# On symmetric polynomials with only real zeros and nonnegative $\gamma$-vectors

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Based on the paper

**On Symmetric polynomials with only real zeros and nonnegative $\gamma$-vectors**

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# Outline

1. Introduction
   - Eulerian polynomials
   - Narayana polynomials
   - Common properties
2. Families of polynomials having the same properties
   - An alternative definition of Eulerian polynomials
   - Generalized Stirling numbers of the second kind
   - The polynomials $A_{r,s,N_r(n)}(z)$
   - Some distinguished bases. Several new generalizations
3. General remarks

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1. Introduction
   - Eulerian polynomials
   - Narayana polynomials
   - Common properties
2. Families of polynomials having the same properties
3. General remarks

# Eulerian polynomials

$$\begin{aligned}
A_1(z) &= z. \\
A_2(z) &= z + z^2. \\
A_3(z) &= z + 4z^2 + z^3. \\
A_4(z) &= z + 11z^2 + 11z^3 + z^4. \\
A_5(z) &= z + 26z^2 + 66z^3 + 26z^4 + z^5. \\
A_6(z) &= z + 57z^2 + 302z^3 + 302z^4 + 57z^5 + z^6. \\
A_7(z) &= z + 120z^2 + 1191z^3 + 2416z^4 + 1191z^5 + 120z^6 + z^7. \\
A_8(z) &= z + 247z^2 + 4293z^3 + 15619z^4 + 15619z^5 + 4293z^6 + 247z^7 + z^8. \\
A_9(z) &= z + 502z^2 + 14608z^3 + 88234z^4 + 156190z^5 + 88234z^6 + 14608z^7 + 502z^8 + z^9. \\
&\vdots
\end{aligned}$$

$$A_n(z) = \sum_{k=1}^{n} A(n,k)\, z^k$$

$$A(n,k) = \sum_{p=0}^{k} (-1)^p \binom{n+1}{p} (k-p)^n$$

# Descent g.f. for permutations in $S_n$

$\forall\, n \in \mathbb{Z}_+$ , $[n] := \{1, \ldots, n\}$ , $S_n$: the symmetric group of degree $n$.

$$w = w_1 w_2 \cdots w_n \in S_n$$

A *descent* in $w \in S_n \iff w_i > w_{i+1}$, where $i \in [n-1]$.

$$\mathrm{des}(w) := \left|\{i \in [n-1] \,:\, w_i > w_{i+1}\}\right|.$$

| $w$ | $\mathrm{des}(w)$ | $z^{\mathrm{des}(w)+1}$ |
|---|---|---|
| 123 | | |
| 132 | | |
| 213 | | |
| 231 | | |
| 312 | | |
| 321 | | |

$$A_n(z) = \sum_{w \in S_n} z^{\mathrm{des}(w)+1}$$

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$$A_3(z) = z + 4z^2 + z^3$$

$$A(n,k) = \left|\{w \in S_n \,:\, \mathrm{des}(w) = k-1\}\right|$$

# Other permutation statistics

$$w = 0 w_1 w_2 \cdots w_n 0 \in S_n$$

A *final descent* in $w \in S_n \iff w_{n-1} > w_n$.

A *double descent* in $w \in S_n \iff w_i > w_{i+1} > w_{i+2}$

A *peak* in $w \in S_n \iff w_{i-1} < w_i > w_{i+1} \quad , \quad 1 \leq i \leq n$

$$\mathrm{peak}(w) := \left|\{i \in [n] \,:\, i \text{ is a peak of } w\}\right|.$$

$$\widehat{S}_n := \{w \in S_n \,:\, w \text{ has no final descent nor double descents}\}$$

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**Theorem** Foata-Schützenberger 1970, Shapiro-Woan-Getu 1983

$$\sum_{w \in S_n} z^{\mathrm{des}(w)+1} = \sum_{w \in \widehat{S}_n} z^{\mathrm{peak}(w)} (z+1)^{n+1-2\,\mathrm{peak}(w)}$$

# Example ($n = 3$)

| $w$ | $\mathrm{des}(w)$ | $\mathrm{peak}(w)$ | $z^{\mathrm{des}(w)+1}$ | $z^{\mathrm{peak}(w)}(1+z)^{n+1-2\,\mathrm{peak}(w)}$ |
|---|---|---|---|---|
| 123 | | | | |
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|---|---|---|---|---|
| 123 | 0 | 1 | $z$ | $z(1+z)^2$ |
| 132 | | | | |
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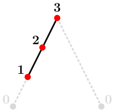

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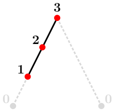


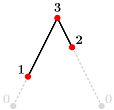

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| 123 | 0 | 1 | $z$ | $z(1+z)^2$ |
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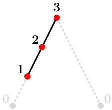


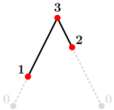


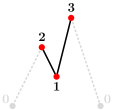

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|---|---|---|---|---|
| 123 | 0 | 1 | $z$ | $z(1+z)^2$ |
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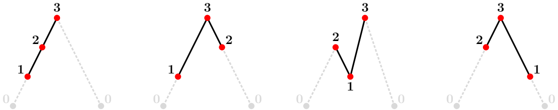

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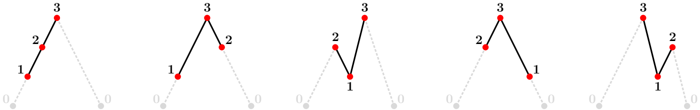

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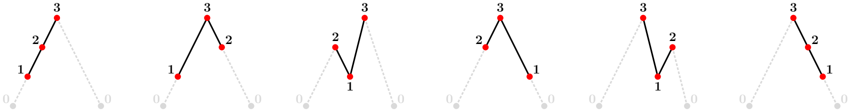

