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Topological sensitivity analysis for elliptic differential operators of order $2m$

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Abstract

The topological derivative is defined as the first term of the asymptotic expansion of a given shape functional with respect to a small parameter that measures the size of a singular domain perturbation. It has applications in many different fields such as shape and topology optimization, inverse problems, image processing and mechanical modeling including synthesis and/or optimal design of microstructures, fracture mechanics sensitivity analysis and damage evolution modeling. The topological derivative has been fully developed for a wide range of second order differential operators. In this paper we deal with the topological asymptotic expansion of a class of shape functionals associated with elliptic differential operators of order $2m$, $m \geqslant 1$. The general structure of the polarization tensor is derived and the concept of degenerate polarization tensor is introduced. We provide full mathematical justifications for the derived formulas, including precise estimates of remainders.

Keywords: Topological sensitivity analysis; Topological derivative; Elliptic operator; Polarization tensor; Degenerate polarization tensor

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1. Introduction

The topological derivative measures the sensitivity of a given shape functional with respect to infinitesimal singular domain perturbations, such as the insertion of holes, inclusions, source-terms or even cracks [12,14,16,25]. Specifically, if the shape functional is denoted by $\mathcal{J}(\Omega)$ and the domain obtained after a perturbation of size ε localized around a point z is denoted by Ω_ε, it is defined by

$$D_T\mathcal{J}(\Omega)=\lim_{\varepsilon\to 0}\frac{\mathcal{J}(\Omega_\varepsilon)-\mathcal{J}(\Omega)}{\varphi(\varepsilon)},$$

for some appropriate scaling function $\varphi(\varepsilon)$. This notion has proved to be a powerful tool for the treatment of different problems such as topology optimization, inverse analysis and image processing (see e.g. [7,11,17–19]), and has become a subject of intensive research. There are also some applications in the multi-scale constitutive modeling context [8], fracture mechanics sensitivity analysis [28] and damage evolution modeling [2]. All these problems share in common to be governed by partial differential equations (PDE's), and the type of PDE obviously impacts drastically on the mathematical analysis involved. Concerning the theoretical developments of the topological asymptotic analysis, the reader may refer to the papers [4,13,22], among others.

According to the literature, the topological derivative concept has been fully developed for a wide range of second order equations, while a forth order equation is addressed in [6]. In this paper, the topological asymptotic expansion of a class of shape functionals associated with an elliptic differential operator of order $2m$, with $m \geqslant 1$, is derived. The topologically perturbed domain is obtained when an arbitrarily shaped hole is introduced inside the initial domain. Then, the resulting void is filled with a phase whose material properties present a contrast with the original ones. The main ingredient arising in the asymptotic formula is the so-called Pólya–Szegő polarization tensor [24] (see also [3]), of which we derive the general structure for the operators under consideration. We also introduce the concept of degenerate polarization tensor, in the sense that it is independent of the shape of the topological perturbation and, at the same time, its entries do not remain bounded when the contrast on the material properties goes to zero. In this particular case it is remarkable that the polarization tensor can be easily obtained in its closed form. We show that this phenomenon of degeneracy occurs when the operator satisfies a particular algebraic property which is easy to check, a typical example being the bi-Laplacian. Let us mention in this respect that the degeneracy of the bi-Laplacian occurs in the context of dislocation modeling [26,27]. It basically means that dislocated regions can be created or annihilated (in the sense of nucleation) with an energetical cost independent of their shapes. The bi-Laplacian also appears in some plate models and thus our results have implications in the optimal design of such thin structures, considering the compliance as objective function, for instance. Specific examples of shape functionals and degenerated operators will be given in Section 4.2.

The paper is organized as follows. Some notation and preliminary statements are introduced in Section 2. Basic properties of the boundary value problems under consideration are collected in Section 3. The topological asymptotic expansion for a class of shape functionals is derived in its general form in Section 4, and the concept of degenerate polarization tensor is introduced in Section 5. Some particular cases of differential operators, including degenerate cases, are presented in Section 6 together with a set of examples with analytical solution. The appropriate estimates of remainders are provided in Section 7, with full mathematical justifications. The extension of the obtained results to elliptic systems is discussed in Section 8. Some concluding remarks and perspectives are given in Section 9. Appendix A is devoted to the proof of a classical coercivity result, however the uniformity with respect to the parameter ε is here highlighted. In Appendix B, the notion of collectively compact operators used throughout the analysis is recalled. Finally, the weighted and quotient Sobolev spaces needed for the formulation of appropriate exterior problems, appearing in particular in the construction of the polarization tensor, are described in Appendix C.

