An umbral symbolic characterization of Riordan arrays

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Abstract

Riordan arrays are **infinite lower triangular complex valued-matrices** that have been applied to a wide range of subjects, from Computer Science to Combinatorial Physics, in connection with **combinatorial identities, recurrence relations, walk problems, asymptotic approximation** and the **problem of normal ordering for boson strings**, among other relevant topics. The traditional way of approaching Riordan arrays is by means of **generating functions**.

I will present in this talk a promising alternative characterization of Riordan arrays based on a symbolic renewed approach to umbral calculus. A deep generalization of an Abel's identity for polynomials is a key tool in this symbolic approach.

This talk is based on joint work with **Ângela Mestre** (CELC), **Pasquale Petrullo** (Università degli studi della Basilicata) and **Maria Manuel Torres** (CELC).

Contents

Outline

Examples

$$\begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ 1 & 5 & 10 & 10 & 5 & 1 \\ & & \cdots & \cdots & & \end{pmatrix}$$

$\boldsymbol{P}$: Pascal array

$$\begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 5 & 5 & 3 & 1 & & \\ 14 & 14 & 9 & 4 & 1 & \\ 42 & 42 & 28 & 14 & 5 & 1 \\ & & \cdots & \cdots & & \end{pmatrix}$$

$\boldsymbol{C}$: Ballot array

$$\begin{pmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 5 & 4 & 1 & & & \\ 14 & 14 & 6 & 1 & & \\ 42 & 48 & 27 & 8 & 1 & \\ 132 & 165 & 110 & 44 & 10 & 1 \\ & & \cdots & \cdots & & \end{pmatrix}$$

$\boldsymbol{C}^{(1)}$: Catalan array

$$\begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 1 & 3 & 1 & & \\ 0 & 1 & 7 & 6 & 1 & \\ 0 & 1 & 15 & 25 & 10 & 1 \\ & & \cdots & \cdots & & \end{pmatrix}$$

$\boldsymbol{S}$: Stirling array of 2nd. kind

Three typical presentations for Riordan arrays

$R = (g, f)$	g	f	$R_{n,k}$
exponential	$1 + g_1 z + g_2 \frac{z^2}{2!} + g_3 \frac{z^3}{3!} + \cdots$	$f_1 z + f_2 \frac{z^2}{2!} + f_3 \frac{z^3}{3!} + \cdots$	$\left[\frac{z^n}{n!}\right]\left(g(z)\frac{f(z)^k}{k!}\right)$
ordinary	$1 + g_1 z + g_2 z^2 + g_3 z^3 + \cdots$	$f_1 z + f_2 z^2 + f_3 z^3 + \cdots$	$\left[z^n\right]\left(g(z)\, f(z)^k\right)$
generalized	$1 + g_1 \frac{z}{w_1} + g_2 \frac{z^2}{w_2} + g_3 \frac{z^3}{w_3} + \cdots$	$f_1 \frac{z}{w_1} + f_2 \frac{z^2}{w_2} + f_3 \frac{z^3}{w_3} + \cdots$	$\left[\frac{z^n}{w_n}\right]\left(g(z)\frac{f(z)^k}{w_k}\right)$

Table : Traditional description of Riordan arrays

umbral view

(w_n): sequence of nonzero numbers.

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Examples (ordinary type presentation)

$$\begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \\ & & \cdots & \cdots \end{pmatrix}$$

$\boldsymbol{P}$: Pascal array $\left(\frac{1}{1-z}, \frac{z}{1-z}\right)$

$$\begin{pmatrix} 1 \\ 1 & 1 \\ 2 & 2 & 1 \\ 5 & 5 & 3 & 1 \\ 14 & 14 & 9 & 4 & 1 \\ 42 & 42 & 28 & 14 & 5 & 1 \\ & & \cdots & \cdots \end{pmatrix}$$

$\boldsymbol{C}$: Ballot array $\left(\frac{1-\sqrt{1-4z}}{2z}, \frac{1-\sqrt{1-4z}}{2}\right)$

