A classical umbral view of the Riordan group and related Sheffer sequences

José Agapito

(CELC - U. de Lisboa)

Algebra and Combinatorics Seminar

Departamento de Matemática da Universidade de Coimbra

November 26, 2010

Abstract

The Riordan group is the set of infinite lower triangular invertible matrices with the group operation given by a matrix multiplication that combines both the usual Cauchy product and the composition of formal power series. It is related to a broad family of polynomial sequences in one variable called Sheffer sequences. Riordan arrays and Sheffer sequences have various applications in Combinatorics, Analysis, Probability, Physics, etc.

In this talk I will present an enlightening symbolic treatment of the Riordan group and related Sheffer sequences based on a renewed approach to umbral calculus initiated by **Gian Carlo Rota** in the 90's and further developed by **Di Nardo** and **Senato** in the first decade of the present century.

Based on joint work with **Ângela Mestre** (CELC), **Pasquale Petrullo** (Università degli studi della Basilicata) and **Maria Manuel Torres** (CELC).

Contents

Contents

Riordan arrays

[Shapiro et al. 1991]

Let g and f be two formal exponential series; namely

$$g(z) = 1 + g_1 z + g_2 \frac{z^2}{2!} + g_3 \frac{z^3}{3!} + \cdots \quad \text{and} \quad f(z) = f_1 z + f_2 \frac{z^2}{2!} + f_3 \frac{z^3}{3!} + \cdots \quad (1)$$

An (exponential) *Riordan array* is an infinite lower triangular matrix $\mathfrak{M} = (\mathfrak{m}_{n,k})_{n,k \geq 0}$, whose entries are generated by g and f as follows

$$\mathfrak{m}_{n,k} = \left[\frac{z^n}{n!}\right] \left(g(z) \frac{f(z)^k}{k!} \right) \quad \text{for} \quad n, k \geq 0 \, .$$

umbral view

We shall write $\mathfrak{M} = \big(g(z), f(z)\big) = (g, f)$.

Some examples

$$\begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \\ & & \cdots & \cdots \end{pmatrix}$$

Pascal array (e^z, z)

$$\begin{pmatrix} 1 \\ g_1 & 1 \\ g_2 & 2g_1 & 1 \\ g_3 & 3g_2 & 3g_1 & 1 \\ g_4 & 4g_3 & 6g_2 & 4g_1 & 1 \\ g_5 & 5g_4 & 10g_3 & 10g_2 & 5g_1 & 1 \\ & & \cdots & \cdots \end{pmatrix}$$

Appell array $(g(z), z)$

$$\begin{pmatrix} 1 \\ 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -3 & 1 \\ 0 & -6 & 11 & -6 & 1 \\ 0 & 24 & -50 & 35 & -10 & 1 \\ & & \cdots & \cdots \end{pmatrix}$$

Stirling array of 1st. kind $(1, \log(1+z))$

$$\begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 7 & 6 & 1 \\ 0 & 1 & 15 & 25 & 10 & 1 \\ & & \cdots & \cdots \end{pmatrix}$$

Stirling array of 2nd. kind $(1, e^z - 1)$

Entries of a general Riordan array for $0 \leq n, k \leq 3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ g_1 & f_1 & 0 & 0 \\ g_2 & f_2 + 2f_1 g_1 & f_1^2 & 0 \\ g_3 & f_3 + 3f_2 g_1 + 3f_1 g_2 & 3f_1^2 g_1 + 3f_1 f_2 & f_1^3 \\ & & \vdots & & \ddots \end{pmatrix}$$

Fundamental theorem of Riordan arrays (FTRA)

Let A and B be two exponential generating functions; that is

$$A(z) = a_0 + a_1 z + a_2 \frac{z^2}{2!} + a_3 \frac{z^3}{3!} + \cdots \quad \text{and} \quad B(z) = b_0 + b_1 z + b_2 \frac{z^2}{2!} + b_3 \frac{z^3}{3!} + \cdots$$

and let $\big(g(z), f(z)\big)$ be a Riordan array. Then

$$\big(g(z), f(z)\big) \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix} \qquad \Longleftrightarrow \qquad g(z)\, A\big(f(z)\big) = B(z).$$

Note that the composition $A\big(f(z)\big)$ is well defined since f has zero constant term.

umbral view

Example (FTRA in action)

Stirling numbers of 2nd. kind $S(n,j)$

$$\begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 1 & 3 & 1 & & \\ 0 & 1 & 7 & 6 & 1 & \\ 0 & 1 & 15 & 25 & 10 & 1 \\ & & & \cdots & & \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 15 \\ 52 \\ \vdots \end{pmatrix}$$

Bell numbers B_n

$$\sum_{j=0}^{n} S(n,j) = B_n \ .$$

$$(1,\ e^z - 1) \cdot e^z = 1 \cdot e^{[e^z - 1]} = e^{[e^z - 1]} = B(z) \ .$$

The (exponential) Riordan group [Shapiro et al. 1991]