2. Preliminaries and notation

Let Ω be an open and bounded subset of $\mathbb{R}^n$ ($n \in \mathbb{N}^*$) and $\hat{x} \in \Omega$ be fixed. Given an open, bounded and smooth subset ω of $\mathbb{R}^n$ containing the origin, we define for every $\varepsilon > 0$ the set

$$\omega_\varepsilon(\hat{x}) = \hat{x} + \varepsilon\omega.$$

Let $\rho_0 \in L^\infty(\Omega)$ be a given function which takes a constant value $\hat{\rho}_0$ in a neighborhood of $\hat{x}$ and such that $\operatorname{essinf}_\Omega \rho_0 > 0$. Moreover, given a constant $\hat{\rho}_1 > 0$, we set for all $\varepsilon \geqslant 0$

$$\rho_\varepsilon(x) = \begin{cases} \rho_0(x) & \text{if } x \in \Omega \setminus \omega_\varepsilon, \\ \hat{\rho}_1 & \text{if } x \in \Omega \cap \omega_\varepsilon. \end{cases} \tag{2.1}$$

For all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and any $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, we denote by

$$|\alpha| = \sum_{i=1}^n \alpha_i, \qquad |\xi| = \left(\sum_{i=1}^n \xi_i^2\right)^{1/2}, \qquad \xi^\alpha = \prod_{i=1}^n \xi_i^{\alpha_i}$$

the length of α, the norm of ξ and the α-power of ξ, respectively. To avoid any ambiguity, all multi-indices will be denoted by the letters α, β or γ. The derivative of order α of a distribution u is defined by

$$D^\alpha u = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} u.$$

Let $m \in \mathbb{N}^*$. We consider a family of real constant coefficients $(a_{\alpha\beta})_{|\alpha|=|\beta|=m}$ satisfying the following properties.

- Symmetry: it holds for every α, β

$$a_{\alpha\beta} = a_{\beta\alpha}. \tag{2.2}$$

- Positivity: for any family of real numbers $(y_\alpha)_{|\alpha|=m}$ it holds

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} y_\alpha y_\beta \geqslant 0. \tag{2.3}$$

- Uniform ellipticity: there exists $\kappa > 0$ such that

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\xi^{\alpha+\beta} \geqslant \kappa|\xi|^{2m} \quad \forall\xi \in \mathbb{R}^n. \tag{2.4}$$

We define the homogeneous operator $A_\varepsilon : H_0^m(\Omega) \to H^{-m}(\Omega)$ by

$$\langle A_\varepsilon u, v\rangle_{H^{-m}(\Omega),H_0^m(\Omega)} = \sum_{|\alpha|=|\beta|=m} \int_\Omega \rho_\varepsilon a_{\alpha\beta} D^\alpha u D^\beta v\,dx \quad \forall u, v \in H_0^m(\Omega). \tag{2.5}$$

We recall that the space $H_0^m(\Omega)$ is defined as the closure in $H^m(\Omega)$ of the set of functions of class $\mathcal{C}^\infty$ in Ω with compact support, and that it is also the set of functions of $H^m(\Omega)$ with vanishing trace on $\partial\Omega$ up to the order $m-1$, see e.g. [1]. We will later argue that A_ε is invertible (see Corollary 3.2).

We further consider coefficients $(b_{\alpha\beta,\varepsilon})$ defined for all $\varepsilon \geqslant 0$ and α, β such that $|\alpha| \leqslant m$ and $|\beta| \leqslant m-1$. We assume that, for ε small enough,

$$b_{\alpha\beta,\varepsilon} - b_{\alpha\beta,0} = q_{\alpha\beta}\chi_{\omega_\varepsilon}$$

for some coefficients $q_{\alpha\beta}$, and with $\chi_{\omega_\varepsilon}$ the characteristic function of ω_ε. We define the operator $B_\varepsilon : H_0^m(\Omega) \to H^{-m}(\Omega)$ by

$$\langle B_\varepsilon u, v\rangle_{H^{-m}(\Omega),H_0^m(\Omega)} = \sum_{\substack{|\alpha|\leqslant m \\ |\beta|\leqslant m-1}} \int_\Omega b_{\alpha\beta,\varepsilon} D^\alpha u D^\beta v\,dx \quad \forall u, v \in H_0^m(\Omega).$$

We assume that, for all ε sufficiently small, the operator $A_\varepsilon + B_\varepsilon$ and its adjoint $A_\varepsilon + B_\varepsilon^*$ are injective. We will infer (see Proposition 3.3) that $A_\varepsilon + B_\varepsilon$ is invertible, as well as its adjoint (the proof is the same). Henceforth ε will always be implicitly assumed to be small enough.

Given a source $f \in H^{-m}(\Omega)$ we denote for every $\varepsilon \geqslant 0$ by $u_\varepsilon \in H_0^m(\Omega)$ the unique solution of

$$(A_\varepsilon + B_\varepsilon)u_\varepsilon = f. \tag{2.6}$$

The goal of this paper is to analyze the asymptotic behavior of a shape functional of the form $j(\varepsilon) = J_\varepsilon(u_\varepsilon)$ when $\varepsilon \to 0$.