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$\boldsymbol{C}^{(1)}$: Catalan array $\left(\frac{1-2z-\sqrt{1-4z}}{2z^2}, \frac{1-2z-\sqrt{1-4z}}{2z}\right)$

$$\begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 7 & 6 & 1 \\ 0 & 1 & 15 & 25 & 10 & 1 \\ & & \cdots & \cdots \end{pmatrix}$$

$\boldsymbol{S}$: Stirling array of 2nd. kind $\left(1, \frac{e^z-1}{z}\right)$

Examples (exponential type presentation)

$$\begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \\ & & \cdots & \cdots \end{pmatrix}$$

$\boldsymbol{P}$: Pascal array (e^z, z)

$$\begin{pmatrix} 1 \\ 1 & 1 \\ 2 & 2 & 1 \\ 5 & 5 & 3 & 1 \\ 14 & 14 & 9 & 4 & 1 \\ 42 & 42 & 28 & 14 & 5 & 1 \\ & & \cdots & \cdots \end{pmatrix}$$

$\boldsymbol{C}$: Ballot array $(?, ?)$

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$\boldsymbol{S}$: Stirling array of 2nd. kind $(1, e^z - 1)$

Fundamental theorem of Riordan arrays (FTRA)

Let A and B be two exponential generating functions; that is

$$A(z) = a_0 + a_1 z + a_2 \frac{z^2}{2!} + a_3 \frac{z^3}{3!} + \cdots \quad \text{and} \quad B(z) = b_0 + b_1 z + b_2 \frac{z^2}{2!} + b_3 \frac{z^3}{3!} + \cdots$$

and let $\big(g(z), f(z)\big)$ be a Riordan array. Then

$$(g(z), f(z)) \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix} \qquad \Longleftrightarrow \qquad g(z)\, A\big(f(z)\big) = B(z).$$

Note that the composition $A\big(f(z)\big)$ is well defined since f has zero constant term: $f(z) = f_1 z + f_2 \frac{z^2}{2!} + f_3 \frac{z^3}{3!} + \cdots$

Fundamental theorem of Riordan arrays (FTRA)

Let A and B be two exponential generating functions; that is

$$A(z) = 1 + a_1 z + a_2 \frac{z^2}{2!} + a_3 \frac{z^3}{3!} + \cdots \quad \text{and} \quad B(z) = 1 + b_1 z + b_2 \frac{z^2}{2!} + b_3 \frac{z^3}{3!} + \cdots$$

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$$(g(z), f(z)) \begin{pmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix} \qquad \Longleftrightarrow \qquad g(z)\, A\big(f(z)\big) = B(z).$$

Note that the composition $A\big(f(z)\big)$ is well defined since f has zero constant term: $f(z) = 1z + f_2 \frac{z^2}{2!} + f_3 \frac{z^3}{3!} + \cdots$

Example (FTRA in action)

Stirling numbers of 2nd. kind $S(n,j)$

$$\begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 1 & 3 & 1 & & \\ 0 & 1 & 7 & 6 & 1 & \\ 0 & 1 & 15 & 25 & 10 & 1 \\ & & & \dots & & \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 15 \\ 52 \\ \vdots \end{pmatrix}$$

Bell numbers B_n

$$\left(1,\ e^z - 1\right) \cdot e^z = 1 \cdot e^{[e^z - 1]} = e^{[e^z-1]} = B(z) .$$

Row sums: $\sum_{j=0}^{n} S(n,j) = B_n$.

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The Riordan group [Shapiro et al. 1991]

Let (g, f) and (h, l) be two Riordan arrays. Consider the *multiplication*

$$\left(g(z), f(z)\right)\left(h(z), l(z)\right) = \left(g(z)\,h\left(f(z)\right), l\left(f(z)\right)\right). \qquad (1)$$

The Riordan array $(1, z)$ is the *identity* with respect to multiplication (1).

$$g_0 \neq 0 \Rightarrow g \text{ has mult. inverse } g^{-1}.$$

$$f_1 \neq 0 \Rightarrow f \text{ has comp. inverse } f^{\langle -1 \rangle}; \text{ i.e, } f\left(f^{\langle -1 \rangle}(z)\right) = f^{\langle -1 \rangle}\left(f(z)\right) = z.$$

A Riordan array (g, f) is invertible w.r.t. (1) and its *inverse* is given by

$$(g(z), f(z))^{-1} = \left(\frac{1}{g\left(f^{\langle -1 \rangle}(z)\right)}, f^{\langle -1 \rangle}(z)\right),$$

The *Riordan group* $\mathfrak{Rio}$ is the set of all invertible Riordan arrays, together with multiplication (1) as the group operation.

