Equational descriptions of recognizable languages and applications

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CAUL - July 11-12, 2013

Topics

- Regular languages

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- Semigroup equations

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- Varieties

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- Lattices of languages closed under quotients
- Polynomial closure of a lattice of languages closed under quotients

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Star:

$$L \longmapsto L^* = \{u_1 \cdots u_n \mid u_1, \ldots, u_n \in L,\ n \in \mathbb{N}_0\}$$

the submonoid of A^* generated by L

Operations on Languages

Quotients ($a \in A$):

$$L \longmapsto a^{-1}L = \{u \mid au \in L\}$$
$$L \longmapsto La^{-1} = \{u \mid ua \in L\}$$

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Finite automaton

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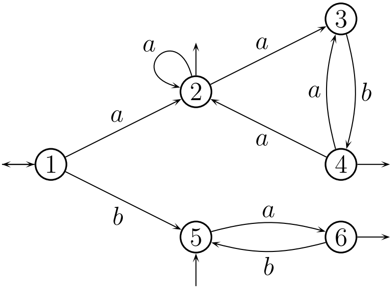
Automaton $\mathcal{A}$:



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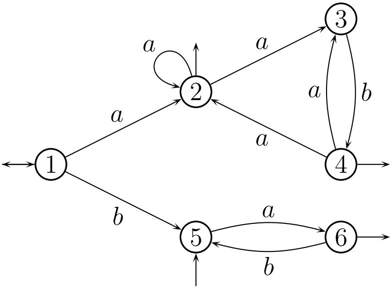

Words recognized by $\mathcal{A}$:

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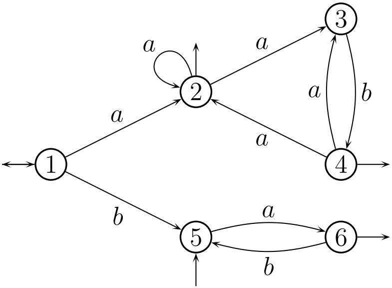

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Alphabet $A = \{0, 1\}$

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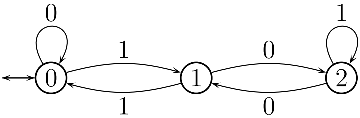

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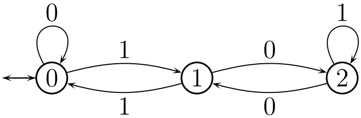

Words recognized by $\mathcal{A}$ are precisely the words that represent the multiples of 3 on base 2, for instance 0, 00, 11, 0011, 1001, 1000110100.

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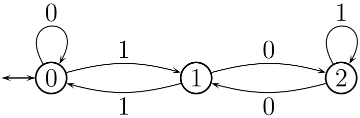

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$\overbrace{\mathrm{SF}(A^*)}^{\text{star-free}}$ is the smallest set of languages over A that has the emptyset and the languages $\{a\}$, with $a \in A$, and is closed under the boolean operations, and product.

Is there an algorithm to test whether a language belongs to $\mathrm{SF}(A^*)$?

Transition monoid

Transition monoid

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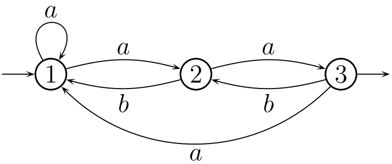
Automaton $\mathcal{A}$:



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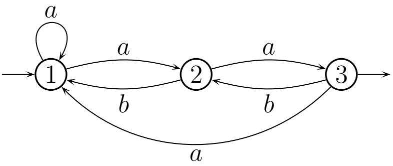

The transitions of $\mathcal{A}$ can be defined by the following two binary relations:

$$a \longmapsto \overline{a} = \{(1,1),\ (1,2),\ (2,3),\ (3,1)\}$$
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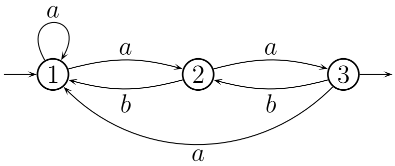

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For words, for instance:

$$babba \longmapsto \overline{babba} = \big\{(3,1),\ (3,2)\big\}$$

Transition monoid

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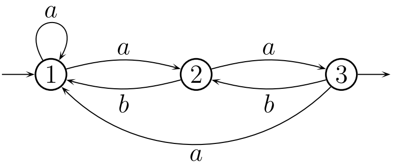

The transitions of $\mathcal{A}$ can be defined by the following two binary relations:

$$a \longmapsto \bar{a} = \{(1,1), (1,2), (2,3), (3,1)\}$$
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For words, for instance:

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For $L \subseteq A^*$, TFAE:

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M recognizes $L \iff M(L)$ is homomorphic image of a submonoid of M.