Let (g, f) and (h, l) be given as in (1). Consider the *multiplication*

$$\big(g(z), f(z)\big)\,\big(h(z), l(z)\big) = \Big(g(z)\,h\big(f(z)\big), l\big(f(z)\big)\Big)\,. \tag{2}$$

The Riordan array $(1, z)$ is the *identity* element with respect to (2). Since $g_0 \neq 0$, it follows that g has multiplicative inverse g^{-1}. Also, if $f_1 \neq 0$, then f has compositional inverse $f^{<-1>}$; that is, $f\big(f^{<-1>}(z)\big) = f^{<-1>}\big(f(z)\big) = z$. In this case, a Riordan array (g, f) is invertible with respect to (2) and its *inverse* is given by

$$\big(g(z), f(z)\big)^{-1} = \left(\frac{1}{g\big(f^{<-1>}(z)\big)}\,,\; f^{<-1>}(z)\right)\,,$$

The *Riordan group* $\mathfrak{Rio}$ is the set of all invertible Riordan arrays, together with multiplication (2) as the group operation.

umbral view

Some distinguished Riordan subgroups

1. *The Appell subgroup*: $\{(g(z), z)\}$.
2. *The Associated subgroup*: $\{(1, f(z))\}$.
3. *The Bell subgroup*: $\{(g(z), zg(z))\}$.
4. *The Stochastic subgroup*:

$$\left\{ (g(z), rz) \in \mathfrak{Rio} \;\middle|\; (g(z), f(z)) \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} \right\}.$$

Contents

The *magic trick*

The Bell numbers $\boldsymbol{B_n}$ are the coefficients in the Taylor series expansion of $e^{[e^z-1]}$; namely

$$e^{[e^z-1]} \;=\; 1+\sum_{n=1}^{\infty} \boldsymbol{B_n}\frac{z^n}{n!} \;\simeq\; 1+\sum_{n=1}^{\infty} B^n\frac{z^n}{n!} \;=\; e^{Bz}$$

The symbol $\simeq$ stresses out the purely formal character of this manipulation.

The classical umbral calculus basic data

1. a commutative integral domain R with identity 1. $R = \mathbb{C}[x, y]$.
2. a set $A = \{\alpha, \beta, \gamma, \ldots\}$ of umbrae, called *alphabet*.
3. a linear functional $E\colon R[A] \to R$ called *evaluation* such that
   - $E[1] = 1$ and $p \in R[A]$ is called umbral polynomial.
   - $E[x^n y^m \alpha^i \beta^j \cdots \gamma^k] = x^n y^m E[\alpha^i]E[\beta^j] \cdots E[\gamma^k]$ (*uncorrelation*)
4. two special umbrae: ε (*augmentation*) and υ (*unity*) such that

$$E[\varepsilon^n] = \delta_{0,n} \qquad \text{and} \qquad E[\upsilon^n] = 1\,,$$

for all $n \geq 0$.

Similarity and generating functions (g.f.)

1. α represents a sequence $(a_n)_{n\geq 1}$ if $E[\alpha^n] = a_n$ for all $n \geq 1$. We say that a_n is the n-th *moment* of α.

   Assume $a_0 = 1$.

2. *umbral equivalence*: $\alpha \simeq \gamma \iff E[\alpha] = E[\gamma]$.

3. *similarity*: $\alpha \equiv \gamma \iff E[\alpha^n] = E[\gamma^n], \ \forall\, n \geq 0$.

4. The generating function of α is the exponential formal series

$$e^{\alpha z} := \upsilon + \sum_{n\geq 1} \alpha^n \frac{z^n}{n!} \in R[A]\,[[z]]\,,$$

so that
$$E[e^{\alpha z}] = 1 + \sum_{n\geq 1} a_n \frac{z^n}{n!} =: f_\alpha(z) \in R[[z]].$$

We shall write $e^{\alpha z} \simeq f_\alpha(z)$. We have $\alpha \equiv \gamma \iff e^{\alpha z} \simeq e^{\gamma z}$.

Some distinguished umbrae

name	α	$e^{\alpha z} \simeq f_\alpha(z)$	$E[\alpha^n] = a_n \ (n \geq 0)$
augmentation	ε	1	$\delta_{0,n}$
Bernoulli	ι	$\frac{z}{e^z-1}$	b_n (n-th Bernoulli number)
unity	υ	e^z	1
singleton	χ	$1+z$	$\delta_{0,n}\ ,\ \ n = 0, 1$
Bell	β	$e^{[e^z-1]}$	B_n (n-th Bell number)

The handling of sequences of binomial type

Let $(a_n)_{n\geq 1}$ and $(l_n)_{n\geq 1}$ be two arbitrary sequences in R represented by umbrae α and λ respectively. Then, the umbra $\alpha + \lambda$ has moments

$$E\left[(\alpha+\lambda)^n\right] = \sum_{j=0}^{n} \binom{n}{j} E\left[\alpha^j \lambda^{n-j}\right] \underset{\text{uncorrelation}}{=} \sum_{j=0}^{n} \binom{n}{j} a_j \, l_{n-j}.$$

Thus, the umbra $\alpha + \lambda$ represents the sequence $\sum_{j=0}^{n} \binom{n}{j} a_j \, l_{n-j}$.