3. Well-posedness

The space $H^m(\Omega)$ is endowed with the standard norm $\|.\|_{H^m(\Omega)}$ and the associated seminorm $|.|_{H^m(\Omega)}$ defined by

$$\|u\|^2_{H^m(\Omega)} = \sum_{|\alpha|\leqslant m} \|D^\alpha u\|^2_{L^2(\Omega)}, \qquad |u|^2_{H^m(\Omega)} = \sum_{|\alpha|=m} \|D^\alpha u\|^2_{L^2(\Omega)}.$$

The expression (2.5) obviously defines a symmetric and continuous bilinear form on $H_0^m(\Omega)$. The coercivity is based on the lemma below, whose proof can be found in Appendix A.

Lemma 3.1. *There exists $c > 0$ independent of ε such that*

$$\langle A_\varepsilon u, u\rangle_{H^{-m}(\Omega), H_0^m(\Omega)} \geqslant c|u|^2_{H^m(\Omega)} \quad \forall u \in H_0^m(\Omega).$$

By the Lax–Milgram theorem and the Poincaré inequality in $H_0^m(\Omega)$ (see [1]) we infer the following result.

Corollary 3.2. *For all $f \in H^{-m}(\Omega)$ and all $\varepsilon \geqslant 0$ there exists a unique $u \in H_0^m(\Omega)$ such that*

$$\langle A_\varepsilon u, \eta\rangle_{H^{-m}(\Omega), H_0^m(\Omega)} = \langle f, \eta\rangle \quad \forall \eta \in H_0^m(\Omega).$$

Moreover, there exists a constant c independent of ε such that $\|u\|_{H^m(\Omega)} \leqslant c\|f\|_{H^m(\Omega)}$.

Proposition 3.3. *For all $f \in H^{-m}(\Omega)$ and all $\varepsilon \geqslant 0$ there exists a unique $u \in H_0^m(\Omega)$ such that*

$$\big\langle (A_\varepsilon + B_\varepsilon)u, \eta\big\rangle_{H^{-m}(\Omega), H_0^m(\Omega)} = \langle f, \eta\rangle \quad \forall \eta \in H_0^m(\Omega).$$

Moreover, there exists a constant c independent of ε such that $\|u\|_{H^m(\Omega)} \leqslant c\|f\|_{H^m(\Omega)}$.

Proof. Since A_ε is invertible, we have

$$(A_\varepsilon + B_\varepsilon)u = f \quad \Leftrightarrow \quad \big(I + B_\varepsilon A_\varepsilon^{-1}\big)A_\varepsilon u = f,$$

where I stands for the identity operator of $H^{-m}(\Omega)$. By Corollary 3.2, $A_\varepsilon^{-1} : H^{-m}(\Omega) \to H_0^m(\Omega)$ is uniformly bounded. Next, the operator B_ε can be decomposed as $B_\varepsilon = J\tilde{B}_\varepsilon$, with J the canonical embedding of $H^{1-m}(\Omega)$ into $H^{-m}(\Omega)$ and $\tilde{B}_\varepsilon$ the operator defined algebraically like B_ε, but acting from $H^m(\Omega)$ into $H^{1-m}(\Omega)$. By construction, $\tilde{B}_\varepsilon$ is uniformly bounded and, by the combination of the Rellich and Schauder theorems, J is compact. It follows that the family of operators $\{B_\varepsilon A_\varepsilon^{-1} : H^{-m}(\Omega) \to H^{-m}(\Omega),\ \varepsilon \geqslant 0\}$ is collectively compact (see Appendix B).

In order to apply Theorem B.1, let us prove that it is also pointwise sequentially compact. Let (ε_k) be a bounded sequence of nonnegative numbers. By the Bolzano–Weierstrass theorem there exists $\varepsilon_\infty \geqslant 0$ such that, for a non-relabeled subsequence, $\varepsilon_k \to \varepsilon_\infty$. Let now $\varphi \in H^{-m}(\Omega)$ be arbitrary and define $\psi_k = A_{\varepsilon_k}^{-1}\varphi \in H_0^m(\Omega)$. Then we have $\rho_{\varepsilon_k} \to \rho_{\varepsilon_\infty}$ almost everywhere, which implies by a standard argument (see e.g. Theorem 16.4.1 of [10]) that $\psi_k \rightharpoonup \psi_\infty := A_{\varepsilon_\infty}^{-1}\varphi$ weakly in $H_0^m(\Omega)$. We now write for any $\eta \in H_0^m(\Omega)$:

$$\big\langle B_{\varepsilon_k} A_{\varepsilon_k}^{-1}\varphi, \eta\big\rangle_{H^{-m}(\Omega), H_0^m(\Omega)} = \big\langle B_{\varepsilon_k}^* \eta, A_{\varepsilon_k}^{-1}\varphi\big\rangle_{H^{-m}(\Omega), H_0^m(\Omega)}.$$

