gives a canonical representative of $\Phi(\alpha)$. This is possible to do in TCA^2 but apparently not in BTFA (because of lack of minimization).

Let Φ be a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$ and α a real number in the domain of Φ. Consider $\varphi(n, r)$ the formula

$$\exists w \exists k\, [\langle w, \alpha(k+1), k, r, n+1\rangle \in \Phi \wedge$$
$$\forall \langle r', w', k'\rangle < \langle r, w, k\rangle\, (\langle w', \alpha(k'+1), k', r', n+1\rangle \notin \Phi)].$$

It can be proved that, given $n \in \mathbb{N}_1$, there exists a unique $r \in \mathbb{D}$ such that $\varphi(n, r)$. The proof of this result (see [19], pages 926–927) uses minimization.

Definition 18. *Given Φ a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$ and α a real number in the domain of Φ, we define $\Phi(\alpha, n) = r :\leftrightarrow \varphi(n, r)$.*

Note that $\Phi(\alpha, n)$ is the unique dyadic rational number such that $\varphi(n, \Phi(\alpha, n))$ and that $\{\langle n, r\rangle : \Phi(\alpha, n) = r\}$ is a set in TCA^2.

Proposition 19. *Let Φ be a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$ and let $\alpha \in \mathrm{dom}(\Phi)$. The function $\lambda : \mathbb{N}_1 \to \mathbb{D}$ defined by $\lambda(n) = \Phi(\alpha, n)$ is a real number. Moreover, for every real number β such that $\Phi(\alpha) = \beta$, we have $|\Phi(\alpha, n) - \beta| \leq \frac{1}{2^n}$. In particular, $\Phi(\alpha) = \lambda$.*

The theory TCA^2 can formalize Riemann integration for functions with a modulus of uniform continuity.

Definition 20. *Let $\Phi : [0,1] \to \mathbb{R}$ be a continuous total function. A* modulus of uniform continuity (m.u.c.) for Φ *is a strictly increasing function h from $\mathbb{N}_1$ to $\mathbb{N}_1$ such that for all $n \in \mathbb{N}_1$ and for all $\alpha, \beta \in [0,1]$, if $|\alpha - \beta| < 2^{-h(n)}$ then $|\Phi(\alpha) - \Phi(\beta)| < 2^{-n}$.*

In order to simplify notation, we start by introducing the notion of Riemann integral for functions restricted to the interval $[0,1]$.

Definition 21. *Take Φ a continuous total function in the interval $[0,1]$, with a modulus of uniform continuity h in that interval. We define* the definite integral *between 0 and 1 of Φ, denoted by $\int_0^1 \Phi(t)\, dt$, in the following way:*

$$\int_0^1 \Phi(t)\, dt :=_{\mathbb{R}} \lim S_n$$

where, for all $n \in \mathbb{N}_1$, $S_n = \sum_{i=0}^{2^{h(n)}-1} \frac{1}{2^{h(n)}} \Phi(\frac{i}{2^{h(n)}}, n)$.

Note that, $f : \mathbb{N}_1 \times \mathbb{N}_2 \to \mathbb{D}$, defined by $f(n,i) = \frac{1}{2^{h(n)}} \Phi(\frac{i}{2^{h(n)}}, n)$ is a function in TCA^2. Also observe that it is possible to consider sums of the form $\sum_{i=0}^{2^{h(n)}-1} f(n,i)$, for f a function from $\mathbb{N}_1 \times \mathbb{N}_2$ to $\mathbb{D}$. In fact, $\sum_{i=0}^{2^{h(n)}-1} f(n,i) = \Sigma_f(n, 2^{h(n)} - 1)$.

Of course, we need to make sense of the limit above. One way to do this is to develop a theory of limits within TCA^2 and prove that Cauchy sequences, with a modulus of Cauchy convergence, have limits. This is done in [21]. In here we rely on the fact (also proved in [21]) that α defined by $\alpha(n) = S_{n+5}$ is a real number. We just let the above limit to be α. It can be proved (see [21], pages 122–123) that the value of the integral (as a real number) does not depend on the function chosen as a modulus of uniform continuity.