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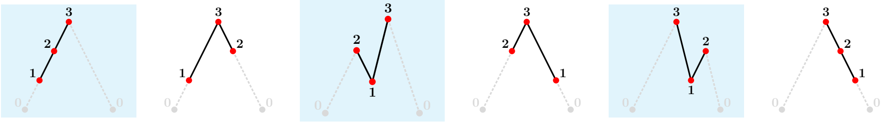

$$\widehat{S}_3 = \{123, 213, 312\} \qquad A_3(z) = z + 4z^2 + z^3 = 1z(1+z)^2 + 2z^2$$

# Example ($n = 3$)

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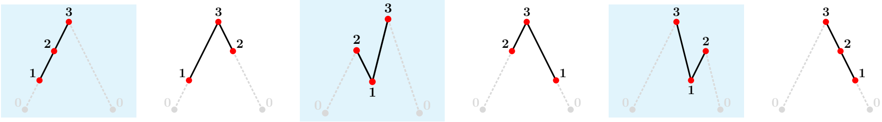


$\widehat{S}_3 = \{123, 213, 312\}$ $\quad A_3(z) = z + 4z^2 + z^3 = 1z(1+z)^2 + 2z^2$

$$\gamma_i(A_n(z)) := \left|\{w \in \widehat{S}_n : \text{peak}(w) = i\}\right|$$

*$\gamma$-numbers of $A_n(z)$*

$$\gamma_1(A_3(z)) = 1\ ,\ \gamma_2(A_3(z)) = 2\ ,\ \gamma_3(A_3(z)) = 0$$

# Narayana polynomials

$$
\begin{aligned}
N_1(z) &= z. \\
N_2(z) &= z + z^2. \\
N_3(z) &= z + 3z^2 + z^3. \\
N_4(z) &= z + 6z^2 + 6z^3 + z^4. \\
N_5(z) &= z + 10z^2 + 20z^3 + 10z^4 + z^5. \\
N_6(z) &= z + 15z^2 + 50z^3 + 50z^4 + 15z^5 + z^6. \\
N_7(z) &= z + 21z^2 + 105z^3 + 175z^4 + 105z^5 + 21z^6 + z^7. \\
N_8(z) &= z + 28z^2 + 196z^3 + 490z^4 + 490z^5 + 196z^6 + 28z^7 + z^8. \\
N_9(z) &= z + 36z^2 + 336z^3 + 1176z^4 + 1764z^5 + 1176z^6 + 336z^7 + 36z^8 + z^9. \\
\vdots & \qquad \vdots
\end{aligned}
$$

$A_{1,2}(z)$

$$N_n(z) = \sum_{k=1}^{n} N(n,k)\, z^k$$

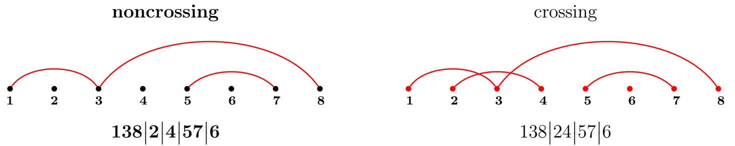
$$N(n,k) = \frac{1}{n}\binom{n}{k-1}\binom{n}{k}$$

# Noncrossing partitions of $[n]$

$P(n)$: set of partitions of $[n]$.

Write $w = w_1 w_2 \cdots w_n \in P(n)$ and $w_i \sim w_j$ if $w_i, w_j$ are in the same block.

$$w \text{ is } \textit{noncrossing} \iff \begin{array}{l} 1 \leq w_i < w_j < w_k < w_l \leq n \quad \text{and} \\ w_i \sim w_k \;,\; w_j \sim w_l \quad \text{then} \quad w_i \sim w_j \sim w_k \sim w_l. \end{array}$$



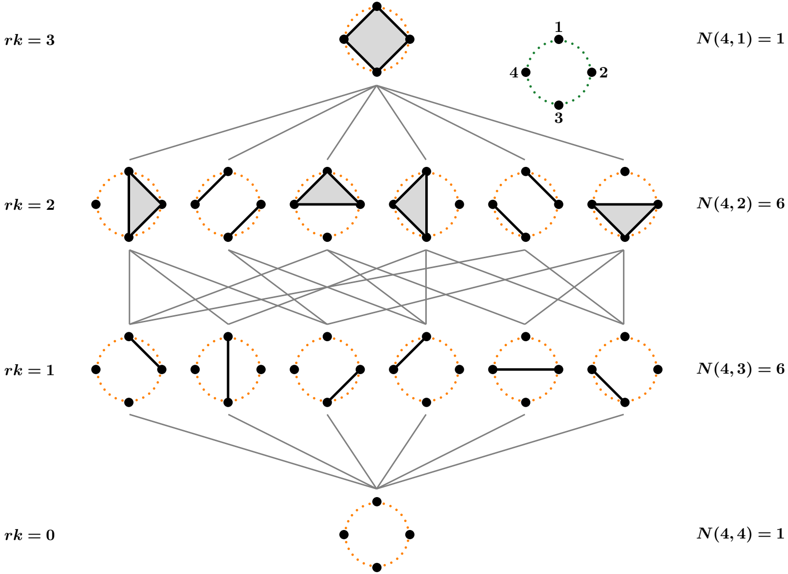

$\mathrm{NC}(n) := \{w \in P(n) \,:\, w \text{ is noncrossing}\}$ $\mathrm{rk}(w) := n - |\textit{blocks of } w|$

$$N_n(z) = \sum_{w \in \mathrm{NC}(n)} z^{\mathrm{rk}(w)+1}$$

$$N(n,k) = \left|\{w \in \mathrm{NC}(n) \,:\, \mathrm{rk}(w) = n-k\}\right|$$

number of noncrossing partitions of $[n]$ into $k$ blocks

# Rank g.f. for noncrossing partitions of $[n]$



# Some common properties

| $A_n(z)$: Eulerian polynomials | $N_n(z)$: Narayana polynomials |
|---|---|
| $A_1(z) = z.$ | $N_1(z) = z.$ |
| $A_2(z) = z + z^2.$ | $N_2(z) = z + z^2.$ |
| $A_3(z) = z + 4z^2 + z^3.$ | $N_3(z) = z + 3z^2 + z^3.$ |
| $A_4(z) = z + 11z^2 + 11z^3 + z^4.$ | $N_4(z) = z + 6z^2 + 6z^3 + z^4.$ |
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| $A_6(z) = z + 57z^2 + 302z^3 + 302z^4 + 57z^5 + z^6.$ | $N_6(z) = z + 15z^2 + 50z^3 + 50z^4 + 15z^5 + z^6.$ |
| $\vdots$ | $\vdots$ |