Some distinguished Riordan subgroups

1. The $\mathfrak{Appell}$ or $\mathfrak{Toeplitz}$ subgroup: $\left\{\left(g(z), z\right)\right\}$.
2. The $\mathfrak{Associatted}$ or $\mathfrak{Lagrange}$ subgroup: $\left\{\left(1, f(z)\right)\right\}$.
3. The $\mathfrak{Bell}$, $\mathfrak{Renewal}$ or $\mathfrak{Rogers}$ subgroup: $\left\{\left(g(z), zg(z)\right)\right\}$.

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$$\begin{aligned} \mathfrak{Rio} &= \mathfrak{Appell} \rtimes \mathfrak{Associatted} \\ \mathfrak{Rio} &= \mathfrak{Appell} \rtimes \mathfrak{Bell} \end{aligned}$$

Outline

Umbrae

1. a commutative integral domain R with identity 1. $R = \mathbb{C}[x, y]$.
2. a set $A = \{\alpha, \gamma, \omega, \ldots\}$ of umbrae, called *alphabet*.
3. a linear functional $E\colon R[A] \to R$ called *evaluation* such that
   - $E[1] = 1$ and $p \in R[A]$ is called umbral polynomial.
   - $E[\alpha^i \gamma^j \cdots \omega^k] = E[\alpha^i]E[\gamma^j] \cdots E[\omega^k]$ (*uncorrelation*)
4. two special umbrae: ε (*augmentation*) and υ (*unity*) such that

$$E[\varepsilon^n] = \delta_{0,n} \qquad \text{and} \qquad E[\upsilon^n] = 1\,,$$

for all $n \geq 0$.

Two equivalence relations

- ω represents a sequence $(w_n)_{n\geq 1}$ if $E[\omega^n] = w_n$ for all $n \geq 1$. We say that w_n is the n-th *moment* of ω. Assume $w_0 = 1$.
- *umbral equivalence*: $\omega \simeq \gamma \iff E[\omega] = E[\gamma]$.
- *similarity*: $\omega \equiv \gamma \iff E[\omega^n] = E[\gamma^n], \ \forall\, n \geq 0$.

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Key feature: It is convenient to assume that each sequence $(w_n)_{n\geq 1}$ in R can be represented by infinitely many similar umbrae. This fact is called saturation.

Generating function

The generating function of ω is the exponential formal series

$$e^{\omega z} := \upsilon + \sum_{n\geq 1} \omega^n \frac{z^n}{n!} \in R[A]\,[[z]]\,,$$

so that $$E[e^{\omega z}] = 1 + \sum_{n\geq 1} w_n \frac{z^n}{n!} =: f_\omega(z) \in R[[z]].$$

We shall write $e^{\omega z} \simeq f_\omega(z)$. We have $\omega \equiv \gamma \iff e^{\omega z} \simeq e^{\gamma z}$.

Some distinguished umbrae

umbra	ω	$e^{\omega z}$	ω^n
augmentation	ε	1	$1, 0, 0, \ldots$
unity	υ	e^z	$1, 1, 1, \ldots$
singleton	χ	$1 + z$	$1, 1, 0, \ldots$
Bell	β	$e^{e^z - 1}$	$1, B_2, B_3, \ldots$ (B_n : Bell numbers)
Bernoulli	ι	$\frac{z}{e^z - 1}$	$1, b_1, b_2, \ldots$ (b_n : Bernoulli numbers)
boolean unity	$\bar{\upsilon}$	$\frac{1}{1-z}$	$1!, 2!, 3!, \ldots$ ($n!$: factorial numbers)
Catalan	ς	$\frac{1-\sqrt{1-4z}}{2z}$	$C_1, 2!C_2, 3!C_3, \ldots$ (C_n : Catalan numbers)

Table : Some distinguished umbrae.