Some of the usual properties of the integral can be established in TCA^2 (see [21], pages 123–126). For instance:

Proposition 22. *Let Φ and Ψ be continuous total functions in the interval $[0,1]$ with a modulus of uniform continuity in that interval and $\gamma \in \mathbb{R}$:*

a) $\int_0^1 \gamma\, dt = \gamma$
b) $\int_0^1 t\, dt = \frac{1}{2}$
c) $\int_0^1 (\Phi + \Psi)(t)\, dt = \int_0^1 \Phi(t)\, dt + \int_0^1 \Psi(t)\, dt$
d) $|\int_0^1 \Phi(t) dt| \leq \int_0^1 |\Phi|(t) dt$
e) If $\Phi(t) = \Psi(t)$ *for all* $t \in [0,1]$ *then* $\int_0^1 \Phi(t)\, dt = \int_0^1 \Psi(t)\, dt$
f) If $\Phi(t) \leq \Psi(t)$ *for all* $t \in [0,1]$ *then* $\int_0^1 \Phi(t)\, dt \leq \int_0^1 \Psi(t)\, dt$
g) $\int_0^1 \gamma\Phi(t)\, dt = \gamma \int_0^1 \Phi(t)\, dt$.

We can (as we sketch next) introduce the notion of Riemann integral with arbitrary dyadic rational limits in an analogous way.

Definition 23. *Take $x, y \in \mathbb{D}$ such that $x < y$ and Φ a continuous total function in the interval $[x, y]$, with a modulus of uniform continuity h in that interval. We define the* integral between x and y *of Φ, we denote by $\int_x^y \Phi(t)\, dt$, in the following way:*

$$\int_x^y \Phi(t)\, dt :=_{\mathbb{R}} \lim S_n$$

where, for all $n \in \mathbb{N}_1$, $S_n = \sum_{i=0}^{2^{h(n)}-1} \frac{y-x}{2^{h(n)}} \Phi(x + \frac{(y-x)i}{2^{h(n)}}, n)$.

With this definition, we have similar properties to those of Proposition 22. Note that S_n is obtained by taking a partition of $[x, y]$ with diameter $\frac{y-x}{2^{h(n)}}$. Such partitions are designated by *standard* partitions. The definition of integral is robust in the following (expected) sense: it does not depend upon the choice of points in the subintervals of the partitions nor on the adjunction of new points to the *standard* partitions. Such is crucial in proving the following property (see [21], pages 128–129):

Proposition 24. *Take z a dyadic rational number such that $x < z < y$ and Φ a continuous total function in $[x, y]$ with a modulus of uniform continuity in that interval, then*

$$\int_x^z \Phi(t)\, dt + \int_z^y \Phi(t)\, dt = \int_x^y \Phi(t)\, dt.$$

Given Φ a continuous total function in $[0, 1]$ with a modulus of uniform continuity h, we will define Ψ, a continuous total function in $[0, 1]$, such that $\Psi(x) = \int_0^x \Phi(t)\, dt$ for all dyadic rational number $x \in [0, 1]$. By Proposition 9, take $m \in \mathbb{N}_1$ such that $\forall \alpha \in [0, 1], |\Phi|(\alpha) \leq 2^m$. Consider $d : \mathbb{D} \to \mathbb{D}$ the function defined by $d(x) = 0$, for $x < 0$; $d(x) = x$, for $0 \leq x \leq 1$; and $d(x) = 1$, for $x > 1$. We define $(x, n)\Psi(y, k)$ as:

$$x, y \in \mathbb{D} \wedge n, k \in \mathbb{N}_1 \wedge \left| \int_0^{d(x)} \Phi(t)\, dt - y \right| < \frac{1}{2^k} - \frac{1}{2^{n-m-1}}.$$

The formula above is equivalent to a $\exists\Sigma_\infty^b$-formula of the form $\exists w \theta'(w, x, n, y, k)$, with θ' bounded. The set $\{\langle w, x, n, y, k\rangle : \theta'(w, x, n, y, k)\}$ is officially the function Ψ.

