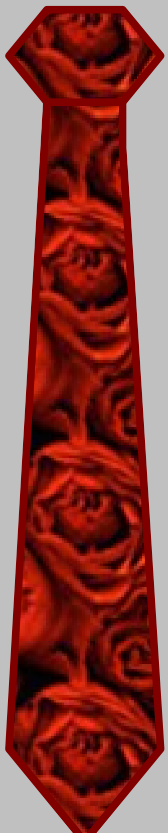
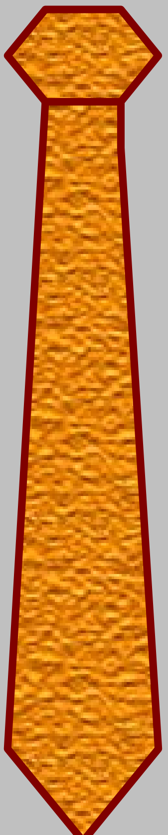
An essay on mathematical fashion: **The theory of tie knots**

Nelson Rei Bernardino





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The Fink-Mao theorem

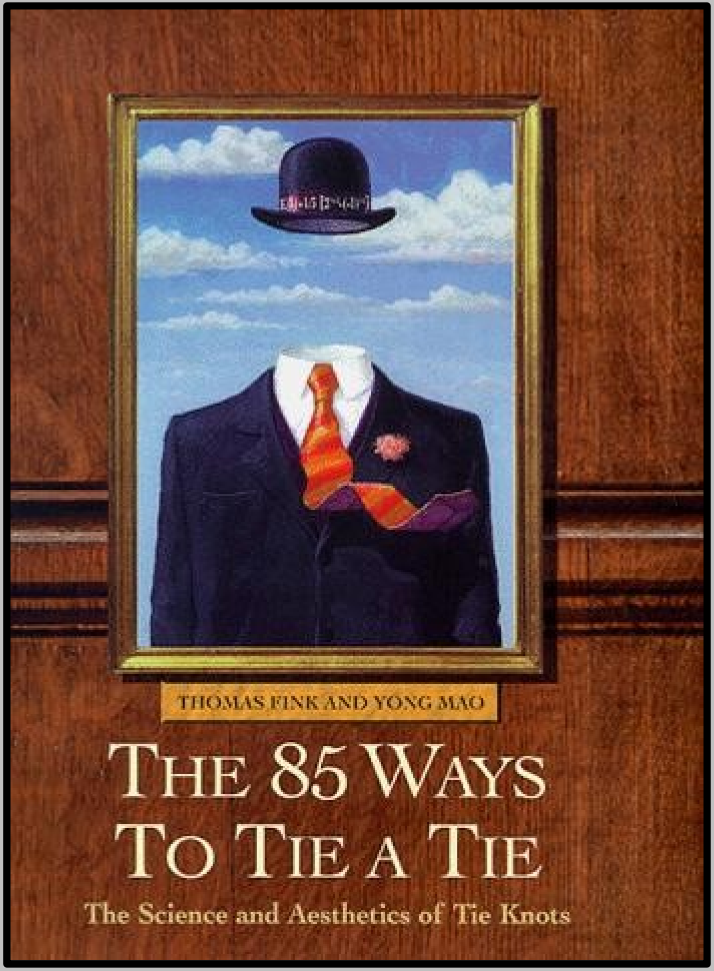
- Theorem 1 (Fink-Mao):

  There are 85 ways to tie a tie.

- Lemma:

  Ties knots are homeomorphic to directed random-walks on a 2-dimensional triangular lattice.

Based on the book:



A pioneer work in mathematical fashion:

Charles E. Siem, *Stress Analysis of a Strapless Evening Gown*, 1956

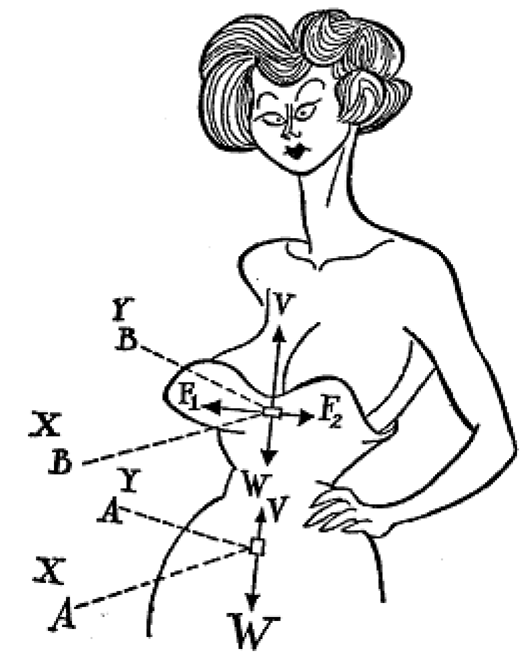


Figure 1.
Forces Acting On Cloth Element.

