Localization formulas and polytope decompositions

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Abstract

Here is an interesting question: **what is the polytope decomposition that a given localization type formula will yield?** Brion [B] uses, for instance, the Lefschetz-Riemann-Roch formula in equivariant K-theory to obtain its famous polytope decomposition. In the same fashion, the Lawrence-Varchenko [L, V] (also called polar) polytope decomposition can be recovered by using the Atiyah-Bott and Berline-Vergne localization in equivariant cohomology (see e.g. [A1]). **Along with Leonor Godinho** [AG1], we use Witten localization in equivariant cohomology, as described by Paradan [P], to find new polytope decompositions. We work with toric manifolds to test and inspire our results.

Two localization formulas

Let (M, ω, μ) be a **toric manifold**. This means (M, ω) is a compact symplectic manifold on which a torus T of half the dimension of M acts in a hamiltonian fashion, where $\mu\colon M \to \mathfrak{t}^*$ is a moment map for the torus action, and where $\mathfrak{t}$ is the Lie algebra of T and $\mathfrak{t}^*$ its dual.

The Atiyah-Bott and Berline-Vergne formula. The fixed point set M^{T} of the action of T on M is equal to the critical set $\mathrm{Crit}(\mu)$ of μ. For a toric manifold, M^{T} is finite and corresponds to the vertices of the moment polytope $\mu(M)$. Let $\alpha \in H^*_{\mathrm{T}}(M)$. Then

$$\int_M \alpha = \sum_{F \in \mathrm{Crit}(\mu)} \frac{\alpha|_F}{\mathrm{eul}(N_F)} \quad ,$$

where $\alpha|_F$ refers to the restriction of α to the fixed point F, and $\mathrm{eul}(N_F)$ is the T-equivariant Euler class of the normal bundle to F. Animated by the theory of **geometric quantization** we choose $\alpha = e^{c_1(\mathbb{L})}\mathbf{Q}_q(M)$, where $e^{c_1(\mathbb{L})}$ is the Chern character of a T-equivariant prequantization line bundle $\mathbb{L} \to M$ and

$$\mathbf{Q}_q(M) = q\mathbf{Td}(M) + (1-q)\mathbf{Td}(-M)$$

is the equivariant Hirzebruch class of M. Here, q is an arbitrary complex parameter and $\mathbf{Td}(M)$ (respectively $\mathbf{Td}(-M)$) denotes the equivariant Todd class of M (and of M with opposite orientation). This choice of α yields a weighted version of the Lawrence-Varchenko polytope decomposition.

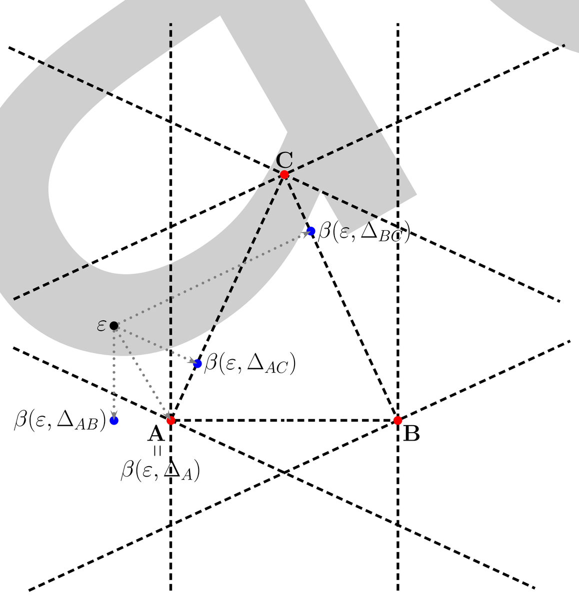

FIG. 1: Open set W for a triangle [ABC].