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The answer is Yes.

Theorem (Schützenberger)

For $L \subseteq A^*$, TFAE:

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$[\Sigma]$ – class of all monoids that satisfy all identities of Σ.

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Then $\mathcal{V}$ is a variety of languages.

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How to caracterize the **M**-varieties by identities?

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- $d(u,v) = 0$ if and only if $u = v$.
- $d(u,v) = d(v,u)$.
- $d(u,w) \leq \max\{d(u,v), d(v,w)\}$.
- $d(uu', vv') \leq \max\{d(u,v), d(u',v')\}$.

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The multiplication on A^* induces, in a natural way, an associative multiplication on $\widehat{A^*}$, which is continuous.

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$((u\hat{\varphi})^{n!})_n$ converges in M (with the discrete topology).
Since M is finite, there exists k s.t. $(u\hat{\varphi})^k = e$, an idempotent.
It follows that if $n \geq k$, then $(u\hat{\varphi})^{n!} = e$, and so $\lim (u\hat{\varphi})^{n!} = e$, the idempotent power of $u\hat{\varphi}$.

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Examples:

M satisfies $xy = yx$ if and only if $\forall s, t \in M,\ st = ts$.

A finite *semigroup* S satisfies $x^\omega y x^\omega = x^\omega$ if and only if $\forall s \in S, e \in E(S),\ ese = e$.

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Examples:

$\mathbf{J_1} = [\![x = x^2,\, xy = yx]\!]$ – finite idempotent and commutative monoids.

$\mathbf{A} = [\![x^\omega = x^{\omega+1}]\!]$ – finite aperiodic monoids.

$\mathbf{LI} = [\![x^\omega y x^\omega = x^\omega]\!]$ – finite locally trivial *semigroups*.

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Variety of languages $\mathcal{V}$:

$$\begin{array}{ccc} A & \longmapsto & (A^*)\mathcal{V} \\ \text{alphabet} & & \text{subset of } \mathrm{Rat}(A^*) \end{array}$$

such that

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How to characterize these classes algebraically?

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Ordered monoid $(M, \leq)$: monoid M equipped with a partial order $\leq$ compatible with the product:

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How to characterize algebraically the classes $\mathcal{V}$ satisfying the following?

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Profinite space or Stone space: topological space that is projective limit of finite topological spaces endowed with the discrete topology.

Proposition

Let X be a topological space. TFAE:

1. *X is profinite.*
2. *X is Hausdorff, compact and totally disconnected.*
3. *X is Hausdorff, compact and admits a base of clopen sets.*

Topological caracterization of the regularity and Stone duality

Stone duality: boolean algebras $\leftrightarrow$ Stone spaces

Boolean algebra $\mathbf{B} \ \mapsto \ \big\{$morphisms from $\mathbf{B}$ to $\{0,1\}\big\}$ with the topology induced by the product topology in $\{0,1\}^{\mathbf{B}}$.

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Boolean algebra $\mathbf{B} \mapsto \{\text{morphisms from } \mathbf{B} \text{ to } \{0,1\}\}$ with the topology induced by the product topology in $\{0,1\}^{\mathbf{B}}$.

Stone space $X \mapsto \{\text{clopen sets of } X\}$.

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Theorem (Pippenger)

The set of recognizable languages of A^* is dual to the Stone space $\widehat{A^*}$.

Satisfaction of an equation by a language

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Let $L \subseteq A^*$.
L is regular if and only if $\overline{L}$ is open.

Proposition (Gehrke, Grigorieff, Pin)

Let $L \subseteq A^*$ regular and $u \in \widehat{A^*}$. TFAE:

1. $u \in \overline{L}$.
2. $\hat{\varphi}(u) \in \varphi(L)$, *for every morphism* $\varphi \colon A^* \to M$, *where M is a finite monoid.*
3. $\hat{\eta}(u) \in \eta(L)$, *where* $\eta \colon A^* \to M(L)$ *is the syntactic morphism of L.*

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$L \subseteq A^*$ regular, $u, v \in \widehat{A^*}$.

L satisfies $u \leq v$ if $\hat{\eta}(u) \leq \hat{\eta}(v)$, where $\eta\colon A^* \to M(L)$ is the syntactic morphism of L.

