A symbolic umbral characterization of Riordan arrays

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Abstract

Riordan arrays are infinite **lower triangular complex valued-matrices** with applications to combinatorial identities, recurrence relations, walk problems, asymptotic approximations, the problem of normal ordering for boson strings, and so on, among other relevant topics. The **traditional** way of approaching Riordan arrays is by means of **generating functions**. I present in this poster a **promising alternative characterization** of Riordan arrays based on a **symbolic** renewed approach to the classical **umbral calculus**. A deep generalization of an **Abel's identity** for polynomials is a key tool in our symbolic approach.

Riordan arrays

Examples

$$\underset{P:\ \text{Pascal}}{\begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \\ & & \cdots & \cdots \end{pmatrix}} \quad \underset{C:\ \text{Ballot}}{\begin{pmatrix} 1 \\ 1 & 1 \\ 2 & 2 & 1 \\ 5 & 5 & 3 & 1 \\ 14 & 14 & 9 & 4 & 1 \\ 42 & 42 & 28 & 14 & 5 & 1 \\ & & \cdots & \cdots \end{pmatrix}} \quad \underset{C^{(1)}:\ \text{Catalan}}{\begin{pmatrix} 1 \\ 2 & 1 \\ 5 & 4 & 1 \\ 14 & 14 & 6 & 1 \\ 42 & 48 & 27 & 8 & 1 \\ 132 & 165 & 110 & 44 & 10 & 1 \\ & & \cdots & \cdots \end{pmatrix}} \quad \underset{S:\ \text{Stirling of 2nd. kind}}{\begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 7 & 6 & 1 \\ 0 & 1 & 15 & 25 & 10 & 1 \\ & & \cdots & \cdots \end{pmatrix}}$$

$R=(g,f)$	g	f	$R_{n,k}$
exponential	$1+g_1z+g_2\frac{z^2}{2!}+g_3\frac{z^3}{3!}+\cdots$	$z+f_2\frac{z^2}{2!}+f_3\frac{z^3}{3!}+\cdots$	$\left[\frac{z^n}{n!}\right]\left(g(z)\frac{f(z)^k}{k!}\right)$
ordinary	$1+g_1z+g_2z^2+g_3z^3+\cdots$	$z+f_2z^2+f_3z^3+\cdots$	$\left[z^n\right]\left(g(z)\,f(z)^k\right)$
generalized	$1+g_1\frac{z}{w_1}+g_2\frac{z^2}{w_2}+g_3\frac{z^3}{w_3}+\cdots$	$\frac{z}{w_1}+f_2\frac{z^2}{w_2}+f_3\frac{z^3}{w_3}+\cdots$	$\left[\frac{z^n}{w_n}\right]\left(g(z)\frac{f(z)^k}{w_k}\right)$

Table: Traditional description | (w_n): sequence of nonzero numbers.

Fundamental theorem of Riordan arrays (FTRA)

(g,f) : Riordan array | $A(z)=1+a_1z+a_2\frac{z^2}{2!}+a_3\frac{z^3}{3!}+\cdots$ | $B(z)=1+b_1z+b_2\frac{z^2}{2!}+b_3\frac{z^3}{3!}+\cdots$

We have

$$\big(g(z),f(z)\big)\begin{pmatrix}1\\a_1\\a_2\\a_3\\\vdots\end{pmatrix}=\begin{pmatrix}1\\b_1\\b_2\\b_3\\\vdots\end{pmatrix}\iff g(z)\,A\big(f(z)\big)=B(z).$$

The Riordan group [Shapiro et al. 1991] Set of all proper Riordan arrays, with group multiplication

$$\big(g(z),f(z)\big)\big(h(z),l(z)\big)=\Big(g(z)\,h\big(f(z)\big),l\big(f(z)\big)\Big)$$

Identity : $(1,z)$ Inverse : $\big(g(z),f(z)\big)^{-1}=\left(\frac{1}{g\big(f^{\langle-1\rangle}(z)\big)},f^{\langle-1\rangle}(z)\right)$

Some distinguished Riordan subgroups

Appell : $\big(g(z),z\big)$ | Associated : $\big(1,f(z)\big)$ | Bell : $\big(g(z),zg(z)\big)$

Umbrae

Basic data. We need:

1. a commutative integral domain R with identity 1.
2. a set $A=\{\alpha,\gamma,\omega,\ldots\}$ of umbrae, called *alphabet*.
3. a linear functional $E: R[A]\to R$ called *evaluation* such that
$$E[1]=1 \quad\text{and}\quad E[x^ny^m\,\alpha^i\gamma^j\cdots\omega^k]=x^ny^m\,E[\alpha^i]E[\gamma^j]\cdots E[\omega^k] \quad (\textit{uncorrelation property})$$
4. two special umbrae: ε (*augmentation*) and υ (*unity*) such that for all $n\geq 0$, we have
$$E[\varepsilon^n]=\delta_{0,n} \quad\text{and}\quad E[\upsilon^n]=1\,,$$

Two equivalence relations. We say that ω represents a sequence $(w_n)_{n\geq1}$ if $E[\omega^n]=w_n$ for all $n\geq1$. We refer to w_n as the n-th *moment* of ω. Assume $w_0=1$.

umbral equivalence : $\omega\simeq\gamma \iff E[\omega]=E[\gamma]$

similarity : $\omega\equiv\gamma \iff E[\omega^n]=E[\gamma^n],\ \forall\, n\geq 0.$

Key feature: Each sequence $(w_n)_{n\geq1}$ in R can be represented by infinitely many similar umbrae. This fact is called *saturation*.

Generating function of an umbra. The generating function of an umbra ω is the exponential formal series

$$e^{\omega z}:=\upsilon+\sum_{n\geq1}\omega^n\frac{z^n}{n!}\in R[A]\,[[z]] \quad\text{so that}\quad E[e^{\omega z}]=1+\sum_{n\geq1}w_n\frac{z^n}{n!}=:f_\omega(z)\in R[[z]]\,.$$

We write $e^{\omega z}\simeq f_\omega(z)$. Note that $\omega\equiv\gamma \iff e^{\omega z}\simeq e^{\gamma z}$.

umbra	ω	$e^{\omega z}$	ω^n
augmentation	ε	1	$1,0,0,\ldots$
unity	υ	e^z	$1,1,1,\ldots$
singleton	χ	$1+z$	$1,1,0,\ldots$
Bell	β	e^{e^z-1}	$1,B_2,B_3,\ldots$ (B_n : Bell numbers)
boolean unity	$\bar{\upsilon}$	$\frac{1}{1-z}$	$1!,2!,3!,\ldots$ ($n!$: factorial numbers)
Catalan	ς	$\frac{1-\sqrt{1-4z}}{2z}$	$C_1,2!C_2,3!C_3,\ldots$ (C_n : Catalan numbers)

Table: Some fundamental umbrae.

Some useful auxiliary umbrae

- The dot product of umbrae $\gamma.\alpha$, $f_{\gamma.\alpha}(z)=f_\gamma\big(\log f_\alpha(z)\big)$

  composition umbrae $\gamma.\beta.\alpha$, $f_{\gamma.\beta.\alpha}=f_\gamma\big(f_\alpha(z)-1\big)$ comp. inverse of an umbra $\gamma^{\langle-1\rangle}$, $\gamma^{\langle-1\rangle}.\beta.\gamma\equiv\chi\equiv\gamma.\beta.\gamma^{\langle-1\rangle}$
- Derivative umbrae $\alpha_{\mathcal{D}}$, $\alpha_{\mathcal{D}}^n\simeq n\,\alpha^{n-1}$, $f_{\alpha_{\mathcal{D}}}(z)=1+z\,f_\alpha(z)$

Remark

One portion of the introduction is especially noteworthy. "When I first encountered umbral notation it seemed to me that this was all there was to it; it was simply a notation for dealing with exponential generating functions, or to put it bluntly, it was a method for avoiding the use of exponential generating functions when they really ought to be used. The point of this paper is that my first impression was wrong; none of the results proved here (...) can be easily proved by straightforward manipulation of exponential generating functions."

Reviewed by George E. Andrews

Symbolic umbral presentation of Riordan arrays

R	notation	$R_{n,k}$
exponential	(γ,α)	$\binom{n}{k}(\gamma+k.\alpha)^{n-k}$
generalized	${}_\omega(\gamma,\alpha)$	$\frac{w_n}{w_k}\frac{(\gamma+k.\alpha)^{n-k}}{(n-k)!}$

Table: Symbolic umbral description | $\omega^n\simeq\frac{n!}{w_n}$, $w_n\neq0$ for all $n\geq0$, $w_0=1$