Why study tie knots?

"the lady beside him fainted, he spoilt a good collar, tie, and shirt, and the engagement is now off"

The 'Major' in *Clothes and the Man* (1900)*

*actually referring to the dangers of failing to correctly insert a tie pin

History of ties

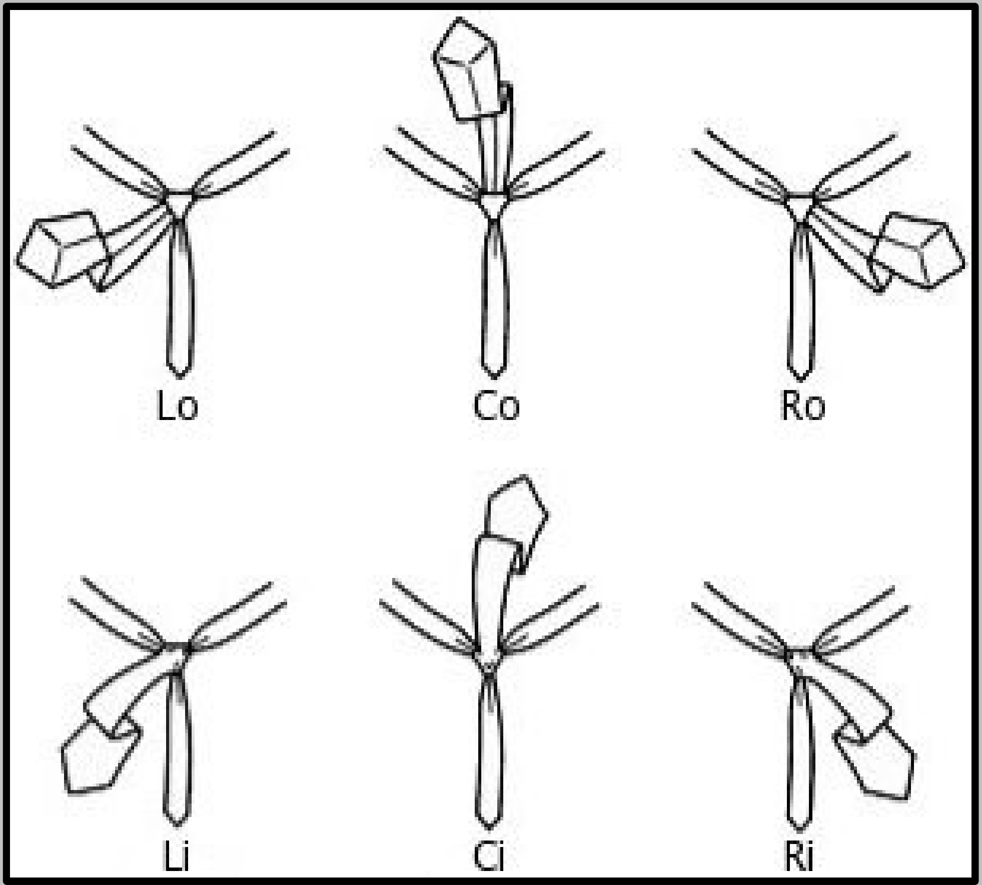
- China's first emperor terracotta army (~210 BC).
- Roman soldiers (~110 AD).
- Croatian mercenaries (~1640).
- French and English courts.
- Modern tie (~1850).

History of knots

- Four-in-hand (~1850).
- Windsor (~1930).
  - Named after the Duke of Windsor (King Edward VIII) which never used it!
- The Half-Windsor (~1930).
  - Actually ¾ of a Windsor!
- Pratt (1989).