- *positive integer coefficients*
- *symmetric (palindromic)*
- *unimodal*
- *log-concave*
- *only real zeros*
- *nonnegative $\gamma$-numbers*

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| $A_4(z) = z + 11z^2 + 11z^3 + z^4.$ | $N_4(z) = z + 6z^2 + 6z^3 + z^4.$ |
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| $\vdots$ | $\vdots$ |

- *positive integer coefficients*
- *symmetric (palindromic)*
- *unimodal*
- *log-concave*
- *only real zeros*
- *nonnegative $\gamma$-numbers*

$$h(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n$$

$$a_k \in \mathbb{Z}_{\geq 0}, \quad \text{for } 1 \leq k \leq n$$

# Some common properties

| $A_n(z)$: Eulerian polynomials | $N_n(z)$: Narayana polynomials |
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| $A_1(z) = z.$ | $N_1(z) = z.$ |
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$$h(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n$$

$$a_k = a_{n+1-k}, \quad \text{for } 1 \leq k \leq \left\lfloor \frac{n+1}{2} \right\rfloor$$

# Some common properties

| $A_n(z)$: Eulerian polynomials | $N_n(z)$: Narayana polynomials |
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$$h(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n$$

if there is an index $1 \leq n^c \leq n$ such that

$$a_k \leq a_{k+1}, \quad \text{for } 1 \leq k \leq n^c - 1$$
$$a_k \geq a_{k+1}, \quad \text{for } n^c \leq k \leq n - 1$$

# Some common properties

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$$h(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n$$

$$a_{k+1}^2 \geq a_k\, a_{k+2}, \quad \text{for } 1 \leq k \leq n-2$$

# Some common properties

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$$h(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n$$

$$h(z) = 0 \iff z \in \mathbb{R} \quad (z \leq 0)$$

# Some common properties

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$$h(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n$$

$$h(z) = \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \gamma_i(h(z))\ z^i (1+z)^{n+1-2i}$$

$$\gamma_i(h(z)) \geq 0$$

# Some common properties

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$$h(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n$$

# Outline

# Euler (1730's)

The classical Eulerian polynomials are given by

$$\sum_{l \geq 0} l^n z^l = \frac{A_n(z)}{(1-z)^{n+1}}$$

For $n = 0, 1, 2, 3$ we get respectively

$$\begin{aligned} 1 + z + z^2 + z^3 + \cdots &= \frac{1}{1-z} \\ z + 2z^2 + 3z^3 + 4z^4 + \cdots &= \frac{z}{(1-z)^2} \\ z + 2^2z^2 + 3^2z^3 + 4^2z^4 + \cdots &= \frac{z + z^2}{(1-z)^3} \\ z + 2^3z^2 + 3^3z^3 + 4^3z^4 + \cdots &= \frac{z + 4z^2 + z^3}{(1-z)^4} \end{aligned}$$

# An alternative definition of Eulerian polynomials

Let us consider the following two well-known linear operators on $\mathbb{C}[z]$:

$$Z: \quad \text{multiplication by } z$$
$$D: \quad \text{the usual derivative}$$

## Eulerian polynomials

$$(ZD)^n \left( \frac{1}{1-z} \right) = \frac{A_n(z)}{(1-z)^{n+1}} .$$

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## Eulerian polynomials

$$(ZD)^n \left(\frac{1}{1-z}\right) = \frac{A_n(z)}{(1-z)^{n+1}}.$$

$$[D, Z] := DZ - ZD = 1$$

# Normal ordering of $(Z^r D^s)^n$

$[D, Z] := DZ - ZD = 1$

$$(ZD)^2 = Z\underline{DZ}D = Z(1+ZD)D = 1ZD + 1Z^2D^2$$

$$(ZD)^3 = Z\underline{DZ}\,\underline{DZ}D = Z(1+ZD)(1+ZD)D = 1ZD + 3Z^2D^2 + 1Z^3D^3$$

$$(ZD)^4 = Z\underline{DZ}\,\underline{DZ}\,\underline{DZ}D = 1ZD + 7Z^2D^2 + 6Z^3D^3 + 1Z^4D^4$$

**Theorem (Stirling numbers of the second kind)** Scherk 1823

$$(ZD)^n = \sum_{k=1}^{n} S(n,k)\, Z^k D^k .$$

# Normal ordering of $(Z^r D^s)^n$

$[D, Z] := DZ - ZD = 1$

$$\begin{aligned}
(ZD)^2 &= Z\underline{DZ}D &&= Z(1+ZD)D = 1ZD + 1Z^2D^2 \\
(ZD)^3 &= Z\underline{DZ}\,\underline{DZ}D &&= Z(1+ZD)(1+ZD)D = 1ZD + 3Z^2D^2 + 1Z^3D^3 \\
(ZD)^4 &= Z\underline{DZ}\,\underline{DZ}\,\underline{DZ}D &&= 1ZD + 7Z^2D^2 + 6Z^3D^3 + 1Z^4D^4
\end{aligned}$$

**Theorem (Stirling numbers of the second kind)** Scherk 1823

$$(ZD)^n = \sum_{k=1}^{n} S(n,k)\, Z^k D^k\,.$$

**Definition (Generalized Stirling numbers of the second kind)** Blasiak et al. 2003

$$(Z^r D^s)^n = \begin{cases} \displaystyle\sum_{k=r}^{rn} S_{r,s}(n,k)\; Z^k D^{n(s-r)+k} & \text{if} \quad s \leq 0 < r \quad \text{or} \quad 0 < r \leq s \\[2ex] \displaystyle\sum_{k=s}^{sn} S_{r,s}(n,k)\; Z^{n(r-s)+k} D^k & \text{if} \quad r \leq 0 < s \quad \text{or} \quad 0 < s \leq r \end{cases}$$

# Generalized Stirling numbers of the second kind

The numbers $S_{r,s}(n,k)$ are symmetric in $(r,s)$; that is,

$$S_{r,s}(n,k) = S_{s,r}(n,k)\,,$$

for any integers $n, k, r, s > 0$.