We can prove (see [21], pages 130–131) the following result:

Proposition 25. *Let Ψ be the set of quintuples, as defined above. This set is a partial continuous function from $\mathbb{R}$ to $\mathbb{R}$. Moreover, if $\alpha \in [0, 1]$ then $\alpha \in \mathrm{dom}(\Psi)$ and, for all dyadic rational number r in $[0, 1]$, $\Psi(\alpha_r) = \int_0^r \Phi(t)\, dt$.*

Given Φ a continuous total function in $[0, 1]$ with a modulus of uniform continuity in that interval, Ψ is the *indefinite integral of* Φ. Note that, although the function Ψ, as a set, depends on the *tally* m chosen, the images of the reals in $[0, 1]$ under Ψ do not depend, as reals, on such a choice. Therefore, although rigorously the indefinite integral of Φ may be given by different sets of quintuples, we can indeed speak of the *indefinite integral function.*

The previous proposition permits to give a meaning to $\int_0^\alpha \Phi(t)dt$ also for real numbers $\alpha \in [0, 1]$. We can easily define $\int_\alpha^\beta \Phi(t)dt$, for $\alpha, \beta \in [0, 1]$ by taking an appropriate difference (the definite integral with real upper and lower limits can, more generally, be defined by approximations, as it is shown in [21]). Propositions 22 and 24 extend to integrals with real limits.

Definition 26. *Let Φ be a continuous total function in $[0,1]$, $\alpha \in [0,1]$ and $\beta \in \mathbb{R}$. β* is the derivative of Φ at α, *written $\Phi'(\alpha) = \beta$, if*

$$\forall n \in \mathbb{N}_1 \exists m \in \mathbb{N}_1 \forall h \neq 0\, (0 \leq \alpha + h \leq 1 \wedge |h| < \frac{1}{2^m} \to |\frac{\Phi(\alpha+h) - \Phi(\alpha)}{h} - \beta| \leq \frac{1}{2^n}).$$

Definition 27. *Let Φ and Ψ be continuous total functions in $[0,1]$. We say that Φ* is the derivative of Ψ *if $\Phi(\alpha) = \Psi'(\alpha)$, $\forall \alpha \in [0,1]$.*

Theorem 28 (The fundamental theorem of calculus)**.** *If Φ is a continuous total function in $[0,1]$ with a m.u.c. and Ψ is such that $\Psi(\alpha) = \int_0^\alpha \Phi(t)\, dt, \forall \alpha \in [0,1]$, then Φ is the derivative of Ψ.*

Proof. The usual proof of the theorem goes through in TCA^2. Take $\alpha \in [0,1]$. Let us prove that, if Φ is a continuous total function in $[0,1]$ with a m.u.c. and Ψ is a continuous total function in $[0,1]$ such that $\Psi(\alpha) = \int_0^\alpha \Phi(t)\, dt, \forall \alpha \in [0,1]$, then $\Phi(\alpha) = \Psi'(\alpha)$, i.e., given $n \in \mathbb{N}_1$ there is $m \in \mathbb{N}_1$ such that

$$\forall h \neq 0 \big(0 \leq \alpha + h \leq 1 \wedge |h| < \frac{1}{2^m} \to |\frac{\Psi(\alpha+h) - \Psi(\alpha)}{h} - \Phi(\alpha)| \leq \frac{1}{2^n}\big).$$

Consider p a m.u.c. for Φ. Given $n \in \mathbb{N}_1$, take $m := p(n)$. Let $h \neq 0$ be such that $0 \leq \alpha + h \leq 1 \wedge |h| < \frac{1}{2^m}$. Since p is a m.u.c. for Φ, we have $|\Phi(\alpha) - \Phi(\alpha+k)| < \frac{1}{2^n}$ for all k such that $|k| \leq |h|$.