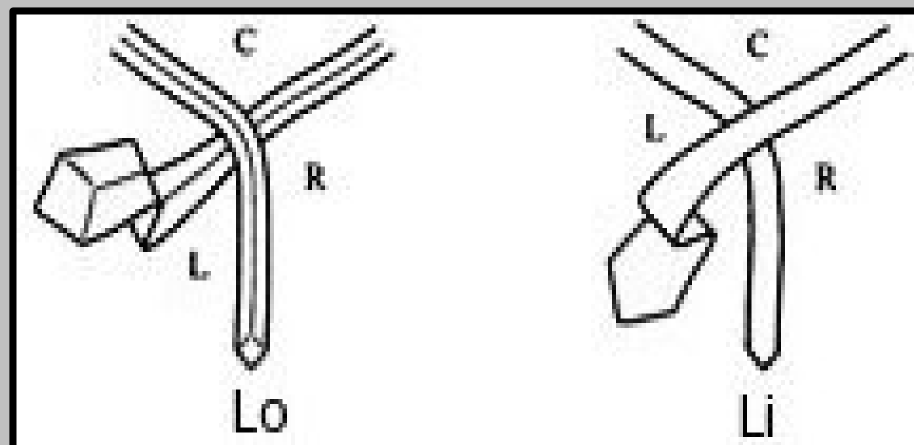
Definition of tie knot I

- Definition: A tie knot is composed of one of the following moves:

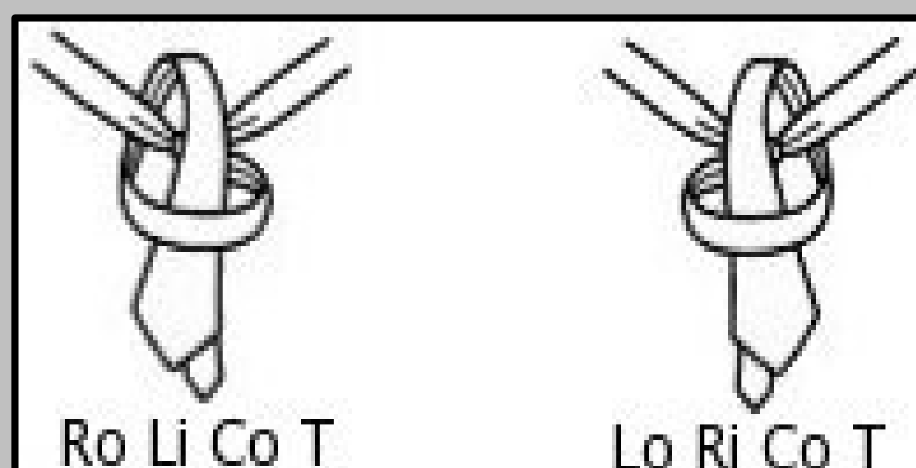


Definition of tie knot II

- ... starting with:



- ... and ending with:



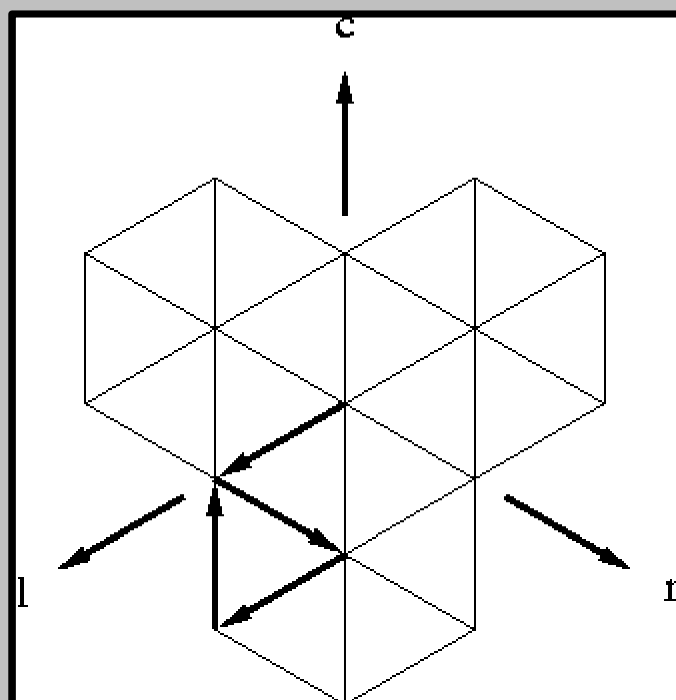
Definition of tie knot III

- No two consecutive moves to the same area are allowed:
  - e.g. *Ro Ri*
- Nor two consequetive moves with the same direction:
  - e.g. *Ro Lo*
- Example: The four-in-hand is described by

Li Ro Li Co T

Mapping to random walks

- Since the direction must alternate, we can describe a knot by a random walk on a triangular lattice:



Counting knots I

- Define *size* of the knot as the number of steps in the walk.

- How many knots of *size s* are there?

  => How many walks of *size s* are there?

