Polytope decompositions and Euler Maclaurin formulas

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Abstract

In the first part of the talk we will present a weighted polytope decomposition that encompasses other known polytope decompositions.

In the second part, we will talk about Euler Maclaurin formulas and explain the role played on them by polytope decompositions.

This presentation is partly based on joint work done and in progress with Leonor Godinho (IST).

Contents

Contents

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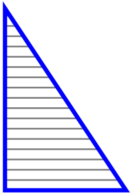
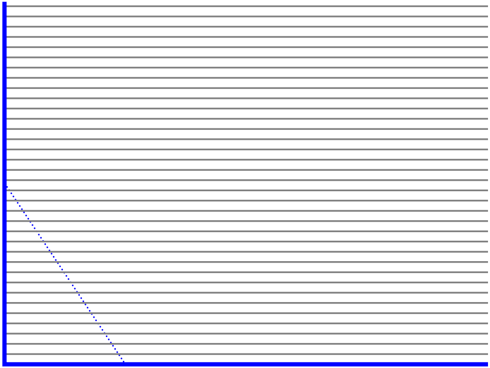
Contents

Contents

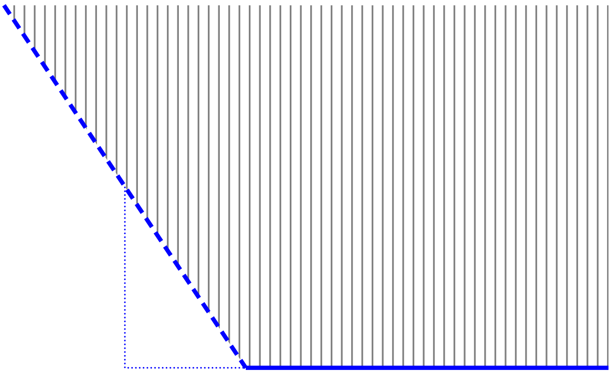
Characteristic function of triangle ABC



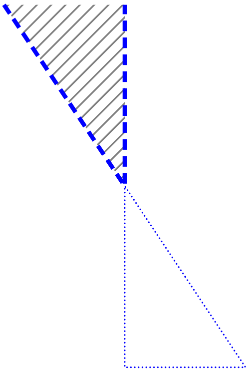
Characteristic function of cone with vertex at A



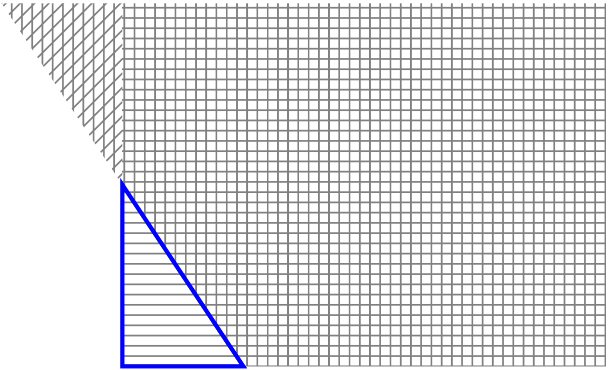
Characteristic function of cone with vertex at B



Characteristic function of cone with vertex at C



Signed sum of characteristic functions



The algebra of polyhedra

V: real vector space of finite dimension.

$A \subset V$: polyhedron

$\mathbf{1}_A$: characteristic function of A.

$\mathcal{P}(V)$: algebra of polyhedra = vector space spanned by the characteristic functions of all polyhedra $A \subset V$.

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Thus,

$$f \in \mathcal{P}(V) \quad \text{means} \quad f = \sum_{i=1}^{m} \alpha_i \mathbf{1}_{A_i},$$

where the A_i are polyhedra in V and α_i are complex numbers.

Polytope decomposition

A polytope decomposition is an element f of $\mathcal{P}(V)$, such that

$$f = \mathbf{1}_P \quad \text{and} \quad f = \sum_{F \in \mathcal{F}} \alpha_F \, \mathbf{1}_{\mathbf{C}_F},$$

where:

- P is a polytope (a bounded polyhedron),
- $\mathcal{F}$ is a collection of faces of P and
- the $\mathbf{C}_F$ are (tangent) cones based at the faces $F \in \mathcal{F}$, oriented towards the polytope P.

Simple, unimodular and lattice polyhedra

A polyhedron C is called simple if each of its vertices v is the intersection of exactly n facets. This is equivalent to saying that the vectors u_j normal to the facets that define v, form a basis of V^*.

We say that C is unimodular if, in addition, for all vertices v, the corresponding vectors u_j normal to the facets that define v form a basis of the lattice $V^*_{\mathbb{Z}}$.

If the vertices of C are in the lattice $V_{\mathbb{Z}}$, we say that C is an integral or lattice polyhedron. If the vertices are in $V_{\mathbb{Q}}$ we say that C is rational. $\left(V_{\mathbb{Q}} = \{x \in V : mx \in V_{\mathbb{Z}},\ \text{for some } m \in \mathbb{Z}\}\right)$

For us, C will be either a polytope or a tangent cone.