FTRA : $(\gamma,\alpha)\eta=\gamma+\eta.\beta.\alpha_{\mathcal{D}}$ **Group multiplication** : $(\gamma,\alpha)(\sigma,\rho)=(\gamma+\sigma.\beta.\alpha_{\mathcal{D}}\,,\,\alpha+\rho.\beta.\alpha_{\mathcal{D}})$

Identity : $(\varepsilon,\varepsilon)$ **Inverse** : $(\gamma,\alpha)^{-1}=(\mathfrak{L}_{\gamma,\alpha},\mathfrak{L}_\alpha)$, $\mathfrak{L}_{\gamma,\alpha}\equiv-1.\gamma.\beta.\alpha_{\mathcal{D}}^{\langle-1\rangle}$

Subgroup	(γ,α)	group multiplication	$(\gamma,\alpha)^{-1}$	$(\gamma,\alpha)_{n,k}$
Appell	(γ,ε)	$(\gamma,\varepsilon)(\sigma,\varepsilon)=(\gamma+\sigma,\varepsilon)$	$(-1.\alpha,\varepsilon)$	$\binom{n}{k}\gamma^{n-k}$
Associated	(ε,α)	$(\varepsilon,\alpha)(\varepsilon,\rho)=(\varepsilon,\alpha+\rho.\beta.\alpha_{\mathcal{D}})$	$(\varepsilon,\mathfrak{L}_\alpha)$	$\binom{n}{k}(k.\alpha)^{n-k}$
Bell	(α,α)	$(\alpha,\alpha)(\sigma,\sigma)=(\alpha+\sigma.\beta.\alpha_{\mathcal{D}},\alpha+\sigma.\beta.\alpha_{\mathcal{D}})$	$(\mathfrak{L}_\alpha,\mathfrak{L}_\alpha)$	$\binom{n}{k}((k+1).\alpha)^{n-k}$

Table: Distinguished Riordan subgroups

Abel's identity

Classical formula $$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}(y+ka)^{n-k}x(x-ka)^{k-1}$$

Umbral formula (Version I) $$(\gamma+\sigma)^n\simeq\sum_{k=0}^{n}\binom{n}{k}(\gamma+k.\alpha)^{n-k}\sigma(\sigma+(-k).\alpha)^{k-1}$$

Umbral formula (Version II) = FTRA $$(\gamma+\eta.\beta.\alpha_{\mathcal{D}})^n\simeq\sum_{k=0}^{n}\binom{n}{k}(\gamma+k.\alpha)^{n-k}\eta^k$$

Known and new recurrence relations

Theorem [A.-Mestre-Petrullo-Torres] *For any umbra λ and any integers m,n,k, with $n\geq k$, it holds*

$${}_\omega(\gamma,\alpha)_{n,k}\simeq\frac{w_n}{w_k}\sum_{i=0}^{n-k}\frac{\big(m.\mathfrak{K}_{\alpha,\lambda}\big)^i}{i!}\,\frac{\big(\gamma+(k-m).\alpha+i.\lambda\big)^{n-k-i}}{(n-k-i)!}\,.$$

Corollary 1. (Horizontal recurrence relation) *For any integer m such that $n\geq m$, it holds*

$${}_\omega(\gamma,\alpha)_{n,k}\simeq\frac{w_n}{w_k\,w_{n-m}}\sum_{i=0}^{n-k}w_{k-m+i}\frac{\big(m.\mathfrak{K}_{\alpha,\alpha}\big)^i}{i!}\;{}_\omega(\gamma,\alpha)_{n-m,k-m+i}\,.$$

Corollary 2. (Vertical recurrence relation) *For any integer m such that $k\geq m$, it holds*

$${}_\omega(\gamma,\alpha)_{n,k}\simeq\frac{w_n\,w_{k-m}}{w_k}\sum_{i=0}^{n-k}\frac{1}{w_{n-m-i}}\frac{\big(m.\alpha\big)^i}{i!}\;{}_\omega(\gamma,\alpha)_{n-m-i,k-m}\,.$$

Corollary 3. (A novel recurrence relation) *For any integer m such that $2k-n\geq m$, it holds*

$${}_\omega(\gamma,\alpha)_{n,k}\simeq\frac{w_n}{w_k}\sum_{i=0}^{n-k}\frac{w_{k-m-i}}{w_{n-m-2i}}\frac{\big(m.\mathfrak{K}_{\alpha,-1.\alpha}\big)^i}{i!}\;{}_\omega(\gamma,\alpha)_{n-m-2i,k-m-i}\,.$$

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