Determinantal Point Processes and Integrable PDEs

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based on joint works with Mattia Cafasso, Christophe Charlier, Tom Claeys, Gabriel Glesner, Sofia Tarricone

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The point process is **determinantal** iff there exists

$$K : E \times E \to \mathbb{R} \qquad \text{(kernel)}$$

such that

$$\mathbb{P}(\lambda_1 \in \Lambda, \dots, \lambda_k \in \Lambda) = \det\big(K(\lambda_i, \lambda_j)\big)_{i,j=1}^{k} \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_k \qquad (\forall k \geq 1)$$

for *distinct* $\lambda_1, \dots, \lambda_k$.

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Why?

- Elegant and rich theory
- Universal structure arising in many examples of repulsively interacting systems (non-interacting trapped fermions, random matrices, random partitions, random tilings, random interface growth models,...)

DPPs from Random Matrix Theory

Gaussian Unitary Ensemble (GUE): random Hermitian matrix M of size n with joint pdf of the entries

$$\frac{1}{Z_n}\,\exp\Big(-\frac{1}{2}\,n\,\mathrm{tr}\,(M^2)\Big)\,\mathrm{d}M.$$

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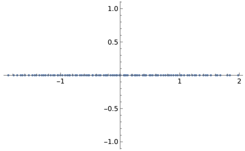
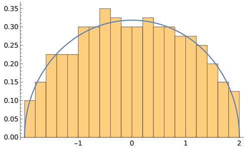
Eigenvalues $\lambda_1, \ldots, \lambda_n$ form a determinantal point process [Mehta, Dyson]

$$K_{\mathrm{GUE}(n)}(\lambda,\mu) \;=\; \frac{1}{\sqrt{2\pi}(n-1)!}\,\frac{\psi_n(\lambda)\psi_{n-1}(\mu)-\psi_{n-1}(\lambda)\psi_n(\mu)}{\lambda-\mu},$$

$$\psi_n(\lambda) \;=\; \mathrm{He}_n(\sqrt{n}\lambda)\,\mathrm{e}^{-\frac{1}{4}n\lambda^2}.$$





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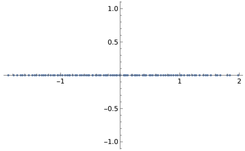

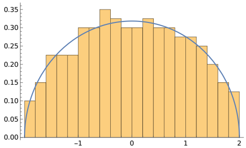

Many generalizations: $\frac{1}{2}M^2 \mapsto$arbitrary potential $V(M)$, circular ensembles, ...

Universal scaling limits in RMT: the bulk

By the Plancherel–Rotach asymptotics

$$\psi_{n+j}(\lambda) \sim \frac{n^{\frac{n}{2}} \mathrm{e}^{-\frac{n}{2}}}{(4-\lambda^2)^{\frac{1}{4}}} \sin\left(\frac{\pi}{4} + (n+j+\tfrac{1}{2}) \arccos\frac{\lambda}{2} - \frac{n}{4}\lambda\sqrt{4-\lambda^2}\right)$$

as $n \to \infty$, $j = O(1)$, $|\lambda| < 2$, the GUE kernel converges to the **sine kernel**:

$$\frac{1}{n\,\delta_0} K_{\mathrm{GUE}(n)}(\lambda_0 + \frac{\lambda}{n\,\delta_0}, \lambda_0 + \frac{\mu}{n\,\delta_0}) \to K_{\mathrm{sine}}(\lambda, \mu) = \frac{\sin\big(\pi(\lambda-\mu)\big)}{\pi(\lambda-\mu)}$$

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It is also found in many other "bulk" distributions: from non-interacting trapped fermions to ... zeros of Riemann's ζ [Montgomery, Odlyzko, Rudnick–Sarnak].