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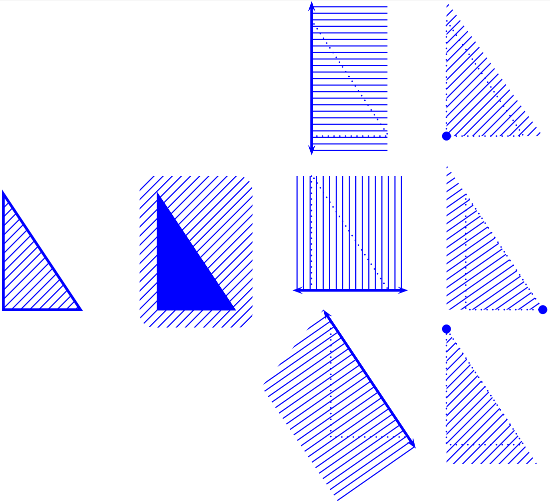
Contents

Some known polytope decompositions

- Brianchon-Gram (1837, 1874).
- Brion (1988).
- Lawrence-Varchenko (1991, 1987).

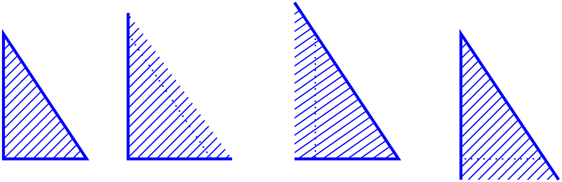
Brianchon-Gram

$$\mathbf{1}_P = \sum_{F \text{ face of } P} (-1)^{\dim F} \mathbf{1}_{\mathbf{C}_F}$$



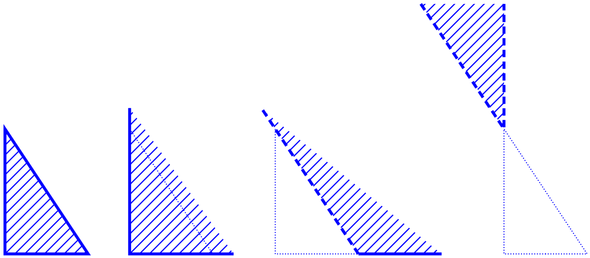
Brion

$$\mathbf{1}_P = \sum_{\mathbf{v} \text{ vertex of } P} \mathbf{1}_{\mathbf{C}_\mathbf{v}} + g \,, \quad \text{where} \quad g = \sum_{F,\, \dim F > 0} (-1)^{\dim F} \mathbf{1}_{\mathbf{C}_F}$$



Lawrence-Varchenko

$$\mathbf{1}_P = \sum_{\mathbf{v} \text{ vertex of } P} (-1)^{m_{\mathbf{v}}} \mathbf{1}_{\mathbf{C}^\sharp_{\mathbf{v}}}$$



Can we cook up other polytope decompositions?

Contents

Open set W

$\mathcal{F}$: collection of all faces of Δ.

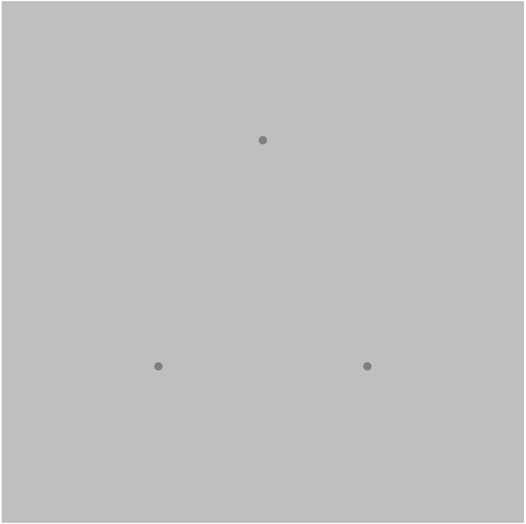
For each $F \in \mathcal{F}$ define:

- $\mathscr{B}_F := \{G \in \mathcal{F} \,:\, G \text{ proper subface of } F\}$
- $\mathrm{aff}(F)_{\mathscr{B}_F} := \mathrm{aff}(F) \setminus \bigcup_{G \in \mathscr{B}_F} G$
- $W_F := \mathrm{aff}(F)_{\mathscr{B}_F} + \mathrm{aff}(F)^{\perp}$