The handling of sequences of binomial type

Let $(a_n)_{n\geq 1}$ and $(l_n)_{n\geq 1}$ be two arbitrary sequences in R represented by umbrae α and λ respectively. Then, the umbra $\alpha+\lambda$ has moments

$$E\left[(\alpha+\lambda)^n\right] = \sum_{j=0}^{n}\binom{n}{j}E\left[\alpha^j\,\lambda^{n-j}\right] \underset{\text{uncorrelation}}{=} \sum_{j=0}^{n}\binom{n}{j}a_j\,l_{n-j}.$$

Thus, the umbra $\alpha+\lambda$ represents the sequence $\displaystyle\sum_{j=0}^{n}\binom{n}{j}a_j\,l_{n-j}$.

Suppose now that $(l_n)=(a_n)$. Does $\alpha+\alpha$ represent the sequence

$$\sum_{j=0}^{n}\binom{n}{j}a_j\,a_{n-j}?$$

The handling of sequences of binomial type

Let $(a_n)_{n\geq 1}$ and $(l_n)_{n\geq 1}$ be two arbitrary sequences in R represented by umbrae α and λ respectively. Then, the umbra $\alpha+\lambda$ has moments

$$E\left[(\alpha+\lambda)^n\right] = \sum_{j=0}^{n}\binom{n}{j}E\left[\alpha^j\,\lambda^{n-j}\right] \underset{\text{uncorrelation}}{=} \sum_{j=0}^{n}\binom{n}{j}a_j\,l_{n-j}.$$

Thus, the umbra $\alpha+\lambda$ represents the sequence $\displaystyle\sum_{j=0}^{n}\binom{n}{j}a_j\,l_{n-j}$.

Suppose now that $(l_n)=(a_n)$. Does $\alpha+\alpha$ represent the sequence

$\displaystyle\sum_{j=0}^{n}\binom{n}{j}a_j\,a_{n-j}$? No, since $E\left[(\alpha+\alpha)^n\right]=E\left[(2\alpha)^n\right]=2^n a_n$.

Auxiliary umbrae

Key feature: Each sequence $(a_n)_{n\geq 1}$ in R can be represented by infinitely many similar (auxiliary) umbrae. This fact is called saturation. The alphabet A will contain all possible auxiliary umbrae.

The dot operations

Let k be a nonnegative integer and let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be k uncorrelated umbra similar to α (with moments a_i). The *dot-product* $\boldsymbol{k \cdot \alpha}$ is an auxiliary umbra defined to satisfy $k \cdot \alpha \equiv \alpha_1 + \alpha_2 + \cdots + \alpha_k$.

Two umbrae α and λ are said to be *inverse* to each other (with respect to addition) when $\alpha + \lambda \equiv \varepsilon$. We shall write $\lambda \equiv -1 \cdot \alpha$. We define $-\boldsymbol{k \cdot \alpha} \equiv -1 \cdot \alpha_1 + \cdots + -1 \cdot \alpha_k$. Also, we set $0 \cdot \alpha \equiv \varepsilon$.

The *dot-power* $\boldsymbol{\alpha^{\cdot k}}$ is an auxiliary umbra such that $\alpha^{\cdot k} \equiv \alpha_1 \alpha_2 \cdots \alpha_k$. We assume $\alpha^{\cdot 0} \equiv \upsilon$. We have $E[(\alpha^{\cdot k})^n] = a_n^k$ for all $n \geq 0$.

G.f. for dot product and dot power [di Nardo & Senato 2001]

For any $k \in \mathbb{Z}$, we can write

$$e^{(k\cdot\alpha)z} = \upsilon + \sum_{n=1}^{\infty} (k\cdot\alpha)^n \frac{z^n}{n!} \simeq f_{k\cdot\alpha}(z) = \left[f_\alpha(z)\right]^k = e^{k \log f_\alpha(z)}.$$

In general, for any umbra γ (with moments g_i), the auxiliary umbra $\gamma \cdot \alpha$ is defined to satisfy

$$e^{(\gamma\cdot\alpha)z} \simeq f_{\gamma\cdot\alpha}(z) := f_\gamma\left(\log f_\alpha(z)\right) = 1 + g_1 \log f_\alpha(z) + g_2 \frac{\left[\log f_\alpha(z)\right]^2}{2!} + \cdots$$

Similarly, for $k \geq 0$, we have $e^{(\alpha^{\cdot k})z} \simeq f_{\alpha^{\cdot k}}(z) := 1 + \sum_{n=1}^{\infty} a_n^k \frac{z^n}{n!}$.