Universal scaling limits in RMT: the edge

Near the edge we instead use the asymptotics

$$\psi_n\left(2+\frac{\lambda}{n^{2/3}}\right)=n^{\frac{n}{2}+\frac{1}{6}}\mathrm{e}^{-\frac{n}{2}}\sqrt{2\pi}(\mathrm{Ai}(\lambda)+O(n^{-2/3})),\quad n\to+\infty,$$

and find the scaling to the **Airy kernel** as $n\to+\infty$

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It is also found in many other "edge" distributions: from non-interacting trapped fermions to ... longest increasing subsequences in a random permutation [Baik–Deift–Johansson, Borodin–Okounkov–Olshanski].

The Tracy–Widom distribution

The **Tracy–Widom distribution** describes large-n fluctuations of the largest GUE eigenvalue λ_{max} around 2:

$$F_{\mathrm{TW}}(s) = \lim_{n\to+\infty} \mathbb{P}\left((\lambda_{max} - 2)\, n^{\frac{2}{3}} \leq s \right)$$

$$= \mathbb{P}(\text{largest particle of Airy process is } \leq s)$$

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A striking example of this universality: let σ be a uniform random permutation of $\{1, \dots, n\}$ and let ℓ_n be the length of the longest increasing subsequence of σ, then

$$\mathbb{P}\Big((\ell_n - 2n^{\frac{1}{2}})n^{-\frac{1}{6}} \leq s \Big) \to F_{\mathrm{TW}}(s)$$

as $n \to +\infty$ [Baik–Deift–Johansson, Borodin–Okounkov–Olshanski].

DPPs and Fredholm determinants

A distinguishing feature of determinantal point processes is that **multiplicative expectations** are Fredholm determinants:

$$\mathbb{E}\left(\prod_i \big(1 - \phi(\lambda_i)\big) \right) = \det_{L^2(\mathbb{R})}\left(1 - \sqrt{\phi} K \sqrt{\phi} \right)$$

$$= 1 + \sum_{n \geq 1} \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \det\big(K(\lambda_i, \lambda_j)\big)_{i,j=1}^n \prod_{i=1}^n \phi(\lambda_i) \mathrm{d}\lambda_i \,.$$

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In particular, **gap probabilities** are expressed as Fredholm determinants: for any Borel $S \subseteq \mathbb{R}$

$$\mathbb{P}\left(\text{no particle in } S \right) = \det\left(1 - 1_S K 1_S \right).$$

In particular:

$$F_{\mathrm{TW}}(s) = 1 + \sum_{n\geq 1} \frac{(-1)^n}{n!} \int_{(s,+\infty)^n} \det\Big(K_{\mathrm{Ai}}(\lambda_i,\lambda_j)\Big)_{i,j=1}^{n} \prod_{i=1}^{n} \mathrm{d}\lambda_i\,.$$

Tracy–Widom distribution and Painlevé II

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Another characterization in terms of integrable systems [Tracy–Widom]: we have

$$F_{\mathrm{TW}}(s) \; = \; \exp\left(- \int_s^{+\infty} (s-t)\, q(t)^2 \,\mathrm{d}t \right)$$

where $q(s)$ is the *Hastings–McLeod solution of Painlevé II*: namely, $q(s)$ is the unique solution to the boundary value problem

$$q''(s) \; = \; s\, q(s) + 2\, q(s)^3 \qquad \textit{(Painlevé II ODE)}$$

$$q(s) \; \sim \; \mathrm{Ai}(s), \quad s \to +\infty.$$

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Note that $\partial_s^2 \log F_{\mathrm{TW}}(s) = q(s)^2$, i.e., F_{TW} is a “tau-function” of Painlevé II.

Painlevé II and Korteweg–de Vries

Painlevé equations also arise as **self-similar reductions of integrable PDEs**.