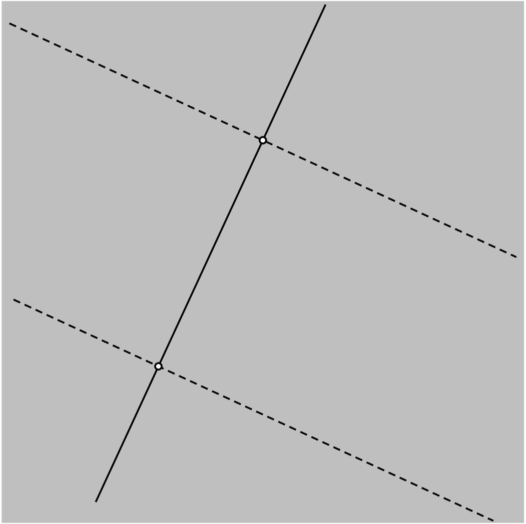
Then W is define by

$$W := \bigcap_{F \in \mathcal{F}} W_F$$

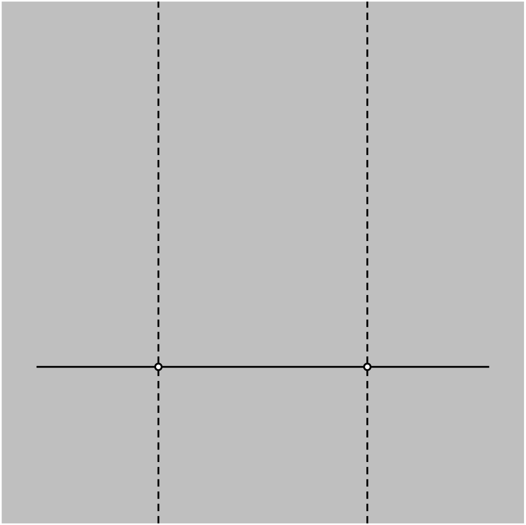
Open regions W_A, W_B, W_C



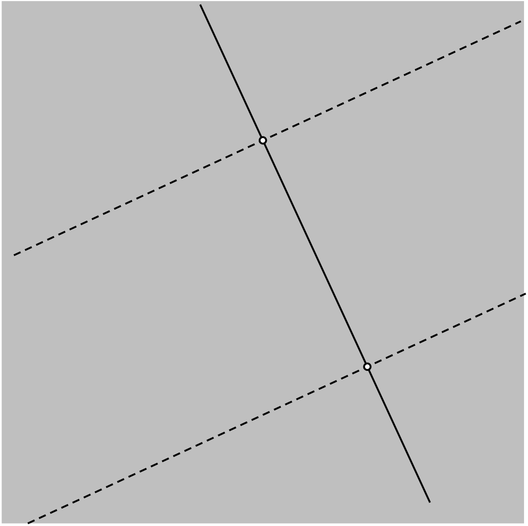
Open region W_{AC}



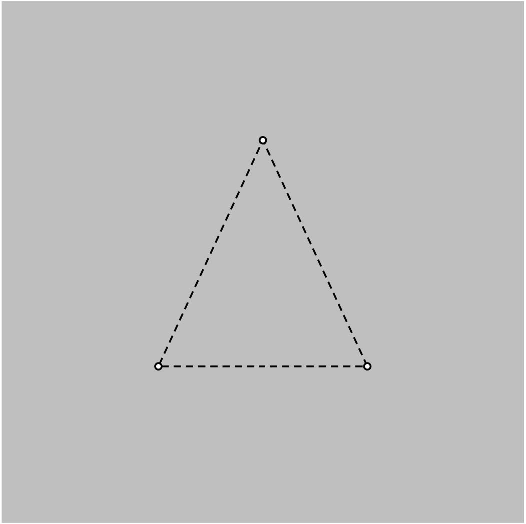
Open region W_{AB}



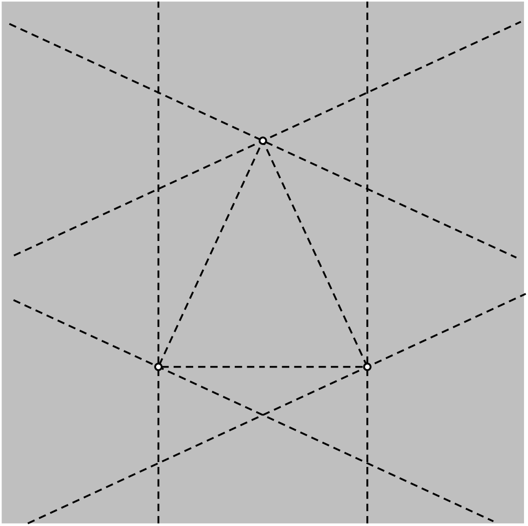
Open region W_{BC}



Open region W_{ABC}



Open set $W = \bigcap_{F \in \mathcal{F}} W_F$



Polarizing vectors

Δ: polytope in V, $\varepsilon \in V$, F: face of Δ.

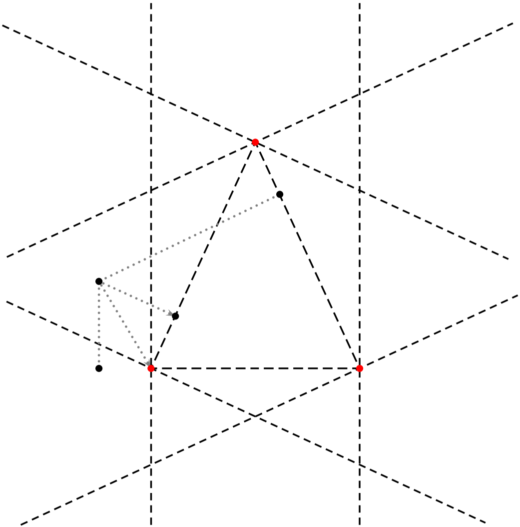
$\text{aff}(F)$: affine space in V generated by the face F.

$\beta\big(\varepsilon, \text{aff}(F)\big)$: orthogonal projection of ε onto $\text{aff}(F)$.

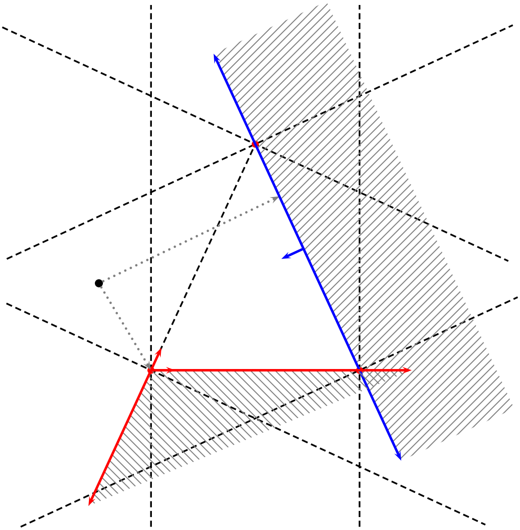
W: the open set previously defined.