Some useful auxiliary umbrae

- The dot product of umbrae $\gamma.\alpha \quad , \quad f_{\gamma.\alpha}(z) = f_\gamma\big(\log f_\alpha(z)\big)$
  - composition umbrae $\gamma.\beta.\alpha \quad , \quad f_{\gamma.\beta.\alpha} = f_\gamma\big(f_\alpha(z) - 1\big)$
  - compositional inverse of an umbra $\gamma^{\langle -1 \rangle} \quad , \quad \gamma^{\langle -1 \rangle}.\beta.\gamma \equiv \chi \equiv \gamma.\beta.\gamma^{\langle -1 \rangle}$
- Derivative umbrae $\alpha_{\mathcal{D}} \quad , \quad \alpha_{\mathcal{D}}^n \simeq n\,\alpha^{n-1} \quad , \quad f_{\alpha_{\mathcal{D}}}(z) = 1 + z\,f_\alpha(z)$

$({}^{A}/_{\equiv}, +, .)$ is almost a right distributive ring

1. $({}^{A}/_{\equiv}, +)$ is an abelian group.

$$\alpha + \eta \equiv \eta + \alpha \qquad \alpha + \varepsilon \equiv \alpha \equiv \varepsilon + \alpha$$

and

$$(\alpha + \eta) + \gamma \equiv \alpha + (\eta + \gamma) \qquad \alpha + (-1.\alpha) \equiv \varepsilon \equiv (-1.\alpha) + \alpha$$

2. $({}^{A}/_{\equiv}, .)$ is a monoid.

$$\alpha.(\eta.\gamma) \equiv (\alpha.\eta).\gamma \quad \text{and} \quad \alpha.\upsilon \equiv \alpha \equiv \upsilon.\alpha$$

3. The scalar product.

$$1\alpha \equiv \alpha \qquad r(\alpha + \eta) \equiv r\alpha + r\eta$$

and

$$r(s\alpha) \equiv (rs)\alpha \qquad (r + s)\alpha \equiv r\alpha + s\alpha$$

4. The right distributive laws. distinct umbrae

$$(\alpha + \eta).\gamma \equiv \alpha.\gamma + \eta.\gamma \qquad \alpha.(r\eta) \equiv r(\alpha.\eta)$$

and

$$\gamma.(\alpha + \eta) \not\equiv \gamma.\alpha + \gamma.\eta \qquad \alpha.(r\eta) \not\equiv (r\alpha).\eta$$

Other useful identities:

$$\alpha.\varepsilon \equiv \varepsilon \equiv \varepsilon.\alpha \quad , \quad \beta.\chi \equiv \upsilon \equiv \chi.\beta \quad , \quad r\alpha \equiv \alpha.(r\upsilon) \quad .$$

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MR2022347 (2004k:05029) 05A40 (05A10 05A19 11B65 11B68 11B73)
Gessel, Ira M. (1-BRND)
Applications of the classical umbral calculus. (English summary)
Dedicated to the memory of Gian-Carlo Rota.
Algebra Universalis **49** (2003), *no. 4,* 397–434.

Umbral calculus has its roots in nineteenth century invariant theory. In the twentieth century, its use spread to combinatorial applications as invariant theory faded.

This thirty-eight page survey provides the reader with a full appreciation of the power and value of the subject. The titles of the sections provide a clear picture of the applications considered: Introduction, The classical umbral calculus, Charlier polynomials, Hermite polynomials, Carlitz and Zeilberger's Hermite polynomials, Bernoulli numbers, Kummer congruences, Median Genocchi numbers and Kummer congruences for Euler and Bell numbers.

The idea of the umbral calculus is deceptively simple. As the author points out, in the umbral calculus, the expression $(a+1)^n$ represents the sum $\sum_{i=0}^{n}\binom{n}{i}a_i$. In the last half of the twentieth century, G.-C. Rota put the whole subject on a rigorous foundation by basing the subject on linear functionals.