The role played by the Bell umbra β [di Nardo & Senato 2001]

Recall that the g.f. of the Bell umbra is $e^{\beta z} \simeq e^{[e^z - 1]}$. Hence, the umbra $\beta \cdot \gamma$ has g.f.

$$e^{(\beta\cdot\gamma)z} \simeq f_\beta\big(\log f_\gamma(z)\big) = e^{\left[e^{\log f_\gamma(z)} - 1\right]} = e^{\left[f_\gamma(z) - 1\right]}.$$

The *composition* umbra of α and γ is the umbra $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} \cdot \boldsymbol{\gamma}$ with g.f.

$$e^{(\alpha\cdot\beta\cdot\gamma)z} \simeq f_\alpha\big(\log f_\beta(\log f_\gamma(z))\big) = f_\alpha\big(f_\gamma(z) - 1\big)$$

If $\gamma \cdot \beta \cdot \alpha \equiv \chi \equiv \alpha \cdot \beta \cdot \gamma$, we say that γ is the *compositional inverse* of α and viceversa. We shall write $\gamma = \boldsymbol{\alpha}^{<-1>}$.

Note that $E[\alpha] \neq 0$ for $\alpha^{<-1>}$ to exist.

Useful identities and dictionary

1. $\alpha \cdot \varepsilon \equiv \varepsilon \equiv \varepsilon \cdot \alpha$.
2. $\beta \cdot \chi \equiv \upsilon \equiv \chi \cdot \beta$.
3. $\alpha\,\eta \equiv \eta\,\alpha$.
4. $r\alpha \equiv \alpha \cdot (r\upsilon)$. (In general, note that $r \cdot \alpha \equiv r\upsilon \cdot \alpha \not\equiv \alpha \cdot r\upsilon \equiv r\alpha$)

Umbrae	formal power series
$\alpha + \eta$	Cauchy product
$\alpha\,\eta$	Hadamard product
$\alpha \dot{+} \eta$	usual addition
$\alpha \cdot \beta \cdot \gamma$	formal composition

$(A, +, \cdot)$ is a left distributive algebra

1. $(A, +)$ is an abelian group.

$$\alpha + \eta \equiv \eta + \alpha \qquad \alpha + \varepsilon \equiv \alpha \equiv \varepsilon + \alpha$$
$$(\alpha + \eta) + \gamma \equiv \alpha + (\eta + \gamma) \quad \text{and} \quad \alpha + (-1 \cdot \alpha) \equiv \varepsilon \equiv (-1 \cdot \alpha) + \alpha$$

2. $(A, \cdot)$ is a monoid.

$$\alpha \cdot (\eta \cdot \gamma) \equiv (\alpha \cdot \eta) \cdot \gamma \quad \text{and} \quad \alpha \cdot \upsilon \equiv \alpha \equiv \upsilon \cdot \alpha$$

3. The scalar product.

$$1\alpha \equiv \alpha \qquad r(\alpha + \eta) \equiv r\alpha + r\eta$$
$$r(s\alpha) \equiv (rs)\alpha \quad \text{and} \quad (r + s)\alpha \equiv r\alpha + s\alpha$$

4. The left distributive laws.

$$(\alpha + \eta) \cdot \gamma \equiv \alpha \cdot \gamma + \eta \cdot \gamma \qquad \alpha \cdot (r\eta) \equiv r(\alpha \cdot \eta)$$
$$\gamma \cdot (\alpha + \eta) \not\equiv \gamma \cdot \alpha + \gamma \cdot \eta \quad \text{and} \quad \alpha \cdot (r\eta) \not\equiv (r\alpha) \cdot \eta$$

The constant umbrae

Let $r \in R$. The *constant* umbra ς_r has moments $E[\varsigma_r^n] = r$ for all $n \geq 1$. We have

1. $\varsigma_0 \equiv \varepsilon$ and $\varsigma_1 \equiv \upsilon$.
2. $E[(\varsigma_r \alpha)^n] = r a_n$ while $E[(r\alpha)^n] = r^n a_n$.
3. $\varsigma_r \, \varsigma_s \equiv \varsigma_s \, \varsigma_r \equiv \varsigma_{rs}$ for any $r, s \in R$.
4. $\varsigma_r \cdot \alpha \equiv \varsigma_r \, \alpha \equiv \alpha \, \varsigma_r$ while $\alpha \cdot \varsigma_r \not\equiv \alpha \, \varsigma_r$.
5. $e^{(\varsigma_r \alpha) z} \simeq 1 + r\big(f_\alpha(z) - 1\big)$ while $e^{(r\alpha)z} \simeq f_\alpha(rz)$.

In fact, we have $\varsigma_r \equiv \chi \cdot r \cdot \beta \cdot \alpha$.

The primitive and derivative umbrae. [di Nardo & Niederhausen & Senato 2001, 2009]

Let α be an umbra with moments a_i. The *derivative* umbra $\boldsymbol{\alpha_{\mathcal{D}}}$ of α is the umbra whose powers satisfy $\alpha_{\mathcal{D}}^n \simeq n\alpha^{n-1}$ for $n \geq 1$. In particular $E[\alpha_{\mathcal{D}}] = 1$ and this implies that $\alpha_{\mathcal{D}}$ has compositional inverse $\alpha_{\mathcal{D}}^{<-1>}$.