Fixing $\varepsilon \in W$, the $\beta_F := \beta\big(\varepsilon, \text{aff}(F)\big) - \varepsilon$ are called **polarizing vectors** of the tangent cones of Δ at its different faces.

Polarizing vectors



Polarized tangent cones vectors



Weights

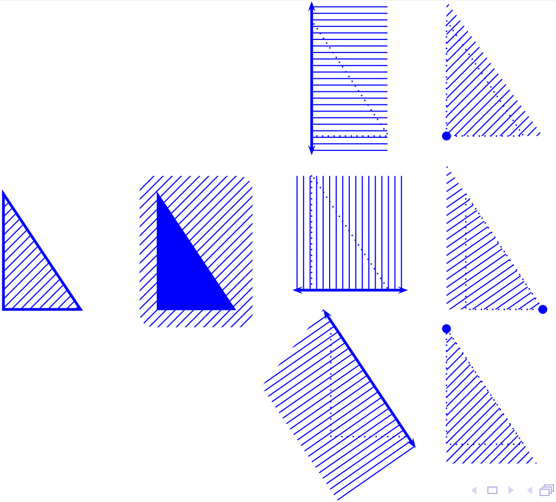
Δ: polytope in V with d facets. F: face of Δ.

$q_1, \ldots q_d$: collection of d arbitrary complex numbers.

We assign $1 - q_j$ or q_j to the j^{th}-facet of Δ (and the corresponding facets of $\mathbf{C}_F$ and $\mathbf{C}_F^\sharp$) according to whether the orientation of the j^{th}-facet changed or not after polarization.

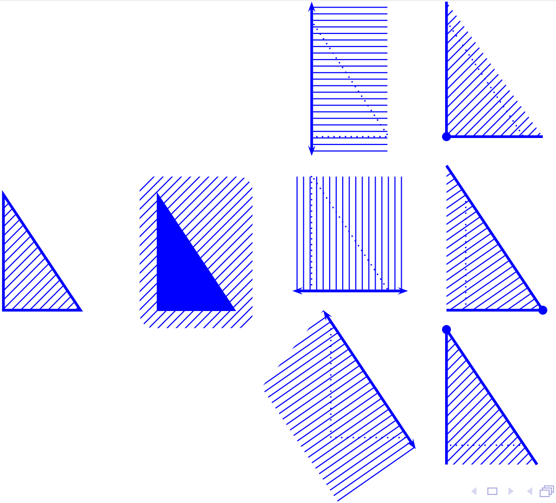
Ordinary Brianchon-Gram

$$\mathbf{1}_P = \sum_{F \text{ face of } P} (-1)^{\dim F} \mathbf{1}_{\mathbf{C}_F}$$



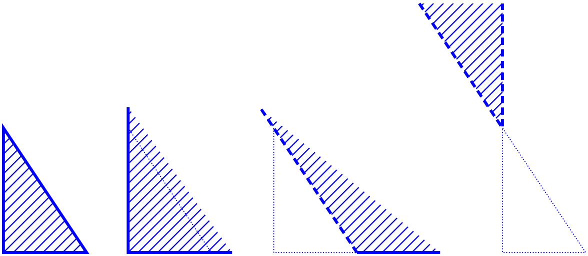
Weighted Brianchon-Gram

$$\mathbf{1}_P^w = \sum_{F \text{ face of } P} (-1)^{\dim F} \mathbf{1}_{\mathbf{C}_F}^w$$



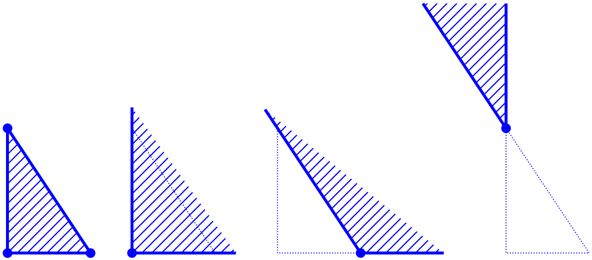
Ordinary Lawrence-Varchenko

$$\mathbf{1}_P = \sum_{\mathbf{v} \text{ vertex of } P} (-1)^{m_{\mathbf{v}}} \mathbf{1}_{\mathbf{C}^\sharp_{\mathbf{v}}}$$



Weighted Lawrence-Varchenko

$$\mathbf{1}_P^w = \sum_{\mathbf{v} \text{ vertex of } P} (-1)^{m_\mathbf{v}} \mathbf{1}^w_{\mathbf{C}^\sharp_\mathbf{v}}$$



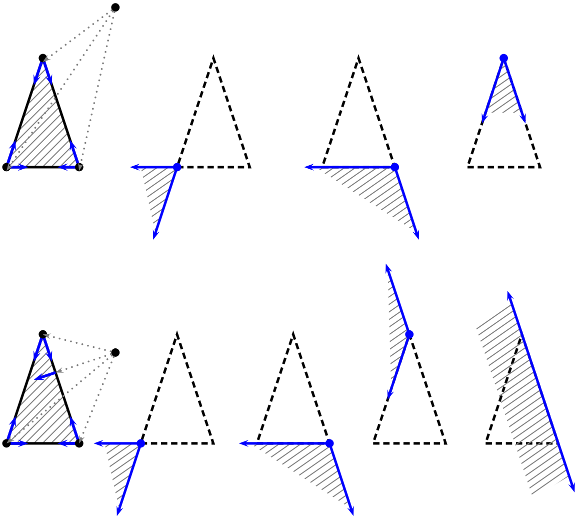
A.-Godinho polytope decomposition (2007)

