QUANTUM INTERMEDIATE LONG WAVE HIERARCHY AND QUASIMODULAR FORMS

Integrable systems, Frobenius manifolds and related topics

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Plan

1) Dispersionless quantum KdV (Eliashberg, Dubrovin)

2) Quantum KdV (Buryak-Rossi)

3) Functions of partitions and quasimodular forms (Bloch-Okounkov, Zagier, van Ittersum)

4) Quantum ILW and quasimodular forms (van Ittersum - R)

Hamiltonian PDEs

Phase space : $\mathcal{M} = \{ u : \mathbb{R} \to \mathbb{C}, \text{ smooth}, \ u(x+2\pi) = u(x) \}$

Poisson bracket : $\{F, G\} = \int_{-\pi}^{\pi} \frac{\delta F}{\delta u(x)} \left(\partial_x \frac{\delta G}{\delta u(x)} \right) \frac{dx}{2\pi}$, $\quad F, G : \mathcal{M} \to \mathbb{C}$

Hamiltonian PDE : $\quad \partial_t u(x) = \{ u(x), G \} = \partial_x \frac{\delta G}{\delta u(x)}$

Fourier coordinates : $\quad \phi = (\phi_k)_{k \in \mathbb{Z}} \longmapsto u(x) = \sum_{k \in \mathbb{Z}} \phi_k e^{ikx}$

$$\phi_k = \int_{-\pi}^{\pi} e^{-ikx} u(x) \frac{dx}{2\pi} \Rightarrow \frac{\delta \phi_k}{\delta u(x)} = e^{-ikx} \Rightarrow \{\phi_k, \phi_l\} = \int_{-\pi}^{\pi} e^{-i(k+l)x} (-il) \frac{dx}{2\pi} = ik\, \delta_{k,-l}$$

(ϕ_0 Casimir , ϕ_k, ϕ_{-k} canonically conjugate)

Dispersionless KdV hierarchy

Commuting Hamiltonian flows (on $\mathcal{M}$)

$$\partial_{t_k} u(x) = \{ u(x), G_k^{Hopf} \} \qquad (k \geq 0)$$

$$G_k^{Hopf} = \int_{-\pi}^{\pi} g_k^{Hopf} \frac{dx}{2\pi} \qquad g_k^{Hopf} = \frac{u(x)^{k+2}}{(k+2)!}$$

$$\{ G_k^{Hopf}, G_l^{Hopf} \} = 0$$

More explicitly

$$\partial_{t_k} u(x) = \frac{1}{k!} u(x)^k u_x(x)$$

($k=1$ $\Rightarrow$ dispersionless KdV equation)

Quantization (I)

Promote function(al)s $F: \mathcal{M} \to \mathbb{C}$ to operators $\hat{F}$ on Λ:

$$\widehat{\{F, G\}} = \frac{1}{i\hbar} [\hat{F}, \hat{G}]$$

Let's fix: $\Lambda = \mathbb{C}[p]$, $\quad p = (p_1, p_2, p_3, \dots)$

$$(\hat{\phi}_k f)(p) = \begin{cases} p_k f(p) & k > 0 \\ c f(p) & k = 0 \\ \hbar |k| \dfrac{\partial f(p)}{\partial p_{|k|}} & k < 0 \end{cases} \qquad \left([\hat{\phi}_k, \hat{\phi}_\ell] = -\hbar k \delta_{k,-\ell} \right)$$

Normal order: $:\hat{\phi}_{a_1} \cdots \hat{\phi}_{a_\ell}: = \prod_{a_i \geq 0} \hat{\phi}_{a_i} \prod_{a_i < 0} \hat{\phi}_{a_i} \Rightarrow$ well-defined...