The g.f. of $\alpha_{\mathcal{D}}$ satisfy $\quad e^{\alpha_{\mathcal{D}} z} \simeq 1 + z\, e^{\alpha z}$.

The *primitive* umbra $\boldsymbol{\alpha_{\mathcal{P}}}$ of α is the umbra whose powers satisfy $\alpha_{\mathcal{P}}^n \simeq \frac{\alpha^{n+1}}{a_1(n+1)}$ for $n \geq 0$. Therefore $a_1 \neq 0$, so that only umbrae with compositional inverse have primitive umbra.

The g.f. of $\alpha_{\mathcal{P}}$ satisfy $\quad e^{\alpha z} \simeq 1 + a_1 z\, e^{\alpha_{\mathcal{P}} z}$.

Straightforward identities

Lemma

Let $\alpha \in A$ be any umbra and let $r \in R$ be a nonzero scalar. Then

1. $(\alpha_{\mathcal{D}})_{\mathcal{P}} \equiv \alpha$. *In addition, if* $a_1 = E[\alpha] \neq 0$ *then* $(\alpha_{\mathcal{P}})_{\mathcal{D}} \equiv \varsigma_{1/a_1} \alpha$.
2. $(\varsigma_r \alpha)_{\mathcal{P}} \equiv \alpha_{\mathcal{P}}$, $(r\alpha)_{\mathcal{P}} \equiv r\alpha_{\mathcal{P}}$, $(\varsigma_r \alpha)_{\mathcal{D}} \equiv \varsigma_r \alpha_{\mathcal{D}}$ *and* $(r\alpha)_{\mathcal{D}} \equiv \varsigma_{1/r}(r\alpha_{\mathcal{D}})$.

Theorem (A.M.P.T. 2010)

Let α and γ be two umbrae with first moments $a_1, g_1 \neq 0$. Then

$$(\alpha \cdot \beta \cdot \gamma)_{\mathcal{P}} \equiv \gamma_{\mathcal{P}} + \alpha_{\mathcal{P}} \cdot \beta \cdot \gamma.$$

Extended Bell subgroup

Corollary

If $g_1 = {}^1/_{a_1}$ *then* $\alpha \cdot \beta \cdot \gamma \equiv (\gamma_{\mathcal{P}} + \alpha_{\mathcal{P}} \cdot \beta \cdot \gamma)_{\mathcal{D}}$.

Applications of Corollary

Taking $\gamma = \alpha^{<-1>}$ yields:

1. $\chi \equiv \alpha \cdot \beta \cdot \alpha^{<-1>} \equiv \left((\alpha^{<-1>})_{\mathcal{P}} + \alpha_{\mathcal{P}} \cdot \beta \cdot \alpha^{<-1>}\right)_{\mathcal{D}}$,
2. $\varepsilon \equiv \chi_{\mathcal{P}} \equiv (\alpha^{<-1>})_{\mathcal{P}} + \alpha_{\mathcal{P}} \cdot \beta \cdot \alpha^{<-1>}$,
3. $(\alpha^{<-1>})_{\mathcal{P}} \equiv -1 \cdot \alpha_{\mathcal{P}} \cdot \beta \cdot \alpha^{<-1>}$,
4. $\alpha_{\mathcal{P}} \equiv -1 \cdot (\alpha^{<-1>})_{\mathcal{P}} \cdot \beta \cdot \alpha$.

If $E[\alpha] = 1$ then

1. $\alpha^{<-1>} \equiv \left(-1 \cdot \alpha_{\mathcal{P}} \cdot \beta \cdot \alpha^{<-1>}\right)_{\mathcal{D}}$
2. $\alpha \equiv \left(-1 \cdot (\alpha^{<-1>})_{\mathcal{P}} \cdot \beta \cdot \alpha\right)_{\mathcal{D}}$.

Lagrange's inversion formula I

Let $f(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}$ and suppose that $a_0 = 0$ and $a_1 \neq 0$. Then $f^{<-1>}$ is well defined. One version of Lagrange's inversion formula states that for any integer $n \geq 1$, it holds

$$\left[\frac{z^n}{n!}\right] f^{<-1>}(z) = \left[\frac{z^{n-1}}{(n-1)!}\right] \left(\frac{f(z)}{z}\right)^{-n} .$$

Let α be an umbra such that $e^{\alpha z} \simeq 1 + f(z)$. Then $f(z) \simeq a_1 z\, e^{\alpha_{\mathcal{P}} z}$ and $e^{\alpha^{<-1>} z} \simeq (a_1 z\, e^{\alpha_{\mathcal{P}} z})^{<-1>}$.