Let $\Delta \subset \mathbb{R}^d$ be a simple polytope of dimension d. Let $\varepsilon \in W$. Then,

$$\mathbf{1}^w_\Delta = \sum_{F \subseteq \Delta} (-1)^{m_F} \mathbf{1}_\Delta(\, \beta(\varepsilon, \mathrm{aff}(F))\,)\, \mathbf{1}^w_{\mathbf{C}^\sharp_F},$$

where the sum is over all faces F of Δ and where m_F is the number of generators of $\mathbf{C}_F$ that are flipped by polarization.

A.-Godinho polytope decomposition (2006)



Localization formulas behind the scenes

The AG polytope decomposition was motivated by the use of two localization formulas in equivariant cohomology due to:

- Atiyah-Bott and Berline-Vergne (abelian case)
- Witten (nonabelian case)

The non-simple case

Using compatible regular triangulations we can extend AG's polytope decomposition to non-simple polytopes for the unweighted case.

When using weights, the extension is only possible if we assign the same weight to all the facets of the polytope.

Contents

Euler-Maclaurin (1730s-40s)

Let $I = [a, b]$ be an interval with $a, b \in \mathbb{Z}$ and let f be a polynomial function with real coefficients. Then

$$\sum_{x \in I \cap \mathbb{Z}} f(x) = \int_a^b f(x)\, dx + \frac{1}{2}\Big[f(b) + f(a)\Big] + \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)!}\Big[f^{(2k-1)}(b) - f^{(2k-1)}(a)\Big],$$

where the b_{2k} $(k \geq 1)$ are the (signed) Bernoulli numbers.

Writing the classical formula in a convenient way

Note that we can write the RHS of the classical formula as

$$\sum_{x\in I\cap\mathbb{Z}} f(x) = \int_a^b f(x)\,dx + \frac{1}{2}\frac{\partial}{\partial h_1}\int_{-\infty}^{b+h_1} f(x)\,dx\Big|_{h_1=0} - \frac{1}{2}\frac{\partial}{\partial h_2}\int_{-\infty}^{a-h_2} f(x)\,dx\Big|_{h_2=0} +$$

$$+\sum_{k=1}^{\infty}\frac{b_{2k}}{(2k)!}\frac{\partial^{2k}}{\partial h_1^{2k}}\int_{-\infty}^{b+h_1} f(x)\,dx\Big|_{h_1=0} - \sum_{k=1}^{\infty}\frac{b_{2k}}{(2k)!}\frac{\partial^{2k}}{\partial h_2^{2k}}\int_{-\infty}^{a-h_2} f(x)\,dx\Big|_{h_2=0}$$

$$= \int_a^b f(x)\,dx + \left(1+\frac{1}{2}\frac{\partial}{\partial h_1}+\sum_{k=1}^{\infty}\frac{b_{2k}}{(2k)!}\frac{\partial^{2k}}{\partial h_1^{2k}}\right)\int_{-\infty}^{b+h_1} f(x)\,dx\Big|_{h_1=0} - \int_{-\infty}^{b} f(x)\,dx +$$

$$-\left(1+\frac{1}{2}\frac{\partial}{\partial h_2}+\sum_{k=1}^{\infty}\frac{b_{2k}}{(2k)!}\frac{\partial^{2k}}{\partial h_2^{2k}}\right)\int_{-\infty}^{a-h_2} f(x)\,dx\Big|_{h_2=0} + \int_{-\infty}^{a} f(x)\,dx$$

Writing the classical formula in a convenient way

For $|z| < 2\pi$, we have

$$\frac{z}{1-e^{-z}} = 1 + \frac{1}{2}z + \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)!} z^{2k}.$$

Writing the classical formula in a convenient way

For $|z| < 2\pi$, we have

$$\mathbf{Td}(z) = \frac{z}{1-e^{-z}} = 1 + \frac{1}{2}z + \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)!} z^{2k}. \quad \text{(Todd function)}$$

Writing the classical formula in a convenient way

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Then

$$\sum_{x \in I \cap \mathbb{Z}} f(x) = \mathbf{Td}\left(\frac{\partial}{\partial h_1}\right) \int_{-\infty}^{b+h_1} f(x)\,dx \Big|_{h_1=0} - \mathbf{Td}\left(\frac{\partial}{\partial h_2}\right) \int_{-\infty}^{a-h_2} f(x)\,dx \Big|_{h_2=0}.$$

Setting $h = (h_1, h_2)$, $I(h) = [a - h_2, b + h_1]$ and $C(x) = (-\infty, x]$, we can write

$$\sum_{x \in I \cap \mathbb{Z}} f(x) = \mathbf{Td}\left(\frac{\partial}{\partial h_1}\right) \mathbf{Td}\left(\frac{\partial}{\partial h_2}\right) \left(\int_{C(b+h_1)} f(x)\,dx - \int_{C(a-h_2)} f(x)\,dx \right) \Big|_{h=0}$$

$$= \mathbf{Td}\left(\frac{\partial}{\partial h_1}\right) \mathbf{Td}\left(\frac{\partial}{\partial h_2}\right) \int_{I(h)} f(x)\,dx \Big|_{h=0}.$$

Classical formula (two presentations)

$$\sum_{x \in I \cap \mathbb{Z}} f(x) = \frac{1}{2}\left[f(b) + f(a)\right] + \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)!}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] + \int_a^b f(x)\, dx$$

$$= \mathbf{Td}\left(\frac{\partial}{\partial h_1}\right) \mathbf{Td}\left(\frac{\partial}{\partial h_2}\right) \int_{I(h)} f(x)\, dx \Bigg|_{h=0}$$

Contents

Two possible extensions to higher dimensions

Δ: polytope in V ($\dim V = n$) of full dimension with d facets.