Quantization (II)

To a differential polynomial

$$g(u) \in \mathbb{C}[u], \qquad u = (u_0, u_1, u_2, \dots) = (u, u_x, u_{xx}, \dots)$$

we associate an operator on Λ

$$\overline{g} = \int_{-\pi}^{\pi} :g(\hat{u}(x)): \frac{dx}{2\pi}, \quad \text{where } \hat{u}(x) = \sum_{k \in \mathbb{Z}} \hat{\phi}_k e^{ikx}$$

E.g. $\frac{1}{2}\overline{u^2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} :\hat{\phi}_k \hat{\phi}_{-k}: = \frac{c^2}{2} + \hbar \sum_{k>0} k p_k \frac{\partial}{\partial p_k}$ (degree operator)

$\left[L, \frac{\overline{u^2}}{2}\right] = 0 \iff L : \Lambda_k \to \Lambda_k, \quad \Lambda = \bigoplus_{k\geq 0} \Lambda_k$ (eigenspaces of $\sum_{k>0} k p_k \frac{\partial}{\partial p_k}$)

Quantization problem for integrable hierarchies

Fact: $\left[\overline{g_k^{\mathrm{Hopf}}}, \overline{g_l^{\mathrm{Hopf}}}\right] = O(\hbar) \neq 0$ in general.

Problem: find $g_k^{q\mathrm{Hopf}} \in \mathbb{C}[u, \hbar]$ s.t.

$$g_k^{q\mathrm{Hopf}}\Big|_{\hbar=0} = g_k^{\mathrm{Hopf}} \quad \& \quad \left[\overline{g_k^{q\mathrm{Hopf}}}, \overline{g_l^{q\mathrm{Hopf}}}\right] = 0$$

Theorem (Eliashberg, 2000 – byproduct of Symplectic Field Theory)

A solution is given explicitly by

$$\sum_{k \geq -2} z^{k+2} g_k^{q\mathrm{Hopf}} = \frac{1}{S(\hbar^{1/2} z)} \exp\left(z S(i \hbar^{1/2} z \partial_x) u_0\right), \qquad S(\xi) = \frac{\sinh(\xi/2)}{\xi/2}$$

Dispersionless spectrum

<u>Theorem</u> (Dubrovin, 2014): We have

$$\overline{g}_k^{\,q\mathrm{Hopf}}\, s_\lambda\left(P/\hbar^{1/2}\right) = \left[\sum_{j=0}^{k+2} \frac{\hbar^{j/2}\, c^{k+2-j}}{(k+2-j)!}\, Q_j(\lambda)\right] \cdot s_\lambda\left(P/\hbar^{1/2}\right)$$

where:

s_λ = Schur functions

$$Q_0(\lambda) = 1, \quad Q_j(\lambda) = \beta_j + \frac{1}{(j-1)!}\sum_{i=1}^{+\infty}\left[\left(\lambda_i - i + \tfrac{1}{2}\right)^j - \left(-i+\tfrac{1}{2}\right)^j\right], \quad \sum_{j=1}^{+\infty} \beta_j \xi^j = \frac{\xi/2}{\sinh(\xi/2)}$$

$\left(\lambda = (\lambda_1, \lambda_2, \dots)\right.$ a "partition": $\lambda_i \in \mathbb{Z}_{\geq 0}$, $\lambda_i \geq \lambda_{i+1}$, $\left.|\lambda| = \sum_{i=1}^{+\infty} \lambda_i < +\infty\right)$

$Q_\lambda \longrightarrow$ "shifted symmetric functions" $\left\{\begin{array}{l}\text{ASYMPTOTIC REP. THEORY} \\ \text{ENUMERATIVE GEOMETRY} \\ \text{(H/GW THEORY,} \\ \text{SIEGEL-VEECH CONSTANTS)}\end{array}\right.$

Buryak-Rossi Double ramification quantum hierarchies (I)

A general construction of quantum integrable hierarchies (for simplicity, here of rank 1) is given by Buryak & Rossi (2015) in Fourier coordinates

$$g_k = \sum_{\substack{g,n\geq 0 \\ 2g-2+n\geq 0}} \frac{\hbar^g}{n!} \sum_{(a_1\ldots a_n)\in\mathbb{Z}^n} \int_{DR_g(-|a|,a_1\ldots a_n)} \psi_1^k \, \Lambda\left(-\frac{\varepsilon^2}{\hbar}\right) c_{g,n+1} \, \phi_{a_1}\cdots\phi_{a_n} e^{i|a|x} \qquad \left(|a| := \sum_{i=1}^n a_i\right)$$

$DR_g(\ldots) \in H_{2g-2+n}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ "double ramification cycle"

$\Lambda_g(\varepsilon) = 1+\varepsilon\lambda_1+\varepsilon^2\lambda_2+\ldots+\varepsilon^g\lambda_g$, $c_{g,n}$ = (values at identity of a) rank 1 CohFT