Proposition (di Nardo & Niederhausen & Senato 2009)

Let α be an umbra with compositional inverse $\alpha^{<-1>}$ (hence $a_1 = E[\alpha] \neq 0$). For any $n \geq 1$, we have $\left(\alpha^{<-1>}\right)^n \simeq \frac{1}{a_1^n}\left(-n \cdot \alpha_{\mathcal{P}}\right)^{n-1}$ or equivalently, $\alpha^{\cdot n}\left(\alpha^{<-1>}\right)^n \simeq \left(-n \cdot \alpha_{\mathcal{P}}\right)^{n-1}$. In particular, $\left(\alpha_{\mathcal{D}}^{<-1>}\right)^n \simeq \left(-n \cdot \alpha\right)^{n-1}$.

Lagrange's inversion formula II

Another version of Lagrange's inversion formula states that for any integer $n \geq 1$ and $f(z)$ as before, it holds

$$\left[\frac{z^n}{n!}\right] \Phi\left(f^{<-1>}(z)\right) = \left[\frac{z^{n-1}}{(n-1)!}\right] D\Phi(z) \left(\frac{f(z)}{z}\right)^{-n} ,$$

where $\Phi(z)$ is any formal exponential series and D is the usual differential operator on formal power series.

Theorem (A.M.P.T. 2010)

Let α be any umbra and γ an umbra with compositional inverse $\gamma^{<-1>}$ (hence $g_1 = E[\gamma] \neq 0$). For any $n \geq 1$ we have

$$(\alpha \cdot \beta \cdot \gamma^{<-1>})^n \simeq \frac{1}{g_1^n} \alpha(\alpha - n \cdot \gamma_{\mathcal{P}})^{n-1} .$$

Abel's identity

The *adjoint* umbra of γ is $\boldsymbol{\gamma}^* = \beta \cdot \gamma^{<-1>}$.

Theorem (A.M.P.T. 2010)

Let $\gamma \in A$ be any umbra with $g_1 = E[\gamma] \neq 0$. For any umbrae α and δ we have

$$(\alpha + \delta)^n \simeq \sum_{k=0}^{n} \binom{n}{k} \gamma^{\cdot k} (\alpha + k \cdot \gamma_{\mathcal{P}})^{n-k} (\delta \cdot \gamma^*)^k.$$

FTRA

Contents

The Riordan group revisited

Definition

Given two umbrae α and γ, we say that (α, γ) represent the Riordan array (g, f) if

$$e^{\alpha z} \simeq g(z) \qquad \text{and} \qquad e^{\gamma z} \simeq 1 + f(z)\ .$$

Thus, the Riordan group is given by

$$\mathfrak{Rio} = \Big\{(\alpha, \gamma) \in A \times A \,:\, E[\gamma] \neq 0\Big\}.$$

The group operation reads as $(\alpha, \gamma)\,(\zeta, \eta) \equiv (\alpha + \zeta\cdot\beta\cdot\gamma\,,\, \eta\cdot\beta\cdot\gamma)$.

The inverse of (α, γ) is $(\alpha, \gamma)^{-1} = (-1\cdot\alpha\cdot\beta\cdot\gamma^{<-1>}, \gamma^{<-1>})$ and the identity is (ε, χ) .

classical view

Entries of an invertible Riordan array

Proposition (A.M.P.T., di Nardo & Niederhausen & Senato 2010, $\gamma \rightsquigarrow \gamma^{<-1>}$)

Let $\mathfrak{M} = (\alpha, \gamma) \in \mathfrak{Rio}$. *Then*

$$\mathfrak{m}_{n,k} \simeq \binom{n}{k} \gamma^{\cdot k} (\alpha + k \cdot \gamma_{\mathcal{P}})^{n-k} \quad \textit{for} \quad n, k \geq 0.$$

▸ classical view
▸ FTRA

Fundamental theorem of Riordan arrays (FTRA)

Let $(\alpha,\gamma) \in \mathfrak{Rio}$ and $\delta \in A$. The group $\mathfrak{Rio}$ acts over A by

$$(\alpha,\gamma) \bullet \delta = \alpha + \delta \cdot \beta \cdot \gamma \, .$$

Given any two umbrae $\delta, \eta \in A$, the FTRA is equivalent to saying that there exists an invertible Riordan array $(\alpha,\gamma) \in \mathfrak{Rio}$ such that $(\alpha,\gamma) \bullet \delta = \eta$. That is, the $\mathfrak{Rio}$-action is transitive. classical view

By replacing δ with $\delta \cdot \beta \cdot \gamma$ in Abel's identiy and using the umbral characterization for the entries of an invertible Riordan array, we obtain

$$(\alpha + \delta \cdot \beta \cdot \gamma)^n \simeq \sum_{k=0}^{n} \mathfrak{m}_{n,k} \, \delta^k \qquad \text{for all} \quad n \geq 0 \, .$$

Some important Riordan subgroups

1. *The Appell subgroup*: $\{(\alpha, \chi)\}$.

$$(\alpha, \chi)\,(\zeta, \chi) \equiv (\alpha + \zeta, \chi) \quad \text{and} \quad (\alpha, \chi)^{-1} \equiv (-1 \cdot \alpha, \chi)\ .$$