One portion of the introduction is especially noteworthy. "When I first encountered umbral notation it seemed to me that this was all there was to it; it was simply a notation for dealing with exponential generating functions, or to put it bluntly, it was a method for avoiding the use of exponential generating functions when they really ought to be used. The point of this paper is that my first impression was wrong; none of the results proved here $(\dots)$ can be easily proved by straightforward manipulation of exponential generating functions."

There has not been a book devoted to the umbral calculus published recently; S. M. Roman's book, [*The umbral calculus*, Academic Press, New York, 1984; MR0741185 (87c:05015)], appeared in 1984. This compelling paper could well be expanded into an appealing book on this topic.

Reviewed by *George E. Andrews*

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Outline

Umbral presentation for Riordan arrays

R	$R_{n,k}$
exponential	$\binom{n}{k}(\gamma + k.\alpha)^{n-k}$
ordinary	$\frac{(\gamma + k.\alpha)^{n-k}}{(n-k)!}$
generalized	$\frac{w_n}{w_k}\,\frac{(\gamma + k.\alpha)^{n-k}}{(n-k)!}$

Table : Umbral description of Riordan arrays

traditional view

ω: umbra with moments $\omega^n \simeq \frac{n!}{w_n}$, $w_n \neq 0$ for all $n \geq 0$, $w_0 = 1$.

Umbral presentation for Riordan arrays

R	notation	$R_{n,k}$
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Umbral coding

FTRA	$(\gamma,\alpha)\eta = \gamma + \eta.\beta.\alpha_{\mathcal{D}}$
Group multiplication	$(\gamma,\alpha)(\sigma,\rho) = (\gamma + \sigma.\beta.\alpha_{\mathcal{D}}\ ,\ \alpha + \rho.\beta.\alpha_{\mathcal{D}})$
Identity	$(\varepsilon,\varepsilon)$
Inverse	$(\gamma,\alpha)^{-1} = (\mathfrak{L}_{\gamma,\alpha}, \mathfrak{L}_{\alpha})\ ,\ \ \mathfrak{L}_{\gamma,\alpha} \equiv -1.\gamma.\beta.\alpha_{\mathcal{D}}^{\langle -1\rangle}\ ,\ \ \mathfrak{L}_{\alpha} := \mathfrak{L}_{\gamma,\alpha}.$

Subgroup	(γ,α)	group multiplication	$(\gamma,\alpha)^{-1}$	$(\gamma,\alpha)_{n,k}$
Appell	(γ,ε)	$(\gamma,\varepsilon)(\sigma,\varepsilon) = (\gamma+\sigma,\varepsilon)$	$(-1.\alpha,\varepsilon)$	$\binom{n}{k}\gamma^{n-k}$
Associated	(ε,α)	$(\varepsilon,\alpha)(\varepsilon,\rho) = (\varepsilon,\alpha+\rho.\beta.\alpha_{\mathcal{D}})$	$(\varepsilon,\mathfrak{L}_{\alpha})$	$\binom{n}{k}(k.\alpha)^{n-k}$
Bell	(α,α)	$(\alpha,\alpha)(\sigma,\sigma) = (\alpha+\sigma.\beta.\alpha_{\mathcal{D}}, \alpha+\sigma.\beta.\alpha_{\mathcal{D}})$	$(\mathfrak{L}_{\alpha},\mathfrak{L}_{\alpha})$	$\binom{n}{k}((k+1).\alpha)^{n-k}$

Table : Some distinguished Riordan subgroups

Abel's identity

Classical formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} (y+ka)^{n-k} x(x-ka)^{k-1}$$

Umbral formula (Version I)

$$(\gamma+\sigma)^n \simeq \sum_{k=0}^{n} \binom{n}{k} (\gamma + k.\alpha)^{n-k} \sigma \left(\sigma + (-k).\alpha\right)^{k-1}$$

Umbral Abel Identity (Version II) = FTRA

$$(\gamma + \eta.\beta.\alpha_{\mathcal{D}})^n \simeq \sum_{k=0}^{n} \binom{n}{k} (\gamma + k.\alpha)^{n-k} \eta^k \ .$$