In particular, for any solution of PII

$$q''(s) = s\,q(s) + 2\,q(s)^3$$

the function

$$u(x,t) = \frac{x}{2t} - t^{-2/3} q\left(-x/t^{1/3}\right)^2$$

satisfies the **Korteweg–de Vries (KdV) equation**

$$u_t + \frac{1}{6}u_{xxx} + 2\,u\,u_x = 0.$$

Theorem (Cafasso–Claeys–R, 2021). *For sufficiently nice $\sigma : \mathbb{R} \to \mathbb{R}$, the Airy multiplicative expectation*

$$F_\sigma(x,t) \;=\; \mathbb{E}_{\mathrm{Ai}}\left(\prod_i \big(1 - \sigma(t^{-\frac{2}{3}}\lambda_i + x t^{-1})\big) \right)$$

is a tau-function of (cylindrical) KdV, namely

$$u_t + \frac{1}{6} u_{xxx} + 2\,u\,u_x \;=\; 0\,, \qquad u = u(x,t) = \frac{x}{2t} + \partial_x^2 \log F_\sigma(x,t)\,.$$

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Such class of KdV solutions is quite general and corresponds to the *cylindrical* KdV equation. The Riemann–Hilbert approach to these solutions is equivalent to the inverse scattering for the cylindrical KdV equation of A. Its and V. Sukhanov, σ being identified with the scattering data.

Applications

- $\sigma = 1_{\mathbb{R}_+}$: $F_\sigma(x,t) = F_{\mathrm{TW}}(-xt^{-\frac{1}{3}})$ [Tracy–Widom]
- σ p.w. const: *generating function for Airy DPP* [Claeys–Doeraene]
- $\sigma(\lambda) = \dfrac{1}{1+\exp(-\lambda)}$: *positive-temperature Airy kernel*
  - *from Gumbel to Tracy–Widom* [Johansson]
  - *narrow wedge KPZ* [Amir–Corwin–Quastel, Borodin–Gorin]
  - *1D trapped free fermions at positive temperature* [Dean–Le Doussal–Majumdar–Schehr]
  - *multiplicative statistics of Hermitian random matrices* [Ghosal–Silva]
- $\sigma(\lambda) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\lambda} \mathrm{e}^{-s^2} \mathrm{d}s$: *edge eigenvalue statistics in the complex elliptic Ginibre Ensemble at weak non-Hermiticity* [Bothner–Little]

The Riemann–Hilbert problem

The main technical tool is a **Riemann–Hilbert** characterization of $F_\sigma(x, t)$ based on the theory of integrable operators [Its–Izergin–Korepin–Slavnov]. Hiding a couple of technical details, it goes as follows.

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There exists a unique $\Psi(\cdot|x,t) : \mathbb{C}\setminus\mathbb{R} \to \mathrm{GL}_2(\mathbb{C})$ such that:

- $\Psi(w|x,t)$ is analytic for $w \in \mathbb{C}\setminus\mathbb{R}$;
- When $\lambda \in \mathbb{R}$, the boundary values $\Psi_\pm(\lambda|x,t) = \lim_{\epsilon\to 0_+}\Psi(\lambda \pm \mathrm{i}\epsilon|x,t)$ exist and are related as

$$\Psi_+(\lambda|x,t) \;=\; \Psi_-(\lambda|x,t)\begin{pmatrix} 1 & 1-\sigma(\lambda) \\ 0 & 1 \end{pmatrix};$$

- As $w \to \infty$ in both half-planes,

$$\Psi(w|x,t) = \left(I + w^{-1}\Psi^{[1]}(x,t) + O(w^{-2})\right) w^{\frac{\sigma_3}{4}} \frac{\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}}{\sqrt{2}} \mathrm{e}^{\left(-\frac{2}{3}tw^{\frac{3}{2}} + x\lambda^{\frac{1}{2}}\right)\sigma_3}.$$

Then,

$$\partial_x \log F_\sigma(x,t) \;=\; -\mathrm{i}\,\Psi^{[1]}_{2,1}(x,t)\,.$$

$t \to 0_+$ ASYMPTOTICS UNIFORM WHEN $x < K$

Theorem (Cafasso–Claeys–R, 2021). *If*

$$\begin{cases} \left|\sigma(\lambda) - \gamma 1_{\mathbb{R}_+}(\lambda)\right| \leq c_1 \mathrm{e}^{-c_2|\lambda|}, & \lambda \in \mathbb{R}, \\ |\sigma'(\lambda)| \leq c_3 |\lambda|^{-2}, & |\lambda| > C, \end{cases}$$

we have the following uniform estimates.