Buryak-Rossi Double ramification quantum hierarchies (II)

Theorem (Buryak – Rossi, 2015):

$$[\bar{g}_k, \bar{g}_j] = 0 \qquad \text{(commutativity)}$$

$$\partial_x (D-1)\, g_{k+1} = \frac{1}{\hbar}[g_k, \bar{g}_1] \qquad \text{(recursion)}$$

$$\left(\text{where } D := \varepsilon \frac{\partial}{\partial \varepsilon} + 2\hbar \frac{\partial}{\partial \hbar} + \sum_{i \geq 0} u_i \frac{\partial}{\partial u_i}\right)$$

Quantum KdV

Theorem (Buryak-Rossi, 2015): The quantization problem for the full dispersive KdV hierarchy is solved by:

$$g_{-2}^{qkdV} = 1, \quad \frac{\partial g_k^{qkdV}}{\partial u_0} = g_{k-1}^{qkdV}, \quad \partial_x (D-1)\, g_k^{qkdV} = (R_1 + R_2)\, g_{k-1}^{qkdV}$$

$$R_1 g := \sum_{\ell=0}^{+\infty} \frac{\partial g}{\partial u_\ell} \partial_x^{\ell+1}\left(\frac{u_0^2}{2} + \varepsilon u_2\right) - \frac{\hbar}{2} \sum_{\ell,m=0}^{+\infty} \frac{\partial^2 g}{\partial u_\ell \partial u_m} \frac{(\ell+1)!\,(m+1)!}{(\ell+m+1)!} u_{\ell+m+3},$$

$$R_2 g := -\frac{\hbar}{2} \sum_{\ell,m,i=0}^{+\infty} \frac{\partial^2 g}{\partial u_\ell \partial u_m} \frac{B_{2i+2}}{2i+2}\left[(-1)^{\ell+i}\binom{m+1}{2i-\ell} + (-1)^{m+i}\binom{\ell+1}{2i-m}\right] u_{\ell+m-2i+1}$$

$$\left(D := \varepsilon\frac{\partial}{\partial\varepsilon} + 2\hbar\frac{\partial}{\partial\hbar} + \sum_{i\geq 0} u_i \frac{\partial}{\partial u_i}, \quad B_i = \text{Bernoulli numbers}, \quad \sum_{i\geq 0}\frac{B_i}{i!} t^i = \frac{t}{e^t - 1}\right)$$

A natural problem: describe spectrum of qKdV.

Modular forms

Holomorphic functions $\phi : \mathbb{H} = \{\mathrm{Im}\, z > 0\} \to \mathbb{C}$ such that

$$\phi\left(\frac{az+b}{cz+d}\right) = (cz+d)^k \phi(z), \quad \forall z \in \mathbb{H} \quad (k = \text{"weight"})$$

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ (+ growth conditions $\begin{cases} \phi(z) = O(1) \text{ at } \mathrm{Im} z \to +\infty \\ \phi(z) = O\left(\frac{1}{(\mathrm{Im}\, z)^k}\right) \text{ at } \mathrm{Im} z \to 0 \end{cases}$)

$$\begin{cases} \phi(z+1) = \phi(z) \\ \phi(-1/z) = z^k \phi(k) \end{cases} \implies \phi(z) = \sum_{n \geq 0} a_n q^n, \quad q = \exp(2\pi i z)$$

$\hookrightarrow$ interesting in various domains of Mathematics

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<u>Fact</u>: space of modular forms $= M = \bigoplus_{k \geq 0} M_k = \mathbb{C}[\underset{\text{wt}=4}{G_4}, \underset{\text{wt}=6}{G_6}]$

where

$$G_k := -\frac{B_k}{2k} + \sum_{n=1}^{+\infty} q^n \sum_{d|n} d^{k-1}$$

("Eisenstein series")

Quasimodular forms

It is natural to add G_2 :

$$\widetilde{M} = \bigoplus_{k\geq 0} \widetilde{M}_k = \mathbb{C}[\underset{\substack{\uparrow \\ \mathrm{wt}=2}}{G_2}, G_4, G_6]$$

G_2 is not modular ("quasimodular")

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 G_2(z) + \frac{ic(cz+d)}{4\pi}, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

$\widetilde{M}$ carries an $\mathfrak{sl}_2$-action; W = weight, $D = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$

∂ derivation on $\widetilde{M}$ defined by $\partial G_2 = -\frac{1}{2}$, $\partial G_4 = \partial G_6 = 0$:

$$[W, D] = 2D, \quad [W, \partial] = -2\partial, \quad [\partial, D] = W$$

Functions of partitions & quasimodularity (I)

Partitions of integers are related to modular forms.