Abel polynomials and Lagrange inversion formula

Definition (Abel polynomials)

Let $\mathfrak{K}_{\sigma,\alpha}$ be the auxiliary umbra whose moments are given by

$$\mathfrak{K}_{\sigma,\alpha}^{n} \simeq \sigma\big(\sigma + (-n).\alpha\big)^{n-1} \quad , \quad n \geq 1. \tag{2}$$

Theorem (Lagrange inversion formula)

For any umbrae α, γ and all integers $n \geq 1$, we have

$$\mathfrak{K}_{\sigma,\alpha}^{n} \simeq (\sigma.\beta.\alpha_{\mathcal{D}}^{\langle -1 \rangle})^{n}. \quad \textit{Equivalently,} \quad \mathfrak{K}_{\sigma,\alpha} \equiv \sigma.\beta.\alpha_{\mathcal{D}}^{\langle -1 \rangle}. \tag{3}$$

In terms of g.f.'s, (2) and (3) give the familiar Lagrange inversion formula

$$n\big[z^{n}\big] f_{\sigma}\big((zf_{\alpha}(z))^{\langle -1 \rangle}\big) = \big[z^{n-1}\big] f'_{\sigma}(z) \left(\frac{1}{f_{\alpha}(z)}\right)^{n}.$$

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$$n\Big[z^{n}\Big] f_{\sigma}\big((zf_{\alpha}(z))^{\langle -1 \rangle}\big) = \Big[z^{n-1}\Big] f'_{\sigma}(z) \left(\frac{1}{f_{\alpha}(z)}\right)^{n}.$$

Some interesting computations

$$f_{\alpha_{\mathcal{D}}}(z) - 1 \;=\; z\, f_\alpha(z) \quad \text{and} \quad \big(z\, f_\alpha(z)\big)^m \;\simeq\; z^m\, e^{(m.\alpha)z}$$

$$e^{(\sigma.\beta.\alpha_{\mathcal{D}})z} \simeq e^{\sigma(zf_\alpha(z))} \simeq \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \sigma^m\,(m.\alpha)^k\,\frac{z^m}{m!}\,\frac{z^k}{k!}$$

Comparing the coefficients of $\frac{z^n}{n!}$, we obtain

$$(\sigma.\beta.\alpha_{\mathcal{D}})^n \simeq \sum_{k=0}^{n}\binom{n}{k}\sigma^k(k.\alpha)^{n-k}\,. \qquad (4)$$

By means of (4) and the umbral Abel identity (version II) it is easy to check that

$$\left(\alpha + \eta.\beta.\alpha_{\mathcal{D}}\right)_{\mathcal{D}} \equiv \eta_{\mathcal{D}}.\beta.\alpha_{\mathcal{D}}\,.$$

A main result

Simple useful umbral trick

$$\gamma + k.\alpha \equiv \gamma + (k-m).\alpha + m.\alpha \qquad \forall\ k, m \in \mathbb{Z}$$

It then follows from the umbral Abel identity (Version I) that

$$\begin{aligned}(\gamma + k.\alpha)^{n-k} &\simeq \sum_{i=0}^{n-k} \binom{n-k}{i} (\gamma + (k-m).\alpha + i.\lambda)^{n-k-i}\ m.\alpha(m.\alpha + (-i).\lambda)^{i-1} \\ &\simeq (n-k)! \sum_{i=0}^{n-k} \frac{(\gamma + (k-m).\alpha + i.\lambda)^{n-k-i}}{(n-k-i)!}\ \frac{m.\alpha(m.\alpha + (-i).\lambda)^{i-1}}{i!}\ .\end{aligned}$$

Theorem (A.-Mestre-Petrullo-Torres)

For any umbra λ and any integers m, n, k, with $n \geq k$, it holds

$$_{\omega}(\gamma, \alpha)_{n,k} \simeq \frac{w_n}{w_k} \sum_{i=0}^{n-k} \frac{(m.\mathfrak{K}_{\alpha,\lambda})^i}{i!}\ \frac{(\gamma + (k-m).\alpha + i.\lambda)^{n-k-i}}{(n-k-i)!}\ .$$