$\boldsymbol{S_{1,1}(n,k)}$

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 0 | 1 | | | | | |
| 2 | 0 | 1 | 1 | | | | |
| 3 | 0 | 1 | 3 | 1 | | | |
| 4 | 0 | 1 | 7 | 6 | 1 | | |
| 5 | 0 | 1 | 15 | 25 | 10 | 1 | |
| 6 | 0 | 1 | 31 | 90 | 65 | 15 | 1 |
| ⋮ | | | | ⋮ | | | |

Classical Stirling numbers 2nd. kind

$\boldsymbol{S_{1,2}(n,k)}$

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 0 | 1 | | | | | |
| 2 | 0 | 2 | 1 | | | | |
| 3 | 0 | 6 | 6 | 1 | | | |
| 4 | 0 | 24 | 36 | 12 | 1 | | |
| 5 | 0 | 120 | 240 | 120 | 20 | 1 | |
| 6 | 0 | 720 | 1800 | 1200 | 300 | 30 | 1 |
| ⋮ | | | | ⋮ | | | |

Unsigned Lah numbers

## Proposition    Blasiak et al. 2003

*For any integers* $r \leq 0 < s$ *or* $0 < s \leq r$, *we have*

$$S_{r,s}(n,k) = \begin{cases} \sum_{p=0}^{s} \binom{s}{p} (n(r-s)+k-r+p)^{\underline{p}} S_{r,s}(n-1, k-s+p) & (\textit{recursive}) \\ \frac{1}{k!} \sum_{p=s}^{k} (-1)^{k-p} \binom{k}{p} \prod_{j=1}^{n} (p+(j-1)(r-s))^{\underline{s}} & (\textit{explicit}) \end{cases}$$

$$x^{\underline{p}} := x(x-1)\cdots(x-p+1)$$

## Proposition

*For any integers* $s \leq 0 < r$ *or* $0 < r \leq s$, *we have*

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## Proposition

*For any integers* $s \leq 0 < r$ *or* $0 < r \leq s$, *we have*

$$S_{r,s}(n,k) = \begin{cases} \sum_{p=0}^{r} \binom{r}{p} (n(s-r)+k-s+p)^{\underline{p}} S_{r,s}(n-1,k-r+p) & (\textit{recursive}) \\ \frac{1}{k!} \sum_{p=r}^{k} (-1)^{k-p} \binom{k}{p} \prod_{j=1}^{n} (p+(j-1)(s-r))^{\underline{r}} & (\textit{explicit}) \end{cases}$$

# The key idea

## Compute

$$\left(Z^r D^s\right)^k \left(\frac{1}{1-z}\right)$$

# A code in *Mathematica* to compute $(z^r D^s)^k \left(\frac{1}{1-z}\right)$ $1 \le k \le n$

```
ZrDsn[r_,s_,n_]:=NestList[Simplify[z^r D[#,{z,s}]] &, 1/(1-z), n];
```

$$\mathbf{ZrDsn[1,1,6]} = \left\{ \frac{1}{1-z}, \frac{z}{(1-z)^2}, \frac{z+z^2}{(1-z)^3}, \frac{z+4z^2+z^3}{(1-z)^4}, \frac{z+11z^2+11z^3+z^4}{(1-z)^5}, \frac{z+26z^2+66z^3+26z^4+z^5}{(1-z)^6}, \frac{z+57z^2+302z^3+302z^4+57z^5+z^6}{(1-z)^7} \right\}.$$

$$\mathbf{ZrDsn[1,2,6]} = \left\{ \frac{1}{1-z}, \frac{2(z)}{(1-z)^3}, \frac{12(z+z^2)}{(1-z)^5}, \frac{144(z+3z^2+z^3)}{(1-z)^7}, \frac{2880(z+6z^2+6z^3+z^4)}{(1-z)^9}, \frac{86400(z+10z^2+20z^3+10z^4+z^5)}{(1-z)^{11}}, \frac{3628800(z+15z^2+50z^3+50z^4+15z^5+z^6)}{(1-z)^{13}} \right\}.$$

▸ Narayana polynomials

$$\mathbf{ZrDsn[1,3,6]} = \left\{ \frac{1}{1-z}, \frac{6(z)}{(1-z)^4}, \frac{360(z+z^2)}{(1-z)^7}, \frac{15120(5z+14z^2+5z^3)}{(1-z)^{10}}, \frac{5443200(7z+37z^2+37z^3+7z^4)}{(1-z)^{13}}, \frac{1796256000(21z+176z^2+334z^3+176z^4+21z^5)}{(1-z)^{16}}, \frac{1961511552000(33z+397z^2+1202z^3+1202z^4+397z^5+33z^6)}{(1-z)^{19}} \right\}.$$