Lebesgue's dominated convergence theorem yields that $B_{\varepsilon_k}^*\eta \to B_{\varepsilon_\infty}^*\eta$ strongly in $H^{-m}(\Omega)$. As a product of weakly and strongly convergent sequences we infer:

$$\big\langle B_{\varepsilon_k} A_{\varepsilon_k}^{-1}\varphi, \eta\big\rangle_{H^{-m}(\Omega), H_0^m(\Omega)} \to \big\langle B_{\varepsilon_\infty}^* \eta, A_{\varepsilon_\infty}^{-1}\varphi\big\rangle_{H^{-m}(\Omega), H_0^m(\Omega)} = \big\langle B_{\varepsilon_\infty} A_{\varepsilon_\infty}^{-1}\varphi, \eta\big\rangle_{H^{-m}(\Omega), H_0^m(\Omega)}.$$

This means that $B_{\varepsilon_k} A_{\varepsilon_k}^{-1}\varphi \rightharpoonup B_{\varepsilon_\infty} A_{\varepsilon_\infty}^{-1}\varphi$ weakly in $H^{-m}(\Omega)$, but the convergence is actually strong by compactness of the sequence. We have thus proved that $B_{\varepsilon_k} A_{\varepsilon_k}^{-1} \to B_{\varepsilon_\infty} A_{\varepsilon_\infty}^{-1}$ pointwise.

By the Fredholm alternative, the operator $I + B_\varepsilon A_\varepsilon^{-1}$ is invertible for each $\varepsilon \geqslant 0$, since it is injective by assumption. Therefore, by virtue of Theorem B.1, the operators $(I + B_\varepsilon A_\varepsilon^{-1})^{-1}$ are uniformly bounded. Writing that $u = A_\varepsilon^{-1}(I + B_\varepsilon A_\varepsilon^{-1})^{-1} f$ and using again Corollary 3.2 provides the desired uniform bound. □

We will later need the following variant of Lemma 3.1. The proof, which is very similar, is left to the reader.

Lemma 3.4. *Let $\underline{\rho}$ be a positive constant. There exists $c > 0$ such that, whenever $\hat{\rho} \in L^\infty(\mathbb{R}^n)$ is essentially bounded from below by $\underline{\rho}$ and $D^\alpha u \in L^2(\mathbb{R}^n)$ for every α with $|\alpha| = m$, we have*

$$\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \hat{\rho}_\varepsilon a_{\alpha\beta} D^\alpha u D^\beta u \, dx \geqslant c|u|^2_{H^m(\mathbb{R}^n)}.$$

As opposed to the previous case where the domain Ω was bounded, in Lemma 3.4 the semi-norm $|u|^2_{H^m(\mathbb{R}^n)}$ is not a norm and the Poincaré inequality does not hold. Hence, in order to prove the existence and uniqueness of a solution in $\mathbb{R}^n$, the Lax–Milgram theorem cannot be applied directly. To address this issue, we introduce in Appendix C a weighted space $W^m(\mathbb{R}^n)$ (cf. Eq. (C.1)) and its quotient space $W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$ where $\mathcal{P}_{m-1}$ is the space of polynomials of degree not greater than $m - 1$. We have the following extension of the Poincaré inequality (cf. Corollary C.5).

Lemma 3.5. *There exists $c > 0$ such that, for all $u \in W^m(\mathbb{R}^n)$,*

$$\|u\|_{W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}} \leqslant c|u|_{H^m(\mathbb{R}^n)}.$$

The combination of Lemmas 3.4 and 3.5 will lead to useful existence and uniqueness results for problems defined in $\mathbb{R}^n$. We recall that the approach with quotient spaces is due to Deny and Lions as reported by Ciarlet in [23] (see e.g. Theorem 14.1 as applied to the Finite Element Methods).

4. Derivation of the general formula

4.1. A preliminary abstract theorem: asymptotic expansion of a cost function

The following theorem provides a general framework for the sensitivity analysis of a cost function associated with a constraint in variational form. It has been introduced in [4], however we give here a short proof for completeness.

Theorem 4.1. *Let $\mathcal{V}$ be a vector space and I be a real interval containing 0. For all $\varepsilon \in I$ consider a vector $u_\varepsilon \in \mathcal{V}$ such that*:

$$a_\varepsilon(u_\varepsilon, \eta) = \langle f_\varepsilon, \eta \rangle \quad \forall \eta \in \mathcal{V}, \tag{4.1}$$

where a_ε and f_ε are a bilinear form on $\mathcal{V} \times \mathcal{V}$ and a linear functional on $\mathcal{V}$, respectively. Consider also a functional $J_\varepsilon : \mathcal{V} \to \mathbb{R}$ and a linear functional $g_\varepsilon \in \mathcal{V}'$. Suppose that the following hypotheses hold:

(1) *For all $\varepsilon \in I$, there exists $v_\varepsilon \in \mathcal{V}$ such that*

$$a_\varepsilon(\eta, v_\varepsilon) = -\langle g_\varepsilon, \eta\rangle \quad \forall \eta \in \mathcal{V}. \tag{4.2}$$

(2) *There exist real numbers δa, δf and a function $\varepsilon \mapsto \varphi(\varepsilon) \in \mathbb{R}$ such that, when $\varepsilon \to 0$,*

$$(a_\varepsilon - a_0)(u_0, v_\varepsilon) = \varphi(\varepsilon)\delta a + o\big(\varphi(\varepsilon)\big), \tag{4.3}$$

$$\langle f_\varepsilon - f_0, v_\varepsilon\rangle = \varphi(\varepsilon)\delta f + o\big(\varphi(\varepsilon)\big). \tag{4.4}$$

(3) *There exist real numbers δJ_1, δJ_2 such that*

$$J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_0) + \langle g_\varepsilon, u_\varepsilon - u_0\rangle + \varphi(\varepsilon)\delta J_1 + o\big(\varphi(\varepsilon)\big), \tag{4.5}$$

$$J_\varepsilon(u_0) = J_0(u_0) + \varphi(\varepsilon)\delta J_2 + o\big(\varphi(\varepsilon)\big). \tag{4.6}$$

Then we have

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = \varphi(\varepsilon)(\delta a - \delta f + \delta J_1 + \delta J_2) + o\big(\varphi(\varepsilon)\big). \tag{4.7}$$

Proof. From (4.5) and (4.6), we obtain

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = \langle g_\varepsilon, u_\varepsilon - u_0\rangle + \varphi(\varepsilon)(\delta J_1 + \delta J_2) + o\big(\varphi(\varepsilon)\big).$$

Taking into account (4.2) and the fact that $u_\varepsilon - u_0 \in \mathcal{V}$, we get

$$\begin{aligned} J_\varepsilon(u_\varepsilon) - J_0(u_0) &= -a_\varepsilon(u_\varepsilon - u_0, v_\varepsilon) + \varphi(\varepsilon)(\delta J_1 + \delta J_2) + o\big(\varphi(\varepsilon)\big) \\ &= -a_\varepsilon(u_\varepsilon, v_\varepsilon) + (a_\varepsilon - a_0)(u_0, v_\varepsilon) + a_0(u_0, v_\varepsilon) + \varphi(\varepsilon)(\delta J_1 + \delta J_2) + o\big(\varphi(\varepsilon)\big). \end{aligned}$$

Then (4.1) yields

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = (a_\varepsilon - a_0)(u_0, v_\varepsilon) + \langle f_\varepsilon - f_0, v_\varepsilon\rangle + \varphi(\varepsilon)(\delta J_1 + \delta J_2) + o\big(\varphi(\varepsilon)\big).$$

Finally, using the hypotheses (4.3) and (4.4), we arrive at

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = \varphi(\varepsilon)(\delta a + \delta f) + \varphi(\varepsilon)(\delta J_1 + \delta J_2) + o\big(\varphi(\varepsilon)\big), \tag{4.8}$$

which leads to the result. □

4.2. A particular class of cost functions

For the sake of simplicity, we focus here on a particular class of cost functions. It should be noted however that the theory developed here can be readily extended to other cost functions, on possibly some additional estimates, similarly to [4,6].

Theorem 4.2. *Suppose that the cost function is of form $J_\varepsilon(u) = J(u)$ with $J : H_0^m(\Omega) \to \mathbb{R}$ satisfying*

$$J(u+h) - J(u) = \langle g, h\rangle_{H^{-m}(\Omega), H^m(\Omega)} + O\left(\|h\|^2\right) \tag{4.9}$$

for all $u, h \in H^m(\Omega)$, where either $\|h\| = \|h\|_{H^m(\Omega\setminus\mathcal{N})}$, with $\mathcal{N}$ an arbitrary neighborhood of $\hat{x}$, or $\|h\| = \|h\|_{H^{m-1}(\Omega)}$. Then (4.5) and (4.6) hold true for $\varphi(\varepsilon) = \varepsilon^n$ and $\delta J_1 = \delta J_2 = 0$.

Proof. We already have (4.6) with $\delta J_2 = 0$, since J_ε is independent of ε. Then (4.9) yields

$$J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) - \langle g, u_\varepsilon - u_0\rangle = O\left(\|u_\varepsilon - u_0\|^2\right). \tag{4.10}$$

It will be proved in Lemma 7.1 that $\|v_\varepsilon - v_0\| = o(\varepsilon^{n/2})$, with a similar estimate holding for the direct state, i.e., $\|u_\varepsilon - u_0\| = o(\varepsilon^{n/2})$. Together with (4.10), this implies (4.5) with $\delta J_1 = 0$. $\square$

For example purposes, two common cost functionals for which the above theorem applies are given below, namely the least-square-type and compliance functionals. We still call $\mathcal{N}$ an arbitrary neighborhood of $\hat{x}$.