Now, the facets of the moment polytope $\mu(M)$ determine a set W of open regions in $\mathfrak{t}^*$ as illustrated in FIG. 1. Let $\varepsilon \in \mathfrak{t}^*$ and let $\mu_\varepsilon := \mu - \varepsilon$ be the perturbed moment map obtained from μ. The critical set of $\|\mu_\varepsilon\|^2$ is given by

$$\mathrm{Crit}(\|\mu_\varepsilon\|^2) = \bigsqcup_{F \text{ face of } \mu(M)} M^{T_F} \cap \mu^{-1}\big(\beta(\varepsilon, \Delta_F)\big) \quad ,$$

where Δ_F is the affine subspace of $\mathfrak{t}^*$ generated by F, the point $\beta(\varepsilon, \Delta_F)$ is the orthogonal projection of ε on Δ_F and M^{T_F} is the fixed point set of the subtorus T_F of T generated by $\exp(\Delta_F^{\perp})$. When $\varepsilon \in W$, the set $M^{T_F} \cap \mu^{-1}\big(\beta(\varepsilon, \Delta_F)\big)$ is a submanifold of M on which T/T_F acts locally freely. For toric manifolds, the set $M^{T_F} \cap \mu^{-1}\big(\beta(\varepsilon, \Delta_F)\big)$ is either a single point or empty.

The Paradan formula (following Witten). Let $\varepsilon \in W$ and let $\eta \in \Omega^{\infty}_T(M)$ be a closed form. Then, on $C^{-\infty}(\mathfrak{t})$ we have

$$\int_M \eta = \sum_{F \in \mathrm{Crit}(\|\mu_\varepsilon\|^2)} I^{\varepsilon}_F(\eta),$$

where $I^{\varepsilon}_F(\eta)$ is a certain generalized function supported on the Lie algebra $\mathfrak{t}_F$ of the subtorus T_F of T (cf. [AG1] for details). Using the equivariant form $\eta = e^{i\omega^\sharp}$, where $\omega^\sharp$ is the equivariant symplectic form on M, we motivate the Agapito-Godinho polytope decomposition formula.

Polytope decompositions

Let P be a convex d-polytope in $\mathbb{R}^d$. The tangent cone $\mathbf{C}_F$ to P at a face F is the minimal affine polyhedron (not necessarily pointed) that contains F and goes in the "direction" of P. We say that P is simple when its edges are generated by vectors in $\mathbb{Z}^d$ that form a basis of $\mathbb{R}^d$; but not necessarily of $\mathbb{Z}^d$. If the latter occurs, P is called Delzant. For instance, the moment polytope $\mu(M)$ of a toric manifold is Delzant. We denote by $\mathbf{1}_P$ and $\mathbf{1}_{\mathbf{C}_F}$ the characteristic functions of P and $\mathbf{C}_F$ respectively.

The Brion formula. For any convex d-polytope in $\mathbb{R}^d$, we have

$$\mathbf{1}_P = g \; + \sum_{\mathrm{v} \text{ vertex of } P} \mathbf{1}_{\mathbf{C}_{\mathrm{v}}} \quad ,$$

where g is a linear combination of characteristic functions of cones with lines (cf. FIG. 2 for an illustration).

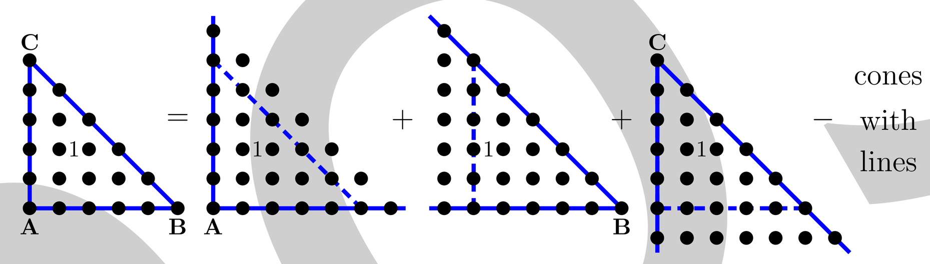

FIG. 2: Example for a lattice triangle [ABC].