# Computing $(z^r D^s)^k \left(\frac{1}{1-z}\right)$ $1 \le k \le n$

$$\textbf{ZrDsn}[2,2,5] = \left\{ \frac{1}{1-z}, \frac{2z(z)}{(1-z)^3}, \frac{4z(z+4z^2+z^3)}{(1-z)^5}, \frac{8z(z+20z^2+48z^3+20z^4+z^5)}{(1-z)^7}, \frac{16z(z+72z^2+603z^3+1168z^4+603z^5+72z^6+z^7)}{(1-z)^9}, \frac{32z(z+232z^2+5158z^3+27664z^4+47290z^5+27664z^6+5158z^7+232z^8+z^9)}{(1-z)^{11}} \right\}.$$

$$\textbf{ZrDsn}[2,3,5] = \left\{ \frac{1}{1-z}, \frac{6z(z)}{(1-z)^4}, \frac{144z(z+3z^2+z^3)}{(1-z)^7}, \frac{8640z(z+10z^2+20z^3+10z^4+z^5)}{(1-z)^{10}}, \frac{1036800z(z+22z^2+113z^3+190z^4+113z^5+22z^6+z^7)}{(1-z)^{13}}, \frac{217728000z(z+40z^2+400z^3+1456z^4+2212z^5+1456z^6+400z^7+40z^8+z^9)}{(1-z)^{16}} \right\}.$$

# Computing $(z^r D^s)^k \left(\frac{1}{1-z}\right)$ $1 \leq k \leq n$

$$\textbf{ZrDsn[2,1,6]} = \left\{ \frac{1}{1-z}, \frac{z^2}{(1-z)^2}, \frac{2z^3}{(1-z)^3}, \frac{6z^4}{(1-z)^4}, \frac{24z^5}{(1-z)^5}, \frac{120z^6}{(1-z)^6}, \frac{720z^7}{(1-z)^7} \right\}.$$

$$\textbf{ZrDsn[3,1,6]} = \left\{ \frac{1}{1-z}, \frac{z^3}{(1-z)^2}, \frac{z^5(3-z)}{(1-z)^3}, \frac{3z^7(5-4z+z^2)}{(1-z)^4}, \frac{3z^9(35-47z+25z^2-5z^3)}{(1-z)^5}, \frac{15z^{11}(63-122z+102z^2-42z^3+7z^4)}{(1-z)^6}, \frac{45z^{13}(231-593z+686z^2-434z^3+147z^4-21z^5)}{(1-z)^7} \right\}.$$

$$\textbf{ZrDsn[3,2,6]} = \left\{ \frac{1}{1-z}, \frac{2z^3}{(1-z)^3}, \frac{12z^4(1+z)}{(1-z)^5}, \frac{144z^5(1+3z+z^2)}{(1-z)^7}, \frac{2880z^6(1+6z+6z^2+z^3)}{(1-z)^9}, \frac{86400z^7(1+10z+20z^2+10z^3+z^4)}{(1-z)^{11}}, \frac{3628800z^8(1+15z+50z^2+50z^3+15z^4+z^5)}{(1-z)^{13}} \right\}.$$

# The polynomials $A_{r,s,N_r(n)}(z)$

## Definition

Let $r, s$ and $n$ be any integers such that $1 \leq r \leq s$ and $n \geq 1$. The formula

$$(Z^r D^s)^n \left( \frac{1}{1-z} \right) = \frac{K_{r,s,n} \; z^{r-1} A_{r,s,N_r(n)}(z)}{(1-z)^{ns+1}} \;,$$

defines a zero constant term polynomial

$$A_{r,s,N_r(n)}(z) = \sum_{k=1}^{N_r(n)} A_{r,s}(n,k) z^k$$

of degree $N_r(n) = r(n-1)+1$, with integer coefficients $A_{r,s}(n,k)$ for every $1 \leq k \leq N_r(n)$. Moreover, $A_{r,s,N_r(n)}(z)$ is symmetric; namely, it holds

$$A_{r,s}(n,k) = A_{r,s}(n, N_r(n)+1-k) \quad , \quad \text{for each } 1 \leq k \leq \lfloor \tfrac{N_r(n)+1}{2} \rfloor \;.$$

The number $K_{r,s,n}$ is a positive integer that in general depends on $r, s$ and $n$.

# The numbers $K_{r,s,n}$

By direct observation/computation, one can identify the integer number $K_{r,s,n}$ for any given values of $r$ and $s$ and every $n \geq 1$. For instance, we have

$$K_{1,1,n} = 1$$

More generally, for any integer $r \geq 1$,

$$K_{r,r,n} = (r!)^n$$

Also,

$$K_{1,2,n} = n!(n+1)!$$

More generally, for any integer $r \geq 1$,

$$K_{r,r+1,n} = \prod_{i=0}^{r} \frac{(n+i)!}{i!}$$

# A novel alternative definition of Narayana polynomials

## Narayana polynomials

$$\left(ZD^2\right)^n\left(\frac{1}{1-z}\right)=\frac{n!(n+1)!\ N_n(z)}{(1-z)^{2n+1}}$$

# A recursion formula

Set $\quad f_n(z) = \prod_{j=1}^{n} \big(z - (j-1)(s-r)\big)^{\underline{s}}.$

## Proposition

properties

$$\sum_{i \geq 0} f_n(i) z^i = \frac{K_{r,s,n}\; z^{n(s-r)+r-1} A_{r,s,N_r(n)}(z)}{(1-z)^{ns+1}}\ .$$

Note that $\quad f_{n+1}(z) = \big(z - n(s-r)\big)^{\underline{s}} f_n(z)$

## Theorem

*The polynomial $A_{r,s,N_r(n)}(z)$ satisfy the following recursive formula.*

$$A_{r,s,N_r(n+1)}(z) = \frac{K_{r,s,n}}{K_{r,s,n+1}} \sum_{j=0}^{s} \sum_{k=0}^{j} \prod_{l=1}^{k} (l+ns) \binom{s}{j} \binom{j}{k} (r-1)^{\underline{s-j}} z^{r-s+j} (1-z)^{s-k} A^{(j-k)}_{r,s,N_r(n)}(z)\,.$$

# Known formulas $(r = 1, s = 1)$

**Corollary** Eulerian polynomials

$$A_{n+1}(z) = z(1 - z)A_n'(z) + (1 + n)zA_n(z)\,,$$

*with initial condition* $A_0(z) = 1$.