- Tracking-type functional

$$J_\varepsilon(u) = \int_D |u - u_d|^2\, dx, \tag{4.11}$$

  with $u_d \in L^2(D)$ and $D \subset \Omega \setminus \mathcal{N}$.
- Compliance functional

$$J_\varepsilon(u) = \int_\Omega f u\, dx, \tag{4.12}$$

  with $f_\varepsilon = f$ independent of ε (see the example in Section 6.2.2).

4.3. Strategy of the proof

We consider in the general setting a family of functionals $J_\varepsilon : H_0^m(\Omega) \to \mathbb{R}$ satisfying (4.5) and (4.6) for some $g_\varepsilon = g \in H^{-m}(\Omega)$, independent of ε, and $\varphi(\varepsilon) = \varepsilon^n$. As announced in Section 2, we also assume for simplicity that $f_\varepsilon = f \in H^{-m}(\Omega)$ is independent of ε, from which Eq. (4.4) is straightforwardly satisfied with $\delta f = 0$.

According to Theorem 4.1, in order to obtain the general expression (4.7) of the topological asymptotic expansion of $J_\varepsilon(u_\varepsilon)$, the main step will be the evaluation of (4.3). This will be achieved in Lemmas 4.3 and 4.4 of Section 4.5. An important part of the analysis, which consists of the estimation of the remainders providing the $o(\varphi(\varepsilon))$-term in (4.3), will be deferred to Section 7.

Let us now start the evaluation of Eq. (4.3). Recall first that in the present context the function spaces are $\mathcal{V} = H_0^m(\Omega)$ and $\mathcal{V}' = H^{-m}(\Omega)$, and the bilinear form a_ε is defined by

$$a_\varepsilon(u,v) = \big\langle (A_\varepsilon + B_\varepsilon)u, v\big\rangle_{H^{-m}(\Omega), H_0^m(\Omega)}.$$

Moreover, the *background state* $u_0 \in H_0^m(\Omega)$ and the *background adjoint state* $v_0 \in H_0^m(\Omega)$ are respectively defined as the solutions of

$$(A_0 + B_0)u_0 = f \quad \text{and} \quad \big(A_0 + B_0^*\big)v_0 = -g, \tag{4.13}$$

with B_0^* the adjoint operator of B_0. Moreover, Eq. (4.2) can be rewritten as

$$\big(A_\varepsilon + B_\varepsilon^*\big)v_\varepsilon = -g.$$

4.4. Preliminary definitions

We introduce the variation

$$\tilde{v}_\varepsilon := v_\varepsilon - v_0, \tag{4.14}$$

and, in order to prove Eq. (4.3), we define the quantity

$$V_a(\varepsilon) := (a_\varepsilon - a_0)(u_0, v_\varepsilon). \tag{4.15}$$

To proceed with the analysis of the asymptotic behavior of $V_a(\varepsilon)$ we shall use the spaces $W^m(\mathbb{R}^n)$ and $W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$ defined in Appendix C. In the course of the analysis we will need some auxiliary functions. They are defined thereafter.

First, we define $h_\varepsilon \in W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$ as the solution of

$$\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \hat{\rho}_\varepsilon a_{\alpha\beta} D^\alpha h_\varepsilon D^\beta \eta \, dx = -(\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega_\varepsilon} a_{\alpha\beta} D^\alpha v_0 D^\beta \eta \, dx \tag{4.16}$$

for all $\eta \in W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$, with

$$\hat{\rho}_\varepsilon(x) = \begin{cases} \hat{\rho}_0 & \text{if } x \in \mathbb{R}^n \setminus \overline{\omega}_\varepsilon, \\ \hat{\rho}_1 & \text{if } x \in \omega_\varepsilon. \end{cases} \tag{4.17}$$

We next set

$$H_\varepsilon(y) = \varepsilon^{-m} h_\varepsilon(\hat{x} + \varepsilon y). \tag{4.18}$$

Then, we define for each γ with $|\gamma| = m$ the function $\Psi_\gamma \in W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$ as the solution of

$$\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \hat{\rho} a_{\alpha\beta} D^\alpha \Psi_\gamma(y) D^\beta \Phi(y) \, dy = -(\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\beta|=m} \int_{\omega} a_{\gamma\beta} D^\beta \Phi(y) \, dy \tag{4.19}$$

for all $\Phi \in W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$, with

$$\hat{\rho}(y) = \begin{cases} \hat{\rho}_0 & \text{if } y \in \mathbb{R}^n \setminus \overline{\omega}, \\ \hat{\rho}_1 & \text{if } y \in \omega. \end{cases} \tag{4.20}$$