<u>Example</u>: $\sum_{n\geq 0} q^n \#\{\text{partitions } \lambda \text{ s.t. } |\lambda| = n\} = \sum_{n\geq 0} q^n \dim \Lambda_n$

$$= \prod_{k\geq 1} (1-q^k)^{-1} =: q^{1/24}/\eta(q) \quad (\text{"Dedekind eta"})$$

and $\eta(q)^{24} \in M_{12}$.

Functions of partitions & quasimodularity (II)

More recently (DIJKGRAAF, BLOCH-OKOUNKOV, KANEKO-ZAGIER):

$$f : \underset{\substack{\uparrow \\ \text{set of all partitions}}}{\mathcal{P}} \to \mathbb{C} \quad \Longrightarrow \quad \langle f \rangle_q = \frac{\sum_{n \geq 0} q^n \sum_{|\lambda| = n} f(\lambda)}{\sum_{n \geq 0} q^n \, \#\{\lambda \in \mathcal{P} : |\lambda| = n\}}$$

Theorem (Bloch-Okounkov, 2000).

Assign $\deg Q_k = k$.

If f is a polynomial in $Q_0, Q_1, Q_2, \ldots$ of homogeneous degree k $\Rightarrow$ $\langle f \rangle_q \in \widetilde{M}_k$.

Functions of partitions & quasimodularity (III)

Even more recently (ZAGIER, VAN ITTERSUM):

$$S_k(\lambda) := -\frac{B_k}{2k} + \sum_{i=1}^{+\infty} \lambda_i^{k-1} \qquad k \geq 2,\ \text{even}$$

("SYMMETRIC FUNCTIONS OF PARTITIONS")

Theorem (van Ittersum, 2020):

Assign $\deg S_k = k$.

If f is a polynomial in $S_2, S_4, S_6, \dots$ of homogeneous degree k $\Rightarrow$ $\langle f \rangle_q \in \tilde{M}_k$.

Back to qKdV

Natural problem : describe eigenvalues of qKdV Hamiltonians

A coarser question : $\langle \text{eigenvalues}(\varepsilon=0) \rangle_q$ is quasimodular of homogeneous weight by the Theorems of Dubrovin and Bloch-Okounkov.

$$\boxed{\text{Does quasimodularity survive the } \varepsilon\text{-deformation?}}$$

A first positive indication: in the $\varepsilon \to \infty$ limit the eigenvalues behave as $\dfrac{\varepsilon^K \, S_{2K+2}(\lambda)}{(-4)^K \, (2K+1)!!}$

$$\varepsilon = 0 : Q_K(\lambda) \xleftrightarrow[\text{qKdV}]{} \varepsilon = \infty : S_K(\lambda)$$

Quasimodularity of differential polynomials

For $g \in \mathbb{C}[u] \Rightarrow \left\{ \bar{g} \right\}_q := \dfrac{\sum_{n \geq 0} q^n \, \mathrm{tr}_{\Lambda_n} \bar{g}}{\sum_{n \geq 0} q^n \dim \Lambda_n}$

<u>Theorem</u> (van Ittersum - R, 2022): Let $\mathcal{B} : \mathbb{C}[u] \to \mathbb{C}[u]$

$$\mathcal{B} := \exp\left(\frac{\hbar}{2} \sum_{i,j=0}^{+\infty} (-1)^{\frac{i-j}{2}} \frac{B_{i+j+2}}{i+j+2} \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} \right)$$

If $\mathcal{B} g$ is homogeneous w.r.t. $\deg u_k = k+1$, $\deg \hbar = 0$,

then: $\left\{ \bar{g} \right\}_q \in \left(\widetilde{M}[c, \hbar] \right)_k$ $\quad (\deg c = +1, \deg \hbar = 0)$.