If $0 < h$, we have $h(\Phi(\alpha) - \frac{1}{2^n}) \leq \int_0^{\alpha+h} \Phi(t)\, dt - \int_0^\alpha \Phi(t)\, dt \leq h(\Phi(\alpha) + \frac{1}{2^n})$ and if $h < 0$ we know that $h(\Phi(\alpha) + \frac{1}{2^n}) \leq \int_0^{\alpha+h} \Phi(t)\, dt - \int_0^\alpha \Phi(t)\, dt \leq h(\Phi(\alpha) - \frac{1}{2^n})$. Therefore, in each case, $\Phi(\alpha) - \frac{1}{2^n} \leq \frac{\int_0^{\alpha+h} \Phi(t)\, dt - \int_0^\alpha \Phi(t)\, dt}{h} \leq \Phi(\alpha) + \frac{1}{2^n}$. We proved that $|\frac{\int_0^{\alpha+h} \Phi(t)\, dt - \int_0^\alpha \Phi(t)\, dt}{h} - \Phi(\alpha)| \leq \frac{1}{2^n}$, i.e., $|\frac{\Psi(\alpha+h) - \Psi(\alpha)}{h} - \Phi(\alpha)| \leq \frac{1}{2^n}$. ☐

6. Three questions in weak analysis

6.1. **Integration in BTFA.** In weak systems of second-order arithmetic the *representation* of analytic notions is of great importance. The notion of continuous function given in Definition 3 is based on the definition which appears in ordinary studies of reverse mathematics. We saw in Section 5 that the theory TCA^2 is able to develop a decent theory of Riemann integration for continuous functions with a modulus of uniform continuity. As explained in the beginning of that section, the availability of counting is also a necessary condition for this development. However, continuous functions can be represented in other ways. These different ways coincide in set theory, but in weak systems they need not be equivalent. For instance, Takeshi Yamazaki presents in [29] an alternative definition. Essentially, a continuous function (on a closed bounded interval) for Yamazaki is defined as the uniform limit (given by a modulus of uniform convergence) of piecewise-linear continuous functions (N.B. With this definition, he proves in the referred paper some theorems of reverse mathematics). Another alternative would be to replace in Yamazaki's definition the piecewise-linear functions by polynomials (see the end of Section 2). It seems to us that with this new definition one could develop a theory of integration in BTFA that encompasses a good class of functions, including the most important transcendental functions (like the sine and cosine functions). Since integration of polynomials can be done via primitivations, the idea is that integration lifts to functions (suitably) approximated by polynomials. The project is clear but its implementation may face some technical difficulties. So, the problem is the following:

Problem: *To develop in* BTFA *a good theory of integration for a sufficiently robust class of continuous functions.*

6.2. **Weierstrass approximation theorem.** In the previous problem, we suggested that continuous functions (on a closed bounded interval) could be defined as uniform approximations of polynomials. Weierstrass approximation theorem ensures that we indeed obtain all the continuous functions (in ordinary set-theoretic mathematics). Is this also the case in BTFA? In other words, can we prove Weierstrass approximation theorem in BTFA? We do not offer a precise statement of the question, but we insist that by a continuous function we mean a total continuous function, as given by Definition 3, with a modulus of uniform continuity. The answer should be negative because, otherwise, this would entail that we could count in BTFA (if the previous problem has a positive solution). We actually have a stronger conjecture:

Conjecture: *Over* BTFA*, Weierstrass approximation theorem is equivalent to the totality of the exponential function.*

The right-to-left direction should be doable by formalizing in $\mathsf{BTFA} + \mathsf{exp}$ a suitable proof of Weierstrass approximation theorem. Note that $\mathsf{BTFA} + \mathsf{exp}$ is equivalent to *Elementary Arithmetic* (cf. [1]). The left-to-right conjecture is based on the following informal considerations. For each tally number n, we can consider in BTFA the (very oscillating) continuous function (in the sense of Definition 3) defined on the interval [0,1] as follows: It is the piecewise-linear (continuous) function Φ_n that, on the numbers of the form $\frac{x}{2^{n+1}}$ (with $0 \leq x \leq 2^{n+1}$), takes the value 0 if $x \equiv 0 \,(\mathrm{mod}\ 4)$, takes the value 1 if $x \equiv 1 \,(\mathrm{mod}\ 4)$ and takes the value -1 if $x \equiv 3 \,(\mathrm{mod}\ 4)$. A polynomial $P_n(X)$ sufficiently close to Φ_n must have at least 2^n roots and, hence, must be of degree at least 2^n. Now, our polynomials are of tally degree. The statement that for each tally number n, the number 2^n is also tally is equivalent to the totality of exponentiation.