Note that the existence and uniqueness of the solutions of (4.16) and (4.19) is a consequence of Lemma 3.4 and Lemma 3.5. We set

$$H = \sum_{|\gamma|=m} D^\gamma v_0(\hat{x}) \Psi_\gamma. \tag{4.21}$$

Finally we define the polarization tensor $(p_{\alpha\beta})$ by its entries

$$p_{\alpha\beta} = |\omega|(r-1)a_{\alpha\beta} + k_{\alpha\beta}, \tag{4.22}$$

with

$$r = \frac{\hat{\rho}_1}{\hat{\rho}_0} \tag{4.23}$$

the contrast and

$$k_{\alpha\gamma} = (r-1) \sum_{|\beta|=m} a_{\alpha\beta} \int_\omega D^\beta \Psi_\gamma(y)\,dy. \tag{4.24}$$

4.5. Asymptotic expansion of the bilinear form

Lemma 4.3. *For ε sufficiently small, the following expression of (4.15) holds true*:

$$\begin{aligned} V_a(\varepsilon) = \varepsilon^n \hat{\rho}_0 \sum_{|\alpha|=|\beta|=m} p_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) \\ + \varepsilon^n |\omega| \sum_{\substack{|\alpha|\leqslant m \\ |\beta|\leqslant m-1}} q_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) + \sum_{i=1}^5 \mathcal{E}_i(\varepsilon), \end{aligned} \tag{4.25}$$

with the remainders

$$\begin{aligned} \mathcal{E}_1(\varepsilon) &= (\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega_\varepsilon} a_{\alpha\beta} \big(D^\alpha u_0 D^\beta v_0 - D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) \big)\,dx \\ &\quad + \sum_{\substack{|\alpha|\leqslant m \\ |\beta|\leqslant m-1}} \int_{\omega_\varepsilon} q_{\alpha\beta} \big(D^\alpha u_0 D^\beta v_0 - D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) \big)\,dx, \\ \mathcal{E}_2(\varepsilon) &= (\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega_\varepsilon} a_{\alpha\beta} \big(D^\alpha u_0 - D^\alpha u_0(\hat{x}) \big) D^\beta \tilde{v}_\varepsilon\,dx, \\ \mathcal{E}_3(\varepsilon) &= \sum_{\substack{|\alpha|\leqslant m \\ |\beta|\leqslant m-1}} \int_{\omega_\varepsilon} q_{\alpha\beta} D^\alpha u_0 D^\beta \tilde{v}_\varepsilon\,dx, \end{aligned}$$

$$\mathcal{E}_4(\varepsilon) = (\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega_\varepsilon} a_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta (\tilde{v}_\varepsilon - h_\varepsilon)(y)\, dy,$$

$$\mathcal{E}_5(\varepsilon) = \varepsilon^n (\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega} a_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta (H_\varepsilon - H)(y)\, dy.$$

Proof. We have by definition

$$V_a(\varepsilon) = \sum_{|\alpha|=|\beta|=m} \int_{\Omega} (\rho_\varepsilon - \rho_0) a_{\alpha\beta} D^\alpha u_0 D^\beta v_\varepsilon\, dx + \sum_{\substack{|\alpha|\leqslant m \\ |\beta|\leqslant m-1}} \int_{\Omega} (a_{\alpha\beta,\varepsilon} - a_{\alpha\beta,0}) D^\alpha u_0 D^\beta v_\varepsilon\, dx,$$

hence, for ε small enough,

$$V_a(\varepsilon) = \sum_{|\alpha|=|\beta|=m} \int_{\omega_\varepsilon} (\hat{\rho}_1 - \hat{\rho}_0) a_{\alpha\beta} D^\alpha u_0 D^\beta v_\varepsilon\, dx + \sum_{\substack{|\alpha|\leqslant m \\ |\beta|\leqslant m-1}} \int_{\omega_\varepsilon} q_{\alpha\beta} D^\alpha u_0 D^\beta v_\varepsilon\, dx.$$

We make the splitting $V_a(\varepsilon) = V_a^1(\varepsilon) + V_a^2(\varepsilon)$ with

$$V_a^1(\varepsilon) = (\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega_\varepsilon} a_{\alpha\beta} D^\alpha u_0 D^\beta v_0\, dx + \sum_{\substack{|\alpha|\leqslant m \\ |\beta|\leqslant m-1}} \int_{\omega_\varepsilon} q_{\alpha\beta} D^\alpha u_0 D^\beta v_0\, dx,$$

$$V_a^2(\varepsilon) = (\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega_\varepsilon} a_{\alpha\beta} D^\alpha u_0 D^\beta \tilde{v}_\varepsilon\, dx + \sum_{\substack{|\alpha|\leqslant m \\ |\beta|\leqslant m-1}} \int_{\omega_\varepsilon} q_{\alpha\beta} D^\alpha u_0 D^\beta \tilde{v}_\varepsilon\, dx.$$