**Corollary** Eulerian numbers

$$A(n,k) = (n - k + 1)A(n - 1, k - 1) + kA(n - 1, k)\,.$$

# New formulas

$(r = 1, s = 2)$

## Corollary — Narayana polynomials

$$N_{n+1}(z) = \frac{z\left((1-z)^2 N_n''(z) + 2(1+2n)(1-z)N_n'(z) + (1+2n)(2+2n)N_n(z)\right)}{(n+2)(n+1)},$$

*with initial conditions* $N_0(z) = 1$ *and* $N_1(z) = z$.

## Corollary — Narayana numbers

$$N(n,k) = \frac{(k-1)(k-2) + 2(2n-1)(n-k+1)}{(n+1)n} N(n-1,k-1) + \frac{2k(2n-k)}{(n+1)n} N(n-1,k) + \frac{(k+1)k}{(n+1)n} N(n-1,k+1).$$

# Rational generating functions

**Theorem** (see e.g. Stanley's Enumerative Combinatorics, Vol 1)

*Let $f(z) \in \mathbb{C}[z]$ of degree $d$. Then*

$$\sum_{n \geq 0} f(n) z^n = \frac{W(z)}{(1-z)^{d+1}}$$

*where $W(z) \in \mathbb{C}[z]$ , $W(1) \neq 0$, and $\deg W \leq d$.*

$$W(z) = w_0 + w_1 z + \cdots + w_d z^d$$

**Definition**

The numbers $w_0, w_1, \ldots, w_d$ are called the *$f$-Eulerian numbers*, and the polynomial $W(z) = w_0 + w_1 z + \cdots + w_d z^d$ is called the *$f$-Eulerian polynomial*.

Denote by $\lambda(f)$ the smallest zero of $f$ and by $\Lambda(f)$ the largest zero of $f$.

# The polynomials $A_{r,s,N_r(n)}(z)$ have the same properties

**Theorem** Brenti 1989

*Let $f(z) \in \mathbb{R}[z]$ and $W(z)$ be its $f$-Eulerian polynomial. Suppose that $f(z)$ has only real zeros and that $f(z) = 0$ for all $z \in ([\lambda(f), -1] \cup [0, \Lambda(f)]) \cap \mathbb{Z}$. Then $W(z)$ has nonnegative coefficients and only real zeros.*

Note that $f_n(z) = \prod_{j=1}^{n} (z - (j-1)(s-r))^{\underline{s}}$ has only integral zeros, the smallest one being $\lambda(f_n) = 0$, and the largest one being $\Lambda(f_n) = (n-1)(s-r) + s - 1$.

**Corollary** ▸ $f$-Eulerian

*The polynomials $A_{r,s,N_r(n)}(z)$ have nonnegative coefficients and only real zeros.*

It is well-known that if a polynomial has nonnegative coefficients and only real zeros, then the zeros must be nonpositive. Furthermore, a polynomial with nonnegative coefficients and only real zeros is log-concave and unimodal.

Recall $N_r(n) = r(n-1)+1$ and set $N_r^c(n) = \lfloor (N_r(n)+1)/2 \rfloor$

## Polynomial bases

$$\mathcal{B}_1^{r,n} = \left\{ z^k \right\}_{k=1}^{N_r(n)}$$ Standard basis

$$\mathcal{B}_2^{r,n} = \left\{ z^k (1-z)^{N_r(n)-k} \right\}_{k=1}^{N_r(n)}$$ Unnormalized Berstein basis

$$\mathcal{B}_3^{r,n} = \left\{ z^k (1+z)^{N_r(n)+1-2k} \right\}_{k=1}^{N_r^c(n)} \cup \left\{ z^k \right\}_{k=N_r^c(n)+1}^{N_r(n)}$$ $\gamma$-basis

Recall that the normal ordering of $(Z^r D^s)^n$ is given by

$$\left(Z^r D^s\right)^n = \sum_{k=r}^{rn} S_{r,s}(n,k)\ Z^k D^{n(s-r)+k} .$$

# Using the unnormalized Berstein basis

## Theorem

$$A_{r,s,N_r(n)}(z) = \frac{1}{K_{r,s,n}} \sum_{k=1}^{N_r(n)} \big(n(s-r)+k+r-1\big)!\, S_{r,s}(n, k+r-1)\ z^k (1-z)^{N_r(n)-k}.$$

## Corollary ($r = s = 1$) Frobenius

$$A_n(z) = \sum_{k=1}^{n} k!S(n,k)\, z^k (1-z)^{n-k}.$$

## Corollary ($s = r+1, r \geq 1$) Sulanke's higher Narayana polynomials

$$A_{r,r+1,N_r(n)}(z) = \prod_{i=0}^{r} \frac{i!}{(n+i)!} \sum_{k=1}^{N_r(n)} \big(n+k+r-1\big)!\, S_{r,r+1}(n, k+r-1)\ z^k (1-z)^{N_r(n)-k}.$$

# By a change of basis

## Theorem

$$A_{r,s}(n,k) = \frac{1}{K_{r,s,n}} \sum_{j=1}^{k} (-1)^{k-j} \binom{N_r(n) - j}{k - j} \big(n(s-r) + j + r - 1\big)!\, S_{r,s}\big(n, j + r - 1\big).$$

## Corollary ($r = s = 1$) Eulerian numbers

$$A(n,k) = \sum_{j=1}^{k} (-1)^{k-j} \binom{n-j}{k-j} j!\, S_{1,1}(n,j) = \sum_{p=0}^{k} (-1)^p \binom{n+1}{p} (k-p)^n .$$