$\{P_F\}_{F\subset\Delta}$: collection of infinite order linear partial differential operators with constant coefficients in n variables, parameterized by the faces F of Δ. Then, for a suitable function f on V,

$$\sum_{x\in\Delta\cap V_{\mathbb{Z}}} f(x) = \sum_{F \text{ face of } \Delta} \int_F P_F f \, d\nu_F \, . \tag{1}$$

$\Delta(h)$: polytope in V obtained from Δ by parallel translation of its facets.

$P\left(\frac{\partial}{\partial h_1}, \cdots, \frac{\partial}{\partial h_d}\right)$: infinite order linear partial differential operator with constant coefficients in d variables. Then, for a suitable function f on V,

$$\sum_{x\in\Delta\cap V_{\mathbb{Z}}} f(x) = P\left(\frac{\partial}{\partial h_1}, \cdots, \frac{\partial}{\partial h_d}\right) \int_{\Delta(h)} f \, d\nu_\Delta \Big|_{h=0} \, . \tag{2}$$

Some known Euler Maclaurin formulas

- Khovanskii and Pukhlikov (1992)
- Cappell and Shaneson (1994)
- Brion and Vergne (1995)
- Guillemin (1997)

Distinguished power series

$$\mathbf{Td}(z) = 1 + \frac{1}{2}z + \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)!} z^{2k} \qquad \text{(Todd function).}$$

$$\mathbf{L}(z) = 1 + \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)!} z^{2k} \qquad \text{(L function).}$$

Distinguished power series

For $|z|$ small enough,

$$\mathbf{Td}(z) = 1 + \frac{1}{2}z + \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)!} z^{2k} = \frac{z}{1 - e^{-z}} \qquad \text{(Todd function).}$$

$$\mathbf{L}(z) = 1 + \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)!} z^{2k} = \frac{z}{2}\coth\left(\frac{z}{2}\right) = \frac{1}{2}\mathbf{Td}(z) + \frac{1}{2}\mathbf{Td}(-z) \qquad \text{(L function).}$$

Distinguished power series

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$$\mathbf{T}(a,z) = \frac{z}{1-ae^{-z}}\ ,\ a \in S^1 \qquad , \qquad \mathbf{L}(a,z) = \frac{1}{2}\mathbf{T}(a,z) + \frac{1}{2}\mathbf{T}(a^{-1},z)\,.$$

Distinguished power series

For $|z|$ small enough,

$$\mathbf{Td}(z) = 1 + \frac{1}{2}z + \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)!}\, z^{2k} = \frac{z}{1-e^{-z}} \qquad \text{(Todd function).}$$

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$$\mathbf{T}(a,z) = \frac{z}{1-ae^{-z}}\ , \ a \in S^1 \qquad , \qquad \mathbf{L}(a,z) = \frac{1}{2}\mathbf{T}(a,z) + \frac{1}{2}\mathbf{T}(a^{-1},z)\,.$$

In general,

$$\mathbf{N}_q^a(z) = q\mathbf{T}(a,z) + (1-q)\mathbf{T}(a^{-1},-z) \quad , \quad a \in S^1 \quad \text{and} \quad q \in \mathbb{C}\,.$$

khovanski-Pukhlikov (1992)

Δ: unimodular lattice polytope in V with d facets.

f: polynomial or exponential function on V.

$$\sum_{x\in\Delta\cap V_{\mathbb{Z}}} f(x) = \prod_{i=1}^{d} \mathbf{Td}\left(\frac{\partial}{\partial h_i}\right) \int_{\Delta(h)} f\, d\nu_{\Delta}\bigg|_{h=0}.$$

Cappell-Shaneson (1994)

Δ: simple lattice polytope in V with d facets.

f: polynomial function on V.

$$\sum_{x \in \Delta \cap V_{\mathbb{Z}}} f(x) = \sum_{F \subset \Delta} \int_F \mathcal{P}_F f \, d\nu_F \, .$$

Brion-Vergne (1995,1997)

Δ: simple lattice polytope in V with d facets.

f: exponential or polynomial function on V.