- $\forall t_0 > 0\ \exists M, c > 0$ *s.t. for* $\boxed{x < -Mt^{\frac{1}{3}},\ 0 < t < t_0}$

$$u(x,t) = \frac{x}{2t} + O(\mathrm{e}^{-c|x|t^{-\frac{1}{3}}}).$$

- $\exists \epsilon > 0$ *s.t.* $\forall M > 0$ *and* $\boxed{|x| \leq Mt^{1/3},\ 0 < t < \epsilon}$

$$u(x,t) = \frac{x}{2t} - t^{-\frac{2}{3}} q_\gamma(-xt^{-\frac{1}{3}})^2 + O(1).$$

- $\boxed{\textit{If } \gamma = 1\textit{:}}$ $\exists \epsilon, M > 0$ *s.t.* $\forall K > 0$ *and* $\boxed{Mt^{1/3} < x < K,\ 0 < t < \epsilon}$

$$u(x,t) = u_0(x)(1 + O(x^{-1}t^{\frac{1}{3}})).$$

This case requires more care.

ASSUMPTIONS: We assume that

$$M(w) := \frac{1}{1-\sigma(w)}$$

is an *entire* function of w satisfying

- $M'(\lambda) \geq 0$ and $(\log M)''(\lambda) \geq 0$ for all $\lambda \in \mathbb{R}$,
- $M(\lambda) = 1 + c'_- e^{-c_-|\lambda|}(1 + o(1))$ as $\lambda \to -\infty$, for some $c_-, c'_- > 0$.
- $M(\lambda) = c'_+ e^{c_+\lambda}(1 + O(e^{-\epsilon\lambda}))$ as $\lambda \to +\infty$, for some $c_+, c'_+, \epsilon > 0$,
- $|M(w)| = O(e^{c_+ \operatorname{Re} w})$ as $\operatorname{Re} w \to +\infty$.

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- $M(\lambda) = c'_{+} e^{c_{+}\lambda}(1 + O(e^{-\epsilon\lambda}))$ as $\lambda \to +\infty$, for some $c_{+}, c'_{+}, \epsilon > 0$,
- $|M(w)| = O(e^{c_{+}\operatorname{Re} w})$ as $\operatorname{Re} w \to +\infty$.

Functions σ with discontinuities like $\sigma = 1_{(0,+\infty)}$ are not considered.

The prototypical example is

$$\sigma(\lambda) = \frac{1}{1 + \mathrm{e}^{-\lambda}}.$$

$t \to 0_+$ ASYMPTOTICS UNIFORM WHEN $|x| > K$

Theorem (Charlier–Claeys–R, 2022). *Under the previous assumptions on σ, for any $t_0 > 0$ exists $K > 0$ such that*

$$u(x,t) = \frac{x}{2t} a_0(y) + \frac{1}{2\sqrt{xt}} a_1(y) + \frac{t^{1/2}}{2x^{3/2}} a_2(y) + O(x^{-2}),$$

with error terms uniform for $x \geq K$, $0 < t \leq t_0$, where $\boxed{y := \frac{\pi^2}{c_+^2} xt}$ *and*

$$\begin{aligned}
a_0 &= \tfrac{2-2\sqrt{1+y}}{y}, \\
a_1 &= -\frac{\log c_+'}{\pi}\sqrt{\tfrac{y}{1+y}}, \\
a_2 &= \tfrac{y^{3/2}}{\sqrt{y+1}(\sqrt{y+1}-1)^2}\left(\tfrac{1}{4\pi c_+}\tfrac{1-2\sqrt{y+1}}{y+1}\log^2 c_+' - j_\sigma\right), \\
j_\sigma &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\log(1-\sigma(\lambda)) + (c_+\lambda + \log c_+')1_{\mathbb{R}_+}(\lambda)\right]\mathrm{d}\lambda.
\end{aligned}$$