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Notice that, by the previous proposition,

$$\begin{aligned} \hat{\eta}(u) \leq \hat{\eta}(v) \quad &\Longleftrightarrow \quad \forall s, t \in M(L) \left(s\hat{\eta}(v)t \in \eta(L) \Rightarrow s\hat{\eta}(u)t \in \eta(L)\right) \\ &\Longleftrightarrow \quad \forall x, y \in A^* \left(\hat{\eta}(xvy) \in \eta(L) \Rightarrow \hat{\eta}(xuy) \in \eta(L)\right) \end{aligned}$$

Lattice of language closed under quotients

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How to characterize algebraically the classes $\mathcal{V}$ satisfying the following?

1. $(A^*)\mathcal{V}$ is closed under finite union and finite intersection.
2. $(A^*)\mathcal{V}$ is closed under quotients: $a^{-1}L$, $La^{-1} \in (A^*)\mathcal{V}$, for any $L \in (A^*)\mathcal{V}$.

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Lattice of languages of A^*: set of languages of A^* closed under finite union and finite intersection.

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Theorem (Gehrke, Grigorieff, Pin)

A set $\mathcal{L}$ of languages of A^* is a lattice of languages closed under quotients if and only if, for some set Σ of equations of the form $u \leq v$, with $u, v \in \widehat{A^*}$, $\mathcal{L}$ is the set of the languages of A^* that satisfy all equations of Σ.

An application: Pol($\mathcal{L}$)

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Joint work with J.-E. Pin.

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Let $\mathcal{L}$ be a set of languages of A^*.

Pol($\mathcal{L}$): the set of languages that are finite union of $L_0 a_1 L_1 \cdots a_n L_n$, with $n \in \mathbb{N}_0$, $L_i \in \mathcal{L}$, $a_j \in A$.

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$\Sigma(\mathcal{L})$: the set of equations of the form $x^\omega y x^\omega \leq x^\omega$, where $x, y \in \widehat{A^*}$ are such that the equations $x = x^2$ and $y \leq x$ are satisfied by $\mathcal{L}$.

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Theorem (BP)

If $\mathcal{L}$ is a lattice of regular languages of A^* closed under quotients, then Pol($\mathcal{L}$) is defined by $\Sigma(\mathcal{L})$.

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How to prove it?

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Define

$$E_L = \left\{(x, y) \in \widehat{A^*} \times \widehat{A^*} \mid L \text{ satisfies } x = x^2 \text{ and } y \leq x\right\}$$

$$F_L = \left\{(x, y) \in \widehat{A^*} \times \widehat{A^*} \mid L \text{ satisfies } x^\omega y x^\omega \leq x^\omega\right\}$$

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E_L and F_L are clopen in $\widehat{A^*} \times \widehat{A^*}$.

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Let $\mathcal{L}$ be a set of regular languages of A^* and K be a regular language of A^*. TFAE:

1. *K satisfies $\Sigma(\mathcal{L})$.*

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1. K *satisfies* $\Sigma(\mathcal{L})$.
2. *The set* $\{F_K\} \cup \{E_L^c \mid L \in \mathcal{L}\}$ *is an open cover of* $\widehat{A^*} \times \widehat{A^*}$.

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A – an alphabet

factorization forest: map

$$A^2A^* \ni x \xrightarrow{F} F(x) = (x_1, x_2, \ldots, x_n),$$

where $n \geq 2$ and $x_i \in A^+$

(a recursive process to factorize words up to products of letters).

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$F(x) = (x_1, x_2, \ldots, x_n) \mapsto$ labeled tree $t(x)$

Height function of F: $h\colon A^* \to \mathbb{N}_0$

$$h(x) = \begin{cases} 0 & \text{if } x \in A \cup \{1\} \\ 1 + \max\{h(x_i) \colon 1 \leq i \leq n\} & \text{if } F(x) = (x_1, x_2, \ldots, x_n) \end{cases}$$

Height of F: $\sup\{h(x) \colon x \in A^*\}$.

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A factorization forest F is said to be Ramseyan modulo φ if
either $F(x) = (x_1, x_2)$
or there exists an idempotent e of M such that $F(x) = (x_1, x_2, \ldots, x_n)$ and $\varphi(x_1) = \varphi(x_2) = \cdots = \varphi(x_n) = e$.

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Theorem (Simon, ...)

There exists a factorization forest F of height $\leq 3|M| - 1$ which is Ramseyan modulo φ.