$$\sum_{x\in\Delta\cap V_{\mathbb{Z}}} f(x) = \sum_{F\subset\Delta} \sum_{\gamma\in\Gamma_F^\flat} e^{2\pi i\langle\gamma,F\rangle} \prod_{j\notin I_F} \mathbf{Td}\left(\frac{\partial}{\partial h_j}\right) \prod_{j\in I_F} \mathbf{T}\left(a_{j,F,\gamma},\frac{\partial}{\partial h_j}\right) \int_{\Delta(h)} f\,d\nu_\Delta\Big|_{h=0}$$

$$= \sum_{F\subset\Delta} \sum_{\gamma\in\Gamma_F^\flat} e^{2\pi i\langle\gamma,F\rangle} \prod_{j=1}^{d} \mathbf{T}\left(a_{j,F,\gamma},\frac{\partial}{\partial h_j}\right) \int_{\Delta(h)} f\,d\nu_\Delta\Big|_{h=0}.$$

$$\sum_{x\in\mathrm{int}(\Delta)\cap V_{\mathbb{Z}}} f(x) = \sum_{F\subset\Delta} \sum_{\gamma\in\Gamma_F^\flat} e^{2\pi i\langle\gamma,F\rangle} \prod_{j=1}^{d} \mathbf{T}\left(a_{j,F,\gamma}^{-1},-\frac{\partial}{\partial h_j}\right) \int_{\Delta(h)} f\,d\nu_\Delta\Big|_{h=0}.$$

Brion-Vergne (1995,1997)

Δ: simple lattice polytope in V with d facets.

f: exponential or polynomial function on V.

$$\sum_{x\in\Delta\cap V_{\mathbb{Z}}} f(x) = \sum_{F\subset\Delta}\sum_{\gamma\in\Gamma_F^\flat} e^{2\pi i\langle\gamma,F\rangle}\prod_{j\notin I_F}\mathbf{Td}\left(\frac{\partial}{\partial h_j}\right)\prod_{j\in I_F}\mathbf{T}\left(a_{j,F,\gamma},\frac{\partial}{\partial h_j}\right)\int_{\Delta(h)} f\,d\nu_\Delta\bigg|_{h=0}$$

$$= \sum_{F\subset\Delta}\sum_{\gamma\in\Gamma_F^\flat} e^{2\pi i\langle\gamma,F\rangle}\prod_{j=1}^{d}\mathbf{T}\left(a_{j,F,\gamma},\frac{\partial}{\partial h_j}\right)\int_{\Delta(h)} f\,d\nu_\Delta\bigg|_{h=0}.$$

$$\sum_{x\in\mathrm{int}(\Delta)\cap V_{\mathbb{Z}}} f(x) = \sum_{F\subset\Delta}\sum_{\gamma\in\Gamma_F^\flat} e^{2\pi i\langle\gamma,F\rangle}\prod_{j=1}^{d}\mathbf{T}\left(a_{j,F,\gamma}^{-1},-\frac{\partial}{\partial h_j}\right)\int_{\Delta(h)} f\,d\nu_\Delta\bigg|_{h=0}.$$

Brion-Vergne (1995,1997)

Δ: simple lattice polytope in V with d facets.

f: exponential or polynomial function on V.

$$\sum_{x\in\Delta\cap V_{\mathbb{Z}}} f(x) = \sum_{F\subset\Delta}\sum_{\gamma\in\Gamma_F^\flat} e^{2\pi i\langle\gamma,F\rangle}\prod_{j\notin I_F}\mathbf{Td}\left(\frac{\partial}{\partial h_j}\right)\prod_{j\in I_F}\mathbf{T}\left(a_{j,F,\gamma},\frac{\partial}{\partial h_j}\right)\int_{\Delta(h)} f\,d\nu_\Delta\Big|_{h=0}$$

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$$= \sum_{F\subset\Delta} \sum_{\gamma\in\Gamma_F^\flat} e^{2\pi i\langle\gamma,F\rangle} \prod_{j=1}^{d} \mathbf{T}\left(a_{j,F,\gamma}, \frac{\partial}{\partial h_j}\right) \int_{\Delta(h)} f\, d\nu_\Delta \Big|_{h=0}.$$

$$\sum_{x\in\mathrm{int}(\Delta)\cap V_{\mathbb{Z}}} f(x) = \sum_{F\subset\Delta} \sum_{\gamma\in\Gamma_F^\flat} e^{2\pi i\langle\gamma,F\rangle} \prod_{j=1}^{d} \mathbf{T}\left(a_{j,F,\gamma}^{-1}, -\frac{\partial}{\partial h_j}\right) \int_{\Delta(h)} f\, d\nu_\Delta \Big|_{h=0}.$$

Guillemin (1997)

Δ: simple polytope in V.

f: exponential or polynomial function on V.

$$\sum_{x\in\Delta\cap V_{\mathbb{Z}}} f(x) = \sum_{F\subset\Delta} \frac{1}{|\Gamma_F|} \sum_{\gamma\in\Gamma_F^\flat} e^{2\pi i\langle\gamma,F\rangle} \prod_{j\notin I_F} \mathbf{Td}\left(\frac{\partial}{\partial h_j}\right) \prod_{j\in I_F} \frac{1}{1-a_{j,F,\gamma}\, e^{-\frac{\partial}{\partial h_j}}} \int_{F(h)} f\, d\nu_F \bigg|_{h=0}.$$

Contents

Weighted Euler-Maclaurin formulas

- Karshon, Sternberg and Weitsman (2005)
- A. and Weitsman (2005)
- A. and Godinho (2007)

Karshon-Sternberg-Weitsman (2005)

Δ: simple lattice polytope in V with d facets.

f: exponential or polynomial function on V.