Connection to narrow-wedge KPZ

We have the identity [Amir–Corwin–Quastel, Borodin–Gorin]

$$\mathbb{E}_{\mathrm{KPZ}}\left[\mathrm{e}^{-\mathrm{e}^{T^{1/3}(\Upsilon(T)-s)}}\right] = F_\sigma(-sT^{-\frac{1}{6}}, T^{-\frac{1}{2}}), \quad \sigma(\lambda) = \frac{1}{1+\mathrm{e}^{-\lambda}},$$

for the **narrow wedge solution** to the **KPZ equation**

$$\Upsilon(T) = \frac{h(2T,0)+\frac{T}{12}}{T^{1/3}}, \quad \begin{cases} \partial_T h(T,X) = \frac{1}{2}\partial_X^2 h(T,X) + \frac{1}{2}\left(\partial_X h(T,X)\right)^2 + \xi(T,X), \\ h(0,X) = \log \delta_{X=0}, \qquad (\xi = \textit{space-time white noise}) \end{cases}$$

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This stochastic PDE was introduced by Kardar, Parisi and Zhang in 1986 and quickly became a **universal model for random interface growth** in physics.
Precise mathematical formulation requires care [Bertini–Giacomin, Hairer].

Theorem (Charlier–Claeys–R, 2021). *Let* $\Upsilon(T) := \frac{h(2T,0)+\frac{T}{12}}{T^{1/3}}$,
$\Phi_1(y) := \frac{4}{15}(1+y)^{5/2} - \frac{4}{15} - \frac{2}{3}y - \frac{1}{2}y^2$, *and*

$$G(s,T) := \frac{T^2}{\pi^6}\Phi_1(\tfrac{\pi^2 s}{T^{2/3}}) + \frac{1}{6}\sqrt{1+\tfrac{\pi^2 s}{T^{2/3}}} + \frac{\log(1+\frac{\pi^2 s}{T^{2/3}})}{48} + \tfrac{1}{8}\log(\sqrt{1+\frac{\pi^2 s}{T^{2/3}}} - 1) + \frac{\log T}{12}.$$

1. *For any* $s_0, T_0 > 0$, *there exists a real constant* $D_+ = D_+(s_0, T_0)$ *such that the inequality*

$$\log \mathbb{P}(\Upsilon_T < -s) \le p - G(s + T^{-1/3}\log p, T) + D_+$$

holds for all $s \ge s_0, T \ge T_0$, *and* $p \ge 1$.

2. *For any* $\epsilon > 0$ *and for any* $s_0, T_0 > 0$ *sufficiently large, there exists a real constant* $D_- = D_-(s_0, T_0)$ *such that the inequality*

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Refines [Corwin–Ghosal, 2018], [Cafasso–Claeys, 2019], [LeDoussal, 2020].

Schrödinger equation, trace-formula

The KdV equation $u_t + \frac{1}{6}u_{xxx} + 2uu_x = 0$ is the compatibility of

$$L\varphi(z|x,t) = z\varphi(z|x,t), \qquad L = \partial_x^2 + 2u(x,t),$$

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$$\varphi(\lambda|x,t) \sim t^{\frac{1}{6}}\,\mathrm{Ai}(t^{\frac{2}{3}}\lambda - xt^{-\frac{1}{3}}), \quad x \to -\infty, \quad \lambda \in \mathbb{R},$$

we have the **trace-formula**

$$u(x,t) = \frac{x}{2t} - \frac{1}{t}\int_{\mathbb{R}} \varphi^2(\lambda|x,t)\mathrm{d}\sigma(\lambda).$$

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⇒ **Amir-Corwin-Quastel integro-differential Painlevé II equation**:

$$\partial_x^2\varphi(z|x,t) = \left(z - \frac{x}{t} + \frac{2}{t}\int_{\mathbb{R}} \varphi^2(\lambda|x,t)\mathrm{d}\sigma(\lambda)\right)\varphi(z|x,t)\,.$$

Darboux transformations of the ODE

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A good hint: **cylindrical KdV N-soliton solutions** (= Darboux transformations of the KdV solution $\frac{x}{2t}$) can be written in terms of the Airy kernel

$$u_{\mathsf{N-sol}}(x,t|\underline{\nu},\underline{\rho}) = \frac{x}{2t} + \partial_x^2 \log\det\left(t^{\frac{2}{3}} K^{\mathrm{Ai}}(t^{\frac{2}{3}}\nu_i - xt^{-\frac{1}{3}}, t^{\frac{2}{3}}\rho_j - xt^{-\frac{1}{3}})\right)_{i,j=1}^{N}$$

with dependence on $2N$ parameters $\underline{\nu} = (\nu_1, \ldots, \nu_N)$, $\underline{\rho} = (\rho_1, \ldots, \rho_N)$.