2. *The Associated subgroup*: $\{(\varepsilon, \gamma)\}$.

$$(\varepsilon, \gamma)\,(\varepsilon, \eta) \equiv (\varepsilon, \eta \cdot \gamma) \quad \text{and} \quad (\varepsilon, \gamma)^{-1} \equiv (\varepsilon, \gamma^{<-1>})\ .$$

3. *The Bell subgroup*: $\{(\alpha, \alpha_{\mathcal{D}})\}$.

$$(\alpha, \alpha_{\mathcal{D}})\,(\zeta, \zeta_{\mathcal{D}}) \equiv (\alpha + \zeta \cdot \beta \cdot \alpha_{\mathcal{D}}\,,\ \zeta_{\mathcal{D}} \cdot \beta \cdot \alpha_{\mathcal{D}})$$

and

$$(\alpha, \alpha_{\mathcal{D}})^{-1} \equiv (-1 \cdot \alpha \cdot \beta \cdot \alpha_{\mathcal{D}}^{<-1>}\,,\ \alpha_{\mathcal{D}}^{<-1>})\ .$$

Note that $\zeta_{\mathcal{D}} \cdot \beta \cdot \alpha_{\mathcal{D}} \equiv (\alpha + \zeta \cdot \beta \cdot \alpha_{\mathcal{D}})_{\mathcal{D}}$ and $\alpha_{\mathcal{D}}^{<-1>} \equiv (-1 \cdot \alpha \cdot \beta \cdot \alpha_{\mathcal{D}}^{<-1>})_{\mathcal{D}}$.

The extended Bell subgroup [A.M.P.T. 2010]

4. *The extended Bell subgroup*: $\{(\alpha, \gamma_{\mathcal{D}})\}$.

$$(\alpha, \gamma_{\mathcal{D}})(\zeta, \eta_{\mathcal{D}}) \equiv (\alpha + \zeta \cdot \beta \cdot \gamma_{\mathcal{D}} \, , \, \eta_{\mathcal{D}} \cdot \beta \cdot \gamma_{\mathcal{D}})$$

and

$$(\alpha, \gamma_{\mathcal{D}})^{-1} \equiv (-1 \cdot \alpha \cdot \beta \cdot \gamma_{\mathcal{D}}^{<-1>} \, , \, \gamma_{\mathcal{D}}^{<-1>}) \, .$$

Note that $\eta_{\mathcal{D}} \cdot \beta \cdot \gamma_{\mathcal{D}} \equiv (\gamma + \eta \cdot \beta \cdot \gamma_{\mathcal{D}})_{\mathcal{D}}$ and $\gamma_{\mathcal{D}}^{<-1>} \equiv (-1 \cdot \gamma \cdot \beta \cdot \gamma_{\mathcal{D}}^{<-1>})_{\mathcal{D}}$.

This subgroup clearly contains the Bell subgroup.

Straightforward identities

The Stabilizer subgroups

5. *The Stabilizer subgroups*: Given any $\delta \in A$, the stabilizer *Stab*(δ) of δ (with respect to the $\mathfrak{Rio}$-action) is

$$Stab(\delta) = \Big\{(\alpha, \gamma) \in \mathfrak{Rio} \, : \, \alpha + \delta \cdot \beta \cdot \gamma \equiv \delta\Big\}.$$

Since $(\alpha + \delta \cdot \beta \cdot \gamma)^n \simeq \sum_{k=0}^{n} \mathfrak{m}_{n,k} \, \delta^k$, the identity $\alpha + \delta \cdot \beta \cdot \gamma \equiv \delta$ is equiv. to

$$\sum_{k=0}^{n-1} \mathfrak{m}_{n,k} \, \delta^k + (\mathfrak{m}_{n,n} - 1) \, \delta^n = 0, \quad \text{for all} \quad n \geq 1.$$

In particular, we have

$$Stab(\varepsilon) = \Big\{(\alpha, \gamma) \in \mathfrak{Rio} \, : \, \alpha \equiv \varepsilon\Big\} = \textit{Associated subgroup}.$$

$$Stab(\upsilon) = \Big\{(\alpha, \gamma) \in \mathfrak{Rio} \, : \, \alpha + \beta \cdot \gamma \equiv \upsilon\Big\} = \textit{Stochastic subgroup}.$$

$$Stab(\chi) = \Big\{(\alpha, \gamma) \in \mathfrak{Rio} \, : \, \alpha + \gamma \equiv \chi\Big\}.$$

Entries for some Riordan subgroups

Subgroup	$\mathfrak{m}_{n,k} \simeq \binom{n}{k} \gamma^{\cdot k} \left(\alpha + k \cdot \gamma_{\mathcal{P}}\right)^{n-k} \quad n, k \geq 0$
Appell (α, χ)	$\binom{n}{k} \alpha^{n-k}$
Associated (ε, γ)	$\binom{n}{k} \gamma^{\cdot k} \left(k \cdot \gamma_{\mathcal{P}}\right)^{n-k}$
Bell $(\alpha, \alpha_{\mathcal{D}})$	$\binom{n}{k} \left((k+1) \cdot \alpha\right)^{n-k}$
extended Bell $(\alpha, \gamma_{\mathcal{D}})$	$\binom{n}{k} (\alpha + k \cdot \gamma)^{n-k}$