6.3. **Cantini's conjecture.** The scheme of *strict* Π^1_1*-reflection*, denoted by $\mathsf{s}\Pi^1_1$-ref is

$$\forall X \exists x\, A(X,x) \to \exists w \forall X \exists x \preceq w\, A(X,x),$$

where $A(X,x)$ is a bounded formula, possibly with first and second-order parameters. This scheme is a set-theoretic truth and is closely related with weak König's lemma. The beginning of chapter VIII of Jon Barwise's book [2] is a good introduction to these matters and, from the arguments in there, one can extract the following fact: over RCA_0, the principles WKL and $\mathsf{s}\Pi^1_1$-ref are equivalent (the equivalence even holds in $\mathsf{BTFA}+\mathsf{exp}$). In [6], Andrea Cantini considered the principle of strict Π^1_1-reflection over the base theory BTFA. He showed that $\mathsf{BTFA} + \mathsf{s}\Pi^1_1$-$\mathsf{ref}$ is a Π^0_2-conservative extension of Σ^b_1-NIA (Fernandes has extended this result in [10]). Therefore, the class of provably total functions (with Σ^b_1-graphs) of $\mathsf{BTFA}+\mathsf{s}\Pi^1_1$-$\mathsf{ref}$ is still the class of polytime computable functions. Cantini also showed that, over BTFA, the principle $\mathsf{s}\Pi^1_1$-ref implies Σ^b_∞-WKL and conjectured the following:

Conjecture: *Over* BTFA*,* Σ^b_∞-WKL *implies* $\mathsf{s}\Pi^1_1$-ref*.*

Our formulation of strict Π^1_1-reflection is different from the one in Cantini but, in fact, it is an equivalent formulation in the presence of bounded collection $\mathsf{B}\Sigma^b_\infty$.

Still, Cantini's conjecture as presented in [6] is not exactly the one given above. Cantini's original formulation is seemingly a bit stronger because it has the theory Σ^b_1-$\mathsf{NIA}+\nabla^b_1$-CA in place of BTFA (see Section 4 for the nabla principle). Do notice, however, that Σ^b_1-$\mathsf{NIA}+\nabla^b_1$-$\mathsf{CA}+\Sigma^b_\infty$-$\mathsf{WKL}$ proves $\mathsf{B}\Sigma^b_\infty$ (cf. [16]). In fact, there is a close relationship between forms of weak König's lemma and forms of bounded collection.

Let the principle of *extended strict* Π^1_1*-reflection* be the generalization of the scheme of $\mathsf{s}\Pi^1_1$-ref by allowing $\Sigma^{1,b}_\infty$-formulas in the schematic position A (see Section 5 for this class of formulas). This is the scheme considered in section 7 of our tryptic paper [13], where it is shown that BTFA together with extended strict Π^1_1-reflection proves $\mathsf{B}^1\Sigma^{1,b}_\infty$ (see Section 5 for this form of bounded collection).

Theorem 29. *The theory* BTFA *with extended strict* Π^1_1*-reflection is conservative over* $\mathsf{BTFA}+\mathsf{B}^1\Sigma^{1,b}_\infty$ *with respect to formulas without second-order unbounded quantifiers.*

A sketch of the proof of this result appears in [13]. It uses a Harrington forcing argument in which the forcing conditions are binary trees, taken in a certain generalized sense. This kind of forcing appeared originally in the doctoral dissertation of Fernandes [8].

The above discussion shows that there is a tight connection between extended strict Π^1_1-reflection and $\mathsf{B}^1\Sigma^{1,b}_\infty$. It is the same sort of connection that exists between Σ^b_∞-WKL and $\mathsf{B}\Sigma^b_\infty$. The principle $\mathsf{s}\Pi^1_1$-ref falls between Σ^b_∞-WKL and extended strict Π^1_1-reflection. In [10], Fernandes was able to isolate a principle (implied by $\mathsf{B}^1\Sigma^{1,b}_\infty$) that must be provable in BTFA if Cantini's conjecture is true. This principle, as opposed to $\mathsf{s}\Pi^1_1$-ref, does not have unbounded second-order quantifiers. The authors believe that Cantini's conjecture is false.

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