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- Perform a **thinning** by $\sigma : \mathbb{R} \to [0,1]$ (sufficiently "nice"): consider the determinantal point process with kernel

$$\sqrt{\sigma(\lambda)} K^{\mathrm{Ai}}_{x,t}(\lambda,\mu) \sqrt{\sigma(\mu)}$$

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- Let $\mathcal{A}^{\sigma}_{x,t}$ be such process. It can be shown that it has a.s. a finite number of particles, such that we can define the **Jánossy densities**

$$F_\sigma(x,t|\underline{\nu}) := \frac{\mathbb{P}\left(\mathcal{A}^{\sigma}_{x,t} \text{ has exactly } N \text{ particles and they are at } \nu_i + \mathrm{d}\nu_i\right)}{\prod_{j=1}^{N} \sigma(\nu_j)\mathrm{d}\nu_j}$$

  for all $\underline{\nu} = (\nu_1, \ldots, \nu_N)$, $\nu_i \in \mathbb{R}$, $\nu_i \neq \nu_j$ for $i \neq j$.

Theorem (Claeys–Glesner–R–Tarricone). *The Jánossy density $F_\sigma(x,t|\underline{\nu})$ of the thinned shifted and rescaled Airy process is a tau-function of (cylindrical) KdV, namely*

$$u_t+\frac{1}{6}u_{xxx}+2\,u\,u_x\;=\;0\,,\qquad u=u(x,t|\underline{\nu})=\frac{x}{2t}+\partial_x^2\log F_\sigma(x,t|\underline{\nu})\,.$$

Moreover: under weak assumptions on σ, for any $t_0>0$ exists $c>0$ such that

$$u(x,t|\underline{\nu})=u_{\mathsf{N-sol}}(x,t|\underline{\nu},\underline{\nu})+O\left(\mathrm{e}^{-c|x|t^{-\frac{1}{3}}}\right),$$

uniformly in $\boxed{t\le t_0\ \text{as}\ xt^{-\frac{1}{3}}\to-\infty}$*; under strong assumptions on σ, for any $t_0>0$, we have*

$$u(x,t|\underline{\nu})=u(x,t)+\sqrt{\frac{1}{xt}}\sum_{j=1}^{N}\cos\left(\frac{4}{3}\sqrt{\frac{x^3}{t}}\big(1+o(1)\big)-2\sqrt{xt}\,\nu_j\big(1+o(1)\big)\right)+O\Big(\frac{t}{x}\Big),$$

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KdV solutions, asymptotics

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uniformly for $\boxed{t \le t_0 \text{ as } \frac{xt}{\log^2(x/t)} \to +\infty}$.

(The last asymptotics is close to a superposition of N cKdV 1-solitons.)

Other models, work in progress

- **Sine process**: Its–Izergin–Korepin–Slavnov (1990, correlation functions of 1d Bose gas), Claeys–Tarricone (2023)
- **(continuous) Bessel process** (hard edge in RMT and fermion models, random partitions): R (2024)

$$(2v_x - t)\, {v_t}^2 + \frac{1}{4}{v_{xt}}^2 - \frac{1}{2}v_{xxt}\, v_t = \frac{\alpha^2}{4} \quad (\alpha > -1)\,.$$

- **(discrete) Bessel process** (Random partitions, polynuclear growth models): Cafasso–R (2022), the integrable equation is 2D Toda (essentially known since Okounkov). Discrete Riemann–Hilbert approach to asymptotics (in progress).