Sheffer umbrae [di Nardo & Niederhausen & Senato 2009, 2010]

Let $\big(g(z), f(z)\big) \in \mathfrak{Rio}$. A polynomial sequence $s_n(x)$ is said to be Sheffer for $\big(g(z), f(z)\big)$ if they satisfy

$$\sum_{n=0}^{\infty} s_n(x) \frac{z^n}{n!} = \frac{1}{g\big(f^{<-1>}(z)\big)} \, e^{x\, f^{<-1>}(z)} \ .$$

Representing $\big(g(z), f(z)\big)$ by the pair of umbrae (α, γ), the *Sheffer umbra* $\boldsymbol{\sigma_x^{(\alpha,\gamma)}}$ for (α, γ) is defined as

$$\sigma_x^{(\alpha,\gamma)} \equiv -1 \cdot \alpha \cdot \beta \cdot \gamma^{<-1>} + x \cdot \upsilon \cdot \beta \cdot \gamma^{<-1>} \equiv (-1 \cdot \alpha + x \cdot \upsilon) \cdot \gamma^* \ .$$

By construction, the g.f. of $\sigma_x^{(\alpha,\gamma)}$ is

$$e^{\sigma_x^{(\alpha,\gamma)} z} \simeq \sum_{n=0}^{\infty} s_n(x) \frac{z^n}{n!} \ ,$$

so that its moments $\left(\sigma_x^{(\alpha,\gamma)}\right)^n \simeq s_n(x)$ form a Sheffer sequence.

Characterization of Sheffer sequences

Proposition (A.M.P.T., di Nardo & Niederhausen & Senato 2010, $-1 \cdot \alpha \cdot \gamma^* \rightsquigarrow \alpha$)

The umbral expression for Sheffer polynomials $s_n(x) \simeq \left(\sigma_x^{(\alpha,\gamma)}\right)^n$ *coming from a Riordan array* (α, γ) *is given by*

$$s_n(x) \simeq \sum_{k=0}^{n} \left[\binom{n}{k} \gamma^{<-1>\cdot k} \Big(-1 \cdot \alpha \cdot \gamma^* + k \cdot (\gamma^{<-1>})_{\mathcal{P}} \Big)^{n-k} \right] x^k$$

$$\simeq \sum_{k=0}^{n} \mathfrak{m}_{n,k}^{-1} \, x^k \, ,$$

where $\mathfrak{m}_{n,k}^{-1}$ *is the* (n,k)*-th entry of* $(\alpha,\gamma)^{-1} = (-1 \cdot \alpha \cdot \gamma^*, \gamma^{<-1>})$.

Some distinguished Sheffer sequences

Name	$s_n(x)$
Appell (α, χ)	$\sum_{k=0}^{n} \binom{n}{k} (-1 \cdot \alpha)^{n-k}\, x^k$
Associated (ε, γ)	$\sum_{k=0}^{n} \binom{n}{k} \left(\gamma^{<-1>}\right)^{\cdot k} \left(k \cdot (\gamma^{<-1>})_{\mathcal{P}}\right)^{n-k}\, x^k$
Bell $(\alpha, \alpha_{\mathcal{D}})$	$\sum_{k=0}^{n} \binom{n}{k} \left((k+1) \cdot (\alpha_{\mathcal{D}}^{<-1>})_{\mathcal{P}}\right)^{n-k}\, x^k$
extended Bell $(\alpha, \gamma_{\mathcal{D}})$	$\sum_{k=0}^{n} \binom{n}{k} \left(-1 \cdot \alpha \cdot \gamma_{\mathcal{D}}^{*} + k \cdot (\gamma_{\mathcal{D}}^{<-1>})_{\mathcal{P}}\right)^{n-k}\, x^k$

Bi-parameterized Sheffer umbrae [A.M.P.T. 2010]

Definition

Let $(\alpha, \gamma) \in \mathfrak{Rio}$ and let $x, y \in A$. The *bi-parameterized Sheffer umbra* corresponding to (α, γ) is given by

$$\sigma_{x,y}^{(\alpha,\gamma)} \equiv (y \cdot \alpha + x \cdot \upsilon) \cdot \gamma^*$$

Note that $\sigma_{x,-1}^{(\alpha,\gamma)} \equiv \sigma_x^{(\alpha,\gamma)}$ (di Nardo and Senato Sheffer umbra), $\sigma_{x,0}^{(\alpha,\gamma)} \equiv x \cdot \gamma^* \equiv \sigma_x^{(\varepsilon,\gamma)}$ (Associated umbra with respect to γ), etc.

Thank you!