Counting knots II

- Walks begin with an *L* move and end with the sequences *RLC* or *LRC*. At step *n*:

$$F_r(n) + F_c(n) + F_l(n) = 2^{n-1}$$

$$F_l(n+2) = F_r(n+1) + F_c(n+1)$$

$$F_l(n+2) = F_r(n) + F_c(n) + 2F_l(n)$$

Counting knots III

- Thus we get:

$$F_{l,r}(n+2) = F_{l,r}(n) + 2^{n-1}$$

- with initial conditions

$$F_l(1)=1,\ F_l(2)=0,\ F_r(1)=0,\ F_r(2)=1$$

- Solved by

$$F_l(n) = 2/3\ (2^{n-2} + (-1)^{n-1})$$

$$F_r(n) = 1/3\ (2^{n-1} - (-1)^{n-1})$$

Counting knots IV

- The number of knots of size s is

$$K(s) = F_l(s-2) + F_r(s-2) = 1/3\ (2^{s-2} - (-1)^{s-2})$$

where $K(1) = K(2) = 0$.

- The total number of knots is

$$\sum_{s=1}^{9} K(s) = 0+0+1+1+3+5+11+21+43 = 85$$

QED

(170 knots with 10 moves)

Classifying knots: shape

- The shape of a knot (broad or narrow) depends on the number of centre moves:

shape = number of centre moves

- Knots with a low ratio

shape/size < 1/6

are too cylindrical and are rejected.

Classifying knots: symmetry

- Symmetry of a knot is defined as the absolute diference between right and left moves:

$$symmetry = |right - left|$$

- The most symmetric knot minimizes the *symmetry*!!!

Classifying knots: balance

- Knots with a good mix of moves are tightly bound and keep their shape. A well balanced knot mixes clockwise with anti-clockwise moves:

$$\textit{balance} = |\textit{clockwise} - \textit{anti-clockwise}|$$

- The better balanced knot minimizes the *balance*!!!

Classifying knots: self-release

- By pulling the passive end through the knot the remaining knot's knoted status depends on the terminal sequence:
  - *Ro Li Co T* is not knoted
  - *Lo Ri Co T* is knoted

- Knots ending with *Ro Li Co T* will undo by pulling the ends of the tie.

Class and aesthetic knots

- Knots are grouped in a class by size and shape, *e.g.* the four-in-hand is of class (4,1).
- After rejecting the knots with a low *size/shape* ratio we have 13 remaining classes.
- The most symmetric and balanced knot in a class is named an *aesthetic knot*.

Table of aesthetic knots

Size	Shape	Sequence	Name	Self-releasing
3	1	*Lo Ri Co T*	Oriental	No
4	**1**	***Li Ro Li Co T***	**Four-in-hand**	**Yes**
5	1	*Lo Ri Lo Ri Co T*	Kelvin	No
5	2	*Lo Ci Ro Li Co T*	Nicky (self-releasing Pratt)	Yes
5	**2**	***Lo Ci Lo Ri Co T***	**Pratt***	**No**
6	1	*Li Ro Li Ro Li Co T*	Victoria	Yes
6	**2**	***Li Ro Ci Lo Ri Co T***	**Half-Windsor**	**No**
6	2	*Li Ro Ci Ro Li Co T*	Half-Windsor variant*	Yes
7	2	*Lo Ri Lo Ci Ro Li Co T*	St Andrew	Yes
7	3	*Lo Ci Ro Ci Lo Ri Co T*	Plattsburgh	No
8	2	*Li Ro Li Co Ri Lo Ri Co T*	Cavendish	No
8	**3**	***Li Co Ri Lo Ci Ro Li Co T***	**Windsor**	**Yes**
9	2	*Lo Ri Lo Ri Co Li Ro Li Co T*	Grantchester	Yes
9	3	*Lo Ri Co Li Ro Ci Lo Ri Co T*	Hanover	No
9	4	*Lo Ci Ro Ci Lo Ci Ro Li Co T*	Balthus	Yes

In **bold** the four commonly used knots.
* Not actually an aesthetic knot as defined.

Conclusions

- There are 85 tie knots...
- ... of which 13 are the most aesthetic.
- Ties knots are random walks on a triangular lattice.
- Mathematical fashion is a promising area of research.
- Nothing is safe from mathematicians/physicists.

References and acknowledgements

- Thomas Fink & Yong Mao, *The 85 ways to tie a tie*, Fourth Estate, London, 1999
- T.M. Fink & Y. Mao, *Designing tie knots by random walks*, Nature, **398** (1999) 31
- T.M. Fink & Y. Mao, *Tie knots, random walks and topology*, Physica A, **276** (2000) 109

I'm very grateful to Dr Thomas Fink for authorisation to use much of the material presented.

The power of a tie (in 1951)

<u>Thank you!</u>