We say that $\xi \in \mathbb{R}^d$ is a "polarizing" vector if it is not normal to any of the "walls" determining the open set W illustrated in FIG. 1. Such a vector is used to (polarize) change the "direction" of the tangent cones $\mathbf{C}_F$ in a systematic way and thus we get the so called polarized tangent cones $\mathbf{C}^{\sharp}_F$ (cf. [AG1] for details and FIG. 3,4 for an illustration). Motivated by our work with toric manifolds and the use of the Atiyah-Bott-Berline-Vergne localization formula as explained before, we assign complex numbers q's and $(1-q)$'s to the facets of P and its tangent cones so that we get the following weighted version of the Lawrence-Varchenko polytope decomposition (which avoids removing facets as occurs with the ordinary - $q = 1$- Lawrence-Varchenko formula) The $\mathbf{1}^w_P$ and $\mathbf{1}^w_{\mathbf{C}^\sharp_F}$ stand for weighted characteristic functions as illustrated in FIG. 3,4.

The (Weighted) Lawrence-Varchenko formula. Let $\xi \in \mathbb{R}^d$ be a polarizing vector. For any convex simple d-polytope P in $\mathbb{R}^d$, we have

$$\mathbf{1}^w_P = \sum_{\mathrm{v} \text{ vertex of } P} (-1)^{m_{\mathrm{v}}} \mathbf{1}^w_{\mathbf{C}^\sharp_{\mathrm{v}}}.$$

where m_{v} is the number of facets of $\mathbf{C}_{\mathrm{v}}$ that change direction with the polarization.

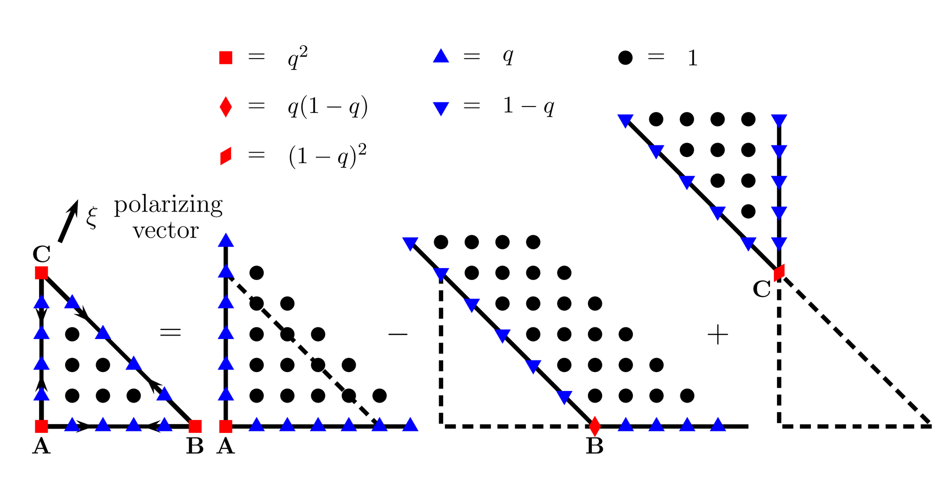

FIG. 3: Example for a lattice triangle [ABC] using ξ.

We can do better, assign arbitrary q_i's and $(1-q_i)$'s to the facets of P and its tangent cones (not just the same q's and $(1-q)$'s) and get a more general weighted L-V formula. Finally, using the Witten localization in equivariant cohomology as explained before, we get

The Agapito-Godinho formula. Let $\varepsilon \in W$. For any convex simple d-polytope P in $\mathbb{R}^d$, we have

$$\mathbf{1}^w_P = \sum_{F \text{ face of } P} (-1)^{m_F} \varphi(\varepsilon, \Delta_F) \mathbf{1}^w_{\mathbf{C}^\sharp_F}.$$

where $\varphi(\varepsilon, \Delta_F) = 1, 0$ whether $\beta(\varepsilon, \Delta_F)$ is in P or not and m_F is the number of facets of $\mathbf{C}_F$ that "flip" with polarization.