Corollaries

1. Horizontal recurrence relation. *For any integer m such that $n \geq m$, it holds*

$$_{\omega}(\gamma,\alpha)_{n,k} \simeq \frac{w_n}{w_k\, w_{n-m}} \sum_{i=0}^{n-k} w_{k-m+i} \frac{\left(m.\mathfrak{K}_{\alpha,\alpha}\right)^i}{i!} \; _{\omega}(\gamma,\alpha)_{n-m,k-m+i}\,.$$

2. Vertical recurrence relation. *For any integer m such that $k \geq m$, it holds*

$$_{\omega}(\gamma,\alpha)_{n,k} \simeq \frac{w_n\, w_{k-m}}{w_k} \sum_{i=0}^{n-k} \frac{1}{w_{n-m-i}} \frac{\left(m.\alpha\right)^i}{i!} \; _{\omega}(\gamma,\alpha)_{n-m-i,k-m}\,.$$

3. A novel recurrence relation. *For any integer m such that $2k - n \geq m$, it holds*

$$_{\omega}(\gamma,\alpha)_{n,k} \simeq \frac{w_n}{w_k} \sum_{i=0}^{n-k} \frac{w_{k-m-i}}{w_{n-m-2i}} \frac{\left(m.\mathfrak{K}_{\alpha,-1.\alpha}\right)^i}{i!} \; _{\omega}(\gamma,\alpha)_{n-m-2i,k-m-i}\,.$$

Some remarks are in order

1. Our horizontal recurrence relation extends to generalized Riordan arrays the horizontal recurrence relation for ordinary Riordan arrays obtained recently by Luzón-Merlini-Morón-Sprugnoli [*Identities induced by Riordan arrays*, Linear Algebra Appl. **436** (3): 631–647 (2012)]. More explicitly, we have $\frac{(m.\mathfrak{K}_{\alpha,\alpha})^i}{i!} \simeq a_i^{(m)}$, where $(a_i^{(m)})$ stands for a sequence that generalizes the classical A-sequence of Rogers [*Pascal triangles, Catalan numbers and Renewal Arrays*, Discrete Math. **22**: 301–310 (1978)],

$$R_{n+1,k+1} = \sum_{j=0}^{n-k} a_j R_{n,k+j} \qquad (A\text{-sequence of a Riordan array})\,.$$

2. Our vertical recurrence relation extends to generalized Riordan arrays the vertical recurrence relation for ordinary Riordan arrays obtained by Luzón-Merlini-Morón-Sprugnoli.

Outline

Pascal, Ballot and Catalan arrays

Pascal array and its inverse

exponential presentation: $\boldsymbol{P} = (\upsilon, \varepsilon) \quad , \quad \boldsymbol{P}^{-1} = (-1.\upsilon, \varepsilon)$

ordinary presentation: $\boldsymbol{P} = {}_{\bar{\upsilon}}(\bar{\upsilon}, \bar{\upsilon}) \quad , \quad \boldsymbol{P}^{-1} = {}_{\bar{\upsilon}}(\bar{\upsilon}. - 1, \bar{\upsilon}. - 1)$

Ballot array and its inverse

ordinary presentation: $\boldsymbol{C} = {}_{\bar{\upsilon}}(\varsigma, \varsigma) \quad , \quad \boldsymbol{C}^{-1} = {}_{\bar{\upsilon}}(-1.\bar{\upsilon}, -1.\bar{\upsilon})$

Catalan array and its inverse

ordinary presentation: $\boldsymbol{C}^{(1)} = {}_{\bar{\upsilon}}(2.\varsigma, 2.\varsigma) \quad , \quad (\boldsymbol{C}^{(1)})^{-1} = {}_{\bar{\upsilon}}(-2.\chi, -2.\chi)$