## Corollary ($r = 1, s = 2$) Narayana numbers

$$N(n,k) = \frac{1}{n!(n+1)!} \sum_{j=1}^{k} (-1)^{k-j} \binom{n-j}{k-j} (n+j)!\, S_{1,2}(n,j) = \frac{1}{n} \binom{n}{k-1} \binom{n}{k} .$$

# Some tables

$A_{1,4}(n,k)$

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 0 | 1 | | | | | |
| 2 | 0 | 1 | 1 | | | | |
| 3 | 0 | 7 | 19 | 7 | | | |
| 4 | 0 | 1 | 5 | 5 | 1 | | |
| 5 | 0 | 91 | 707 | 1311 | 707 | 91 | |
| 6 | 0 | 52 | 570 | 1655 | 1655 | 570 | 52 |

$A_{1,5}(n,k)$

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 0 | 1 | | | | | |
| 2 | 0 | 1 | 1 | | | | |
| 3 | 0 | 3 | 8 | 3 | | | |
| 4 | 0 | 13 | 63 | 63 | 13 | | |
| 5 | 0 | 663 | 4938 | 9038 | 4938 | 663 | |
| 6 | 0 | 17 | 177 | 502 | 502 | 177 | 17 |

$A_{1,6}(n,k)$

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 0 | 1 | | | | | |
| 2 | 0 | 1 | 1 | | | | |
| 3 | 0 | 11 | 29 | 11 | | | |
| 4 | 0 | 4 | 19 | 19 | 4 | | |
| 5 | 0 | 44 | 319 | 579 | 319 | 44 | |
| 6 | 0 | 13 | 131 | 366 | 366 | 131 | 13 |

$A_{1,7}(n,k)$

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 0 | 1 | | | | | |
| 2 | 0 | 1 | 1 | | | | |
| 3 | 0 | 13 | 34 | 13 | | | |
| 4 | 0 | 19 | 89 | 89 | 19 | | |
| 5 | 0 | 1235 | 8788 | 15858 | 8788 | 1235 | |
| 6 | 0 | 155 | 1527 | 4222 | 4222 | 1527 | 155 |

# Some tables

$A_{2,2}(n,k)$

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | |
| 1 | 0 | 1 | | | | | | | | | | |
| 2 | 0 | 1 | 4 | 1 | | | | | | | | |
| 3 | 0 | 1 | 20 | 48 | 20 | 1 | | | | | | |
| 4 | 0 | 1 | 72 | 603 | 1168 | 603 | 72 | 1 | | | | |
| 5 | 0 | 1 | 232 | 5158 | 27664 | 47290 | 27664 | 5158 | 232 | 1 | | |
| 6 | 0 | 1 | 716 | 37257 | 450048 | 1822014 | 2864328 | 1822014 | 450048 | 37257 | 716 | 1 |

$A_{2,3}(n,k)$

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | |
| 1 | 0 | 1 | | | | | | | | | | |
| 2 | 0 | 1 | 3 | 1 | | | | | | | | |
| 3 | 0 | 1 | 10 | 20 | 10 | 1 | | | | | | |
| 4 | 0 | 1 | 22 | 113 | 190 | 113 | 22 | 1 | | | | |
| 5 | 0 | 1 | 40 | 400 | 1456 | 2212 | 1456 | 400 | 40 | 1 | | |
| 6 | 0 | 1 | 65 | 1095 | 7095 | 20760 | 29484 | 20760 | 7095 | 1095 | 65 | 1 |

# Some tables

$A_{2,4}(n,k)$

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | |
| 1 | 0 | 1 | | | | | | | | | | |
| 2 | 0 | 3 | 8 | 3 | | | | | | | | |
| 3 | 0 | 1 | 8 | 15 | 8 | 1 | | | | | | |
| 4 | 0 | 1 | 16 | 70 | 112 | 70 | 16 | 1 | | | | |
| 5 | 0 | 3 | 80 | 630 | 2016 | 2940 | 2016 | 630 | 80 | 3 | | |
| 6 | 0 | 1 | 40 | 495 | 2640 | 6930 | 9504 | 6930 | 2640 | 495 | 40 | 1 |

$A_{3,4}(n,k)$

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | | | |
| 1 | 0 | 1 | | | | | | | | | | | | |
| 2 | 0 | 1 | 6 | 6 | 1 | | | | | | | | | |
| 3 | 0 | 1 | 22 | 113 | 190 | 113 | 22 | 1 | | | | | | |
| 4 | 0 | 1 | 53 | 710 | 3548 | 7700 | 7700 | 3548 | 710 | 53 | 1 | | | |
| 5 | 0 | 1 | 105 | 2856 | 30422 | 151389 | 385029 | 523200 | 385029 | 151389 | 30422 | 2856 | 105 | 1 |

# The $\gamma$-basis

Let $h(z) = a_1 z + \cdots + a_n z^n$ , $a_1, \ldots, a_n \in \mathbb{R}$.
Set $n^c = \left\lfloor \frac{n+1}{2} \right\rfloor$, and let $\mathcal{C}_{\mathcal{B}_3^n \mathcal{B}_1^n}$ be the transition matrix from the standard basis $\mathcal{B}_1^n = \{z^k\}_{k=1}^n$ to the basis $\mathcal{B}_3^n = \{z^k(1+z)^{n+1-2k}\}_{k=1}^{n^c} \cup \{z^k\}_{k=n^c+1}^n$

## Definition

The $\gamma$-*vector* of $h(z)$ is given by

$$\gamma(h) = (\gamma_1(h), \ldots, \gamma_n(h)) = (\mathcal{C}_{\mathcal{B}_3^n \mathcal{B}_1^n}(a_1, \ldots, a_n)^T)^T .$$

Sometimes we shall write $\gamma(n)$ instead of $\gamma(h)$, for the sake of brevity. The components $\gamma_k(h)$ of $\gamma(h)$ are called the $\gamma$-*numbers* of $h(z)$.