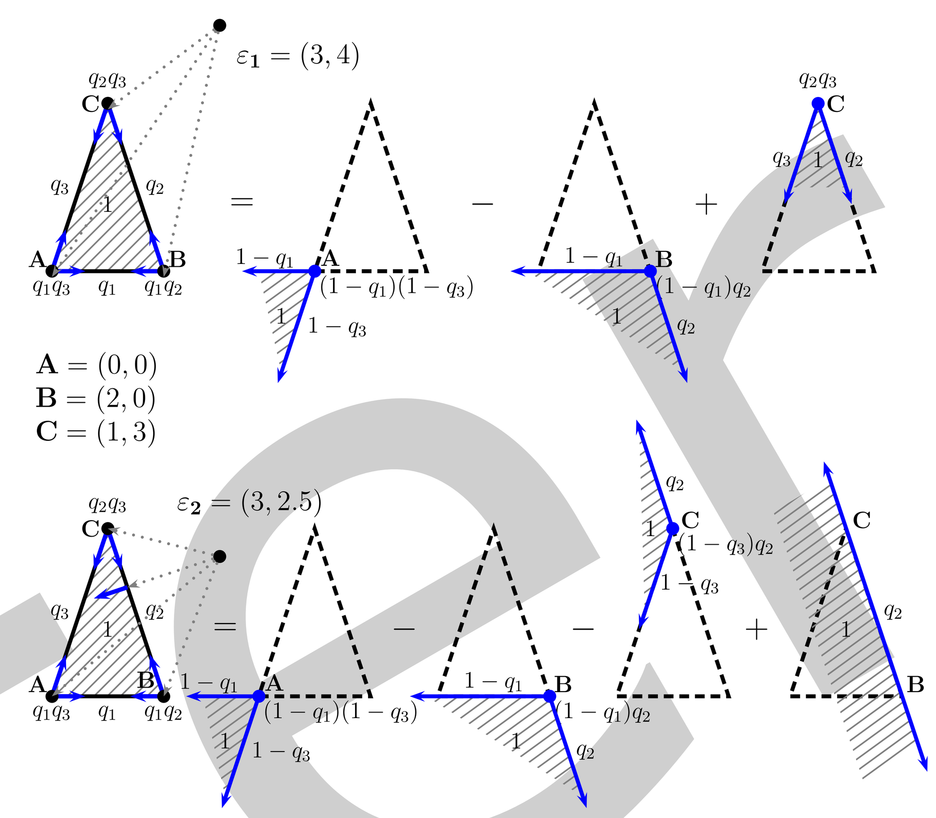

FIG. 4: Example for a triangle [ABC] using ε_1 and ε_2.

Here, as opposed to the L-V formula, a **different polarizing vector** $(\beta(\varepsilon, \Delta_F) - \varepsilon)$ is assigned to each face of the polytope and the facets of the corresponding tangent cones are "flipped" accordingly (see FIG. 4 for an example). Using **regular triangulations**, we can extend all these decompositions to non-simple polytopes (cf. [AG2] for details). The L-V and A-G formulas are independent of polarization (although the polarized tangent cones may change, of course).

Final remarks

As an application of our polytope decompositions, we can obtain Euler Maclaurin formulas on simple lattice polytopes that generalize previous results (cf. [AG1] for details).
In [AG2], we comment on the relation between Brion's formula and the previously mentioned polytope decompositions, using generating functions. We can also ask: **given a polytope decomposition, is there a localization formula that applied to a toric manifold will imply that polytope decomposition?** For example, Harada and Karshon [HK] have developed new localization formulas for the Duistermaat- Heckman measure that yield the Brianchon-Gram decomposition for **all** toric manifolds. This well known classical decomposition (see e.g. [AG2] and references therein) states that

$$\mathbf{1}_P = \sum_{F \text{ face of } P} (-1)^{\dim F} \mathbf{1}_{\mathbf{C}_F}.$$

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