(Proof largely based on previous work by van Ittersum)

Application to qKdV (I)

k	g_k^{qKdV} vs $\mathcal{B}\, g_k^{qKdV}$
-2	1
-1	u_0
0	$\frac{u_0^2}{2} - \frac{\hbar}{2} + \varepsilon u_2$
1	$\frac{u_0^3}{6} - \frac{\hbar}{24} u_0 - \frac{\hbar}{24} u_2 + \varepsilon\left(u_0 u_2 - \frac{\hbar}{120}\right) + \varepsilon^2 \frac{u_4}{2}$
2	$\frac{u_0^4}{24} - \frac{\hbar}{24} u_0 u_2 - \frac{\hbar}{48} u_0^2 + \frac{7\hbar}{5760} + \varepsilon\left(\frac{1}{2} u_0^2 u_2 - \frac{\hbar}{30} u_4 - \frac{\hbar}{24} u_2 - \frac{\hbar}{120} u_0\right) + \varepsilon^2\left(\frac{1}{2} u_0 u_4 + \frac{7}{10} u_2^2 - \frac{\hbar}{240}\right) + \varepsilon^3 \frac{u_6}{6}$

Application to qKdV (II)

We can answer our question!

<u>Theorem</u> (van Ittersum - R, 2022): We have

$$\left\{ \overline{g_k^{qKdV}}(\varepsilon) \right\} \in \left(\widetilde{M}[c, \varepsilon, \hbar] \right)_{k+2}$$

$(\deg c = +1,\ \deg \varepsilon = -1,\ \deg \hbar = 0)$

<u>Proof</u>: use criterion in the last slide. The recursion for $\mathcal{B} g_k^{qKdV}(\varepsilon)$ simplifies: it only involves $\mathcal{R}_1$! ∎

Quantum ILW hierarchy

Hodge CohFT: $c_{g,n} = 1 + \mu\lambda_1 + \dots + \mu^g\lambda_g$. Buryak-Rossi recursion:

$$g_{-2}^{qILW} = 1, \quad \frac{\partial g_k^{qILW}}{\partial u_0} = g_{k-1}, \quad \partial_x (D-1) g_k^{qILW} = (R_1 + R_2) g_{k-1}^{qILW}$$

$$R_1 g := \sum_{\ell=0}^{+\infty} \frac{\partial g}{\partial u_\ell} \partial_x^{\ell+1}\left(\frac{u_0^2}{2} + (\varepsilon - \hbar\mu)\sum_{\gamma\geq 1}\frac{(\varepsilon\mu)^{\gamma-1} B_{2\gamma}}{2(2\gamma)!} u_{2\gamma}\right) - \frac{\hbar}{2}\sum_{\ell,m=0}^{+\infty}\frac{\partial^2 g}{\partial u_\ell \partial u_m}\frac{(\ell+1)!\,(m+1)!}{(\ell+m+1)!} u_{\ell+m+3},$$

$$R_2 g := -\frac{\hbar}{2}\sum_{\ell,m,i=0}^{+\infty}\frac{\partial^2 g}{\partial u_\ell \partial u_m}\frac{B_{2i+2}}{2i+2}\left[(-1)^{\ell+i}\binom{m+1}{2i-\ell} + (-1)^{m+i}\binom{\ell+1}{2i-m}\right] u_{\ell+m-2i+1}.$$

Again: it simplifies for $Bg_k^{qILW} \Rightarrow Bg_k^{ILW}$ homogeneous $\Rightarrow$

$$\left\{g_k^{qILW}(\varepsilon,\mu)\right\}_q \in \left(\tilde{M}[c,\varepsilon,\mu,\hbar]\right)_{k+2} \qquad \begin{cases} \deg c = +1 \\ \deg \varepsilon = -1 = \deg \mu \\ \deg \hbar = 0 \end{cases}$$

Outlook

→ Quasimodularity of homogeneous weight constrains differential polynomials. Quantum KdV (& ILW) satisfy the constraint. Simplification of hamiltonian densities.

→ Quantum KdV interpolates between shifted symmetric and symmetric functions of partitions, preserving quasimodularity.

Future directions

→ Conjecture: eigenvalues of qKdV are Taylor series in ε with shifted symmetric coefficients (i.e. coefficients are polynomial in $Q_0, Q_1, Q_2, \dots$)
(or, equivalently?, in ε^{-1} with coefficients polynomial in $S_0, S_1 \dots$)

→ What for other Cohomological Field Theories of rank 1? Higher rank?

Thank you!