We have

$${}_{\bar{\upsilon}}(2.\varsigma, 2.\varsigma) \equiv {}_{\bar{\upsilon}}(\varsigma, \varsigma)\ {}_{\bar{\upsilon}}(\bar{\upsilon}, \bar{\upsilon}); \quad \text{that is,} \quad \boldsymbol{C}^{(1)} = \boldsymbol{C}\boldsymbol{P}$$

$${}_{\bar{\upsilon}}(-2.\chi, -2.\chi) \equiv {}_{\bar{\upsilon}}(\bar{\upsilon}. - 1, \bar{\upsilon}. - 1)\ {}_{\bar{\upsilon}}(-1.\bar{\upsilon}, -1.\bar{\upsilon}); \quad \text{that is,} \quad (\boldsymbol{C}^{(1)})^{-1} = \boldsymbol{P}^{-1}\boldsymbol{C}^{-1}$$

A family of Catalan arrays

Note that $\boldsymbol{P}^q = (q, \varepsilon) = {}_{\bar{\upsilon}}(\bar{\upsilon}.q, \bar{\upsilon}.q)$ for any $q \in \mathbb{C}$. Define

$$\boldsymbol{C}^{(q)} = {}_{\bar{\upsilon}}(\varsigma, \varsigma)\ {}_{\bar{\upsilon}}(\bar{\upsilon}.q, \bar{\upsilon}.q); \quad \text{that is,} \quad \boldsymbol{C}^{(q)} = \boldsymbol{C}\boldsymbol{P}^q$$

The entries of $\boldsymbol{C}^{(q)}$ are given explicitly by the following formula:

$$\boldsymbol{C}^{(q)}_{n,k} = \sum_{i=0}^{n-k} \binom{i+k}{k} \boldsymbol{C}_{n,i+k}\ q^i.$$

Thus, the first rows of $\boldsymbol{C}^{(q)}$ are

$$\boldsymbol{C}^{(q)} = \begin{pmatrix} 1 & & & & & \\ 1+q & 1 & & & & \\ 2+2q+q^2 & 2+2q & 1 & & & \\ 5+5q+3q^2+q^3 & 5+6q+3q^2 & 3+3q & 1 & & \\ 14+14q+9q^2+4q^3+q^4 & 14+18q+12q^2+4q^3 & 9+12q+6q^2 & 4+4q & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

A family of Catalan arrays $\boldsymbol{C}^{(q)} = \boldsymbol{C}\boldsymbol{P}^q$

Correspondingly,

$$\left(\boldsymbol{C}^{(q)}\right)^{-1} = {}_{\bar{\upsilon}}(-1.\bar{\upsilon}, -1.\bar{\upsilon})\ {}_{\bar{\upsilon}}(\bar{\upsilon}. - q, \bar{\upsilon}. - q)$$

The entries of $(\boldsymbol{C}^{(q)})^{-1}$ are given explicitly by the following formula:

$$(\boldsymbol{C}^{(q)})^{-1}_{n,k} = (-1)^{n-k} \sum_{i=0}^{n-k} \binom{i+k+1}{n-i-k} \binom{i+k}{k} q^i.$$

Properties of $\boldsymbol{C}^{(q)}$ and $(\boldsymbol{C}^{(q)})^{-1}$:

- $\boldsymbol{C}^{(q)}_{n,k}$ and $(-1)^{n-k}(\boldsymbol{C}^{(q)})^{-1}_{n,k}$ are polyn. in q with positive integer coeffs.
- $\boldsymbol{C}^{(q)} = \boldsymbol{C}$ when $q = 0$, and $\boldsymbol{C}^{(q)} = \boldsymbol{C}^{(1)}$ when $q = 1$.
- $\forall\ n \geq 0$, $\boldsymbol{C}^{(q)}_{n+1,1}$ has deg $\boldsymbol{C}^{(q)}_{n+1,1} = n$, constant term C_{n+1} and leading coeff. $n+1$.
- $(\boldsymbol{C}^{(q)})^{-1}$ is the coefficient matrix of Chebyshev polynomials of 2nd. kind $(U_n(\frac{x-2}{2}))$ when $q = 1$. Recall, $\sum_{n\geq 0} U_n(x)z^n = \frac{1}{1 - xz + z^2}$

Open question

Find a combinatorial description of $\boldsymbol{C}^{(q)}$ and $(\boldsymbol{C}^{(q)})^{-1}$.

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Thank you!