$$\sum_{x\in\Delta\cap V_{\mathbb{Z}}}^{\frac{1}{2}} f(x) = \sum_{F\subset\Delta}\sum_{\gamma\in\Gamma_F^\flat}\prod_{j\notin I_F}\mathbf{L}\left(\frac{\partial}{\partial h_j}\right)\prod_{j\in I_F}\mathbf{L}\left(a_{j,F,\gamma},\frac{\partial}{\partial h_j}\right)\int_{\Delta(h)} f\,d\nu_\Delta\Big|_{h=0}$$

$$= \sum_{F\subset\Delta}\sum_{\gamma\in\Gamma_F^\flat}\prod_{j=1}^{d}\mathbf{L}\left(a_{j,F,\gamma},\frac{\partial}{\partial h_j}\right)\int_{\Delta(h)} f\,d\nu_\Delta\Big|_{h=0}.$$

Karshon-Sternberg-Weitsman (2005)

Δ: simple lattice polytope in V with d facets.

f: exponential or polynomial function on V.

$$\sum_{x \in \Delta \cap V_{\mathbb{Z}}}^{\frac{1}{2}} f(x) = \sum_{F \subset \Delta} \sum_{\gamma \in \Gamma_F^{\flat}} \prod_{j \notin I_F} \mathbf{L}\left(\frac{\partial}{\partial h_j}\right) \prod_{j \in I_F} \mathbf{L}\left(a_{j,F,\gamma}, \frac{\partial}{\partial h_j}\right) \int_{\Delta(h)} f \, d\nu_{\Delta} \Big|_{h=0}$$

$$= \sum_{F \subset \Delta} \sum_{\gamma \in \Gamma_F^{\flat}} \prod_{j=1}^{d} \mathbf{L}\left(a_{j,F,\gamma}, \frac{\partial}{\partial h_j}\right) \int_{\Delta(h)} f \, d\nu_{\Delta} \Big|_{h=0} .$$

A.-Godinho (2007)

Advances in Mathematics, 214, No. 1, pp. 379-416
Preprint version at ArXiV: math.CO/0512475.

Δ: simple lattice polytope in V with d facets.

f: exponential or polynomial function on V.

$$\sum_{x\in\Delta\cap V_{\mathbb{Z}}}^{q} f(x) = \sum_{F\subset\Delta} \sum_{\gamma\in\Gamma_F^\flat} e^{2\pi i\langle\gamma,F\rangle} \prod_{j=1}^{d} \mathbf{N}_{q_j}^{a_j,F,\gamma} \left(\frac{\partial}{\partial h_j}\right) \int_{\Delta(h)} f\, d\nu_\Delta \Big|_{h=0}, \qquad \text{(AG)}$$

where $q=(q_1,\ldots,q_d)\in\mathbb{C}^d$.

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where $q=(q_1,\ldots,q_d)\in\mathbb{C}^d$.

All but Cappell-Shaneson's Euler Maclaurin formula are particular cases of (AG). For instance, when all $q_j=1$ we obtain the unweighted formulas. When Δ is integral and all $q_j=\frac{1}{2}$, we get KSW's formula, and when all $q_j=q$, we get AW's formula.

Final comments

How is CS's formula related to the other ones?

Recall that CS's formula was presented in a way different from the other EM formulas we have talked about.

Euler Maclaurin formulas (two presentations)

Δ: polytope in V ($\dim V = n$) of full dimension with d facets.

$\{P_F\}_{F\subset\Delta}$: collection of infinite order linear partial differential operators with constant coefficients in n variables, parameterized by the faces F of Δ. Then, for a suitable function f on V,

$$\sum_{x\in\Delta\cap V_{\mathbb{Z}}} f(x) = \sum_{F \text{ face of } \Delta} \int_F P_F f \, d\nu_{_F} \,. \tag{1}$$

$\Delta(h)$: polytope in V obtained from Δ by parallel translation of its facets.

$P\left(\frac{\partial}{\partial h_1}, \cdots, \frac{\partial}{\partial h_d}\right)$: infinite order linear partial differential operator with constant coefficients in d variables. Then, for a suitable function f on V,

$$\sum_{x\in\Delta\cap V_{\mathbb{Z}}} f(x) = P\left(\frac{\partial}{\partial h_1}, \cdots, \frac{\partial}{\partial h_d}\right) \int_{\Delta(h)} f \, d\nu_{_\Delta} \Bigg|_{h=0} \,. \tag{2}$$

How is CS's formula related to the other ones?

Using an algebraic formalism due precisely to Cappell and Shaneson, we can relate all these formulas (in fact, KSW have done this in a rather recent paper (2007)).

However, KSW did not work out the relation of CS's formula with the other ones for the case of a simple polytope.

Our approach, using weights and working with a sort of intermediate algebraic formalism, does the job.

How is CS's formula related to the other ones?

Within these algebraic formalisms is easy to check the equivalence of the two presentations for Euler Maclaurin formulas mentioned previously.

We can show that the CS formula (for simple polytopes) can be grouped into two sums, one of which corresponds to the AG formula for the case when all $q_j = \frac{1}{2}$.

About Berline-Vergne's local formula

There is in the literature a rather recent paper on a local Euler Maclaurin formula for polytopes by N. Berline and M. Vergne (2007) which has a presentation similar to CS's formula. The corresponding differential operators P_F in this formula enjoy two essential properties: they are **local** and **computable**, as opposed to CS and BV formulas.

We would like to use our approach in order to include this formula as well.

About the weights

We assign arbitrary complex numbers to the facets of a polytope. These weights have no relation among them whatsoever.

We would like to explore the use of other type of weights or putting some relations on them in the context of both polytope decompositions and Euler Maclaurin formulas.

Thank you!