$$\gamma(5) = (a_1, -4a_1 + a_2, 2a_1 - 2a_2 + a_3, -a_2 + a_4, -a_1 + a_5),$$
$$\gamma(6) = (a_1, -5a_1 + a_2, 5a_1 - 3a_2 + a_3, -a_3 + a_4, -a_2 + a_5, -a_1 + a_6)$$

# A well-known Catalan matrix

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 5 & 5 & 3 & 1 & 0 & 0 & \cdots \\ 0 & 14 & 14 & 9 & 4 & 1 & 0 & \cdots \\ 0 & 42 & 42 & 28 & 14 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Define $\mathbf{C}[n]$ to be the submatrix formed by the first $n$ columns and rows of $\mathbf{C}$.

$$\mathbf{C}[7] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 5 & 5 & 3 & 1 & 0 & 0 \\ 0 & 14 & 14 & 9 & 4 & 1 & 0 \\ 0 & 42 & 42 & 28 & 15 & 5 & 1 \end{pmatrix}, \quad \mathbf{C}[8] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 5 & 3 & 1 & 0 & 0 & 0 \\ 0 & 14 & 14 & 9 & 4 & 1 & 0 & 0 \\ 0 & 42 & 42 & 28 & 14 & 5 & 1 & 0 \\ 0 & 132 & 132 & 90 & 48 & 20 & 6 & 1 \end{pmatrix}.$$

# $\mathbf{P}[n]$: Pascal triangle, $\mathbf{P}[n]_{k,j} = \binom{n-j}{k-j}$.

$$\mathcal{C}_{\mathcal{B}_3^7 \mathcal{B}_1^7} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 1 & 0 & 0 & 0 & 0 & 0 \\ 9 & -4 & 1 & 0 & 0 & 0 & 0 \\ -2 & 2 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{C}[7]\,\mathbf{P}[7]^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 1 & 0 & 0 & 0 & 0 & 0 \\ 9 & -4 & 1 & 0 & 0 & 0 & 0 \\ -2 & 2 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 2 & 1 & 0 \\ -1 & -4 & -5 & 0 & 5 & 4 & 1 \end{pmatrix},$$

$$\mathcal{C}_{\mathcal{B}_3^8 \mathcal{B}_1^8} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14 & -5 & 1 & 0 & 0 & 0 & 0 & 0 \\ -7 & 5 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{C}[8]\,\mathbf{P}[8]^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14 & -5 & 1 & 0 & 0 & 0 & 0 & 0 \\ -7 & 5 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & -2 & 2 & 3 & 1 & 0 \\ -1 & -5 & -9 & -5 & 5 & 9 & 5 & 1 \end{pmatrix}.$$

## Theorem

*The $\gamma$-vector of any symmetric polynomial $h(z) = a_1 z + \cdots + a_n z^n$ is given by*

$$\gamma(h) = (\gamma_1(h), \ldots, \gamma_{n^c}(h), 0, \ldots, 0) = (\mathbf{C}[n]\,\mathbf{P}[n]^{-1}(a_1, \ldots, a_n)^T)^T\,.$$

It is easy to obtain a handy formula for the entries of $\mathbf{C}[n]\,\mathbf{P}[n]^{-1}$. It is given by

$$(\mathbf{C}[n]\,\mathbf{P}[n]^{-1})_{k,j} = \begin{cases} \displaystyle\sum_{i=2}^{k}(-1)^{i-j}\frac{i-1}{k-1}\binom{2k-i-2}{k-2}\binom{n-j}{i-j} & , \quad 1 \le j \le k \le n \quad , \quad k \neq 1, \\ 0 & , \quad 1 \le k < j \le n, \\ \delta_{1,j} & , \quad 1 \le j \le n. \end{cases}$$

## Corollary

*The $\gamma$-numbers of any symmetric polynomial $h(z)$ are given by*

$$\gamma_1(h) = a_1 \quad \textit{and} \quad \gamma_k(h) = \sum_{j=1}^{k}\left[\sum_{i=2}^{k}(-1)^{i-j}\frac{i-1}{k-1}\binom{2k-i-2}{k-2}\binom{n-j}{i-j}\right]a_j \quad \textit{for} \quad 2 \le k \le n^c\,.$$

# An important Lemma

Note that if $h(z) = a_1 z + \cdots + a_n z^n$ is symmetric, then the last $n - n^c$ components of $\gamma(h)$ are zero. The other components of $\gamma(h)$ though may be positive, negative or zero, even if the coefficients of $h(z)$ (in its standard expansion) are nonnegative.

**Example** $h(z) = z + 3z^2 + 7z^3 + 3z^4 + z^5$

$$\gamma(h) = (1, -1, 3, 0, 0).$$

**Lemma**

*If $h(z) = a_1 z + \cdots + a_n z_n$ is a symmetric polynomial with nonnegative coefficients and only real zeros then the $\gamma$-vector of h is nonnegative.*

# The $\gamma$-vector of $A_{r,s,N_r(n)}(z)$

## Theorem

*The polynomial* $A_{(r,s),N_r(n)}(z)$ *belongs to the cone*

$$\mathrm{span}^+ \left\{ z^k (1+z)^{N_r(n)+1-2k} \right\}_{k=1}^{N_r^c(n)}$$

*and its* $\gamma$*-vector is given by*

$$\gamma(A_{r,s,N_r(n)}(z)) = (\mathbf{C}[N_r(n)]\, \mathbf{P}[N_r(n)]^{-1}\, \mathbf{P}[N_r(n)]^{-1} (b_1^{r,s,n}, \ldots, b_{N_r(n)}^{r,s,n})^T)^T ,$$

*where* $b_k^{r,s,n} = \frac{1}{K_{r,s,n}} \big(n(s-r)+k+r-1\big)!\, S_{r,s}(n, k+r-1)$ *for* $1 \le k \le N_r(n)$.

# Outline

# Thank you!