• *First approximation*. With the help of the splitting $D^\alpha u_0 D^\beta v_0 = D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) + [D^\alpha u_0 D^\beta v_0 - D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x})]$ we obviously get

$$V_a^1(\varepsilon) = |\omega_\varepsilon| (\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) + |\omega_\varepsilon| \sum_{\substack{|\alpha|\leqslant m \\ |\beta|\leqslant m-1}} q_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) + \mathcal{E}_1(\varepsilon). \tag{4.26}$$

• *Second and third approximations*. Similarly we have

$$V_a^2(\varepsilon) = \sum_{|\alpha|=|\beta|=m} \int_{\omega_\varepsilon} (\hat{\rho}_1 - \hat{\rho}_0) a_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta \tilde{v}_\varepsilon\, dx + \mathcal{E}_2(\varepsilon) + \mathcal{E}_3(\varepsilon). \tag{4.27}$$

• *Fourth approximation*. We now approximate $\tilde{v}_\varepsilon$. We have for any $\eta \in H_0^m(\Omega)$:

$$\begin{aligned} a_\varepsilon(\eta, \tilde{v}_\varepsilon) &= a_\varepsilon(\eta, v_\varepsilon) - a_\varepsilon(\eta, v_0) = -\langle g, \eta \rangle - (a_\varepsilon - a_0)(\eta, v_0) - a_0(\eta, v_0) \\ &= -(a_\varepsilon - a_0)(\eta, v_0). \end{aligned} \tag{4.28}$$

For ε small enough, we therefore have

$$\sum_{|\alpha|=|\beta|=m} \int_{\Omega} \rho_\varepsilon a_{\alpha\beta} D^\alpha \eta D^\beta \tilde{v}_\varepsilon \, dx + \sum_{\substack{|\alpha|\leqslant m \\ |\beta|\leqslant m-1}} \int_{\Omega} b_{\alpha\beta,\varepsilon} D^\alpha \eta D^\beta \tilde{v}_\varepsilon \, dx$$
$$= -(\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega_\varepsilon} a_{\alpha\beta} D^\alpha \eta D^\beta v_0 \, dx - \sum_{\substack{|\alpha|\leqslant m \\ |\beta|\leqslant m-1}} \int_{\omega_\varepsilon} q_{\alpha\beta} D^\alpha \eta D^\beta v_0 \, dx, \tag{4.29}$$

which suggests to approximate $\tilde{v}_\varepsilon$ in (4.27) by h_ε, solution of (4.16). We arrive at

$$V_a^2(\varepsilon) = (\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega_\varepsilon} a_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta h_\varepsilon \, dx + \mathcal{E}_2(\varepsilon) + \mathcal{E}_3(\varepsilon) + \mathcal{E}_4(\varepsilon).$$

By the change of variable $x = \hat{x} + \varepsilon y$ this can be rewritten ass

$$V_a^2(\varepsilon) = \varepsilon^n (\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega} a_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta H_\varepsilon(y) \, dy$$
$$+ \mathcal{E}_2(\varepsilon) + \mathcal{E}_3(\varepsilon) + \mathcal{E}_4(\varepsilon). \tag{4.30}$$

• *Fifth approximation*. On one hand, plugging $h_\varepsilon(x) = \varepsilon^m H_\varepsilon(\varepsilon^{-1}(x - \hat{x}))$ and $\eta(x) = \varepsilon^m \phi_\varepsilon(\varepsilon^{-1}(x - \hat{x}))$ in (4.16) yields after the change of variable $x = \hat{x} + \varepsilon y$ in both integrals of (4.16)

$$\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \hat{\rho}(y) a_{\alpha\beta} D^\alpha H_\varepsilon(y) D^\beta \phi_\varepsilon(y) \, dy$$
$$= -(\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega} a_{\alpha\beta} D^\alpha v_0(\hat{x} + \varepsilon y) D^\beta \phi_\varepsilon(y) \, dy \tag{4.31}$$

for every $\phi_\varepsilon \in W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$, with $\hat{\rho}$ defined by (4.20). On the other hand, combining (4.19) and (4.21) results in

$$\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \hat{\rho}(y) a_{\alpha\beta} D^\alpha H(y) D^\beta \Phi(y) \, dy$$
$$= -(\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega} a_{\alpha\beta} D^\alpha v_0(\hat{x}) D^\beta \Phi(y) \, dy \tag{4.32}$$

for every $\Phi \in W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$. On replacing H_ε by $H + (H_\varepsilon - H)$ in (4.30) we obtain

$$V_a^2(\varepsilon) = \varepsilon^n (\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=m} \int_{\omega} a_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta H(y) \, dy + \sum_{i=2}^{5} \mathcal{E}_i(\varepsilon). \tag{4.33}$$

Plugging (4.21) into the above yields

$$V_a^2(\varepsilon) = \varepsilon^n(\hat{\rho}_1 - \hat{\rho}_0) \sum_{|\alpha|=|\beta|=|\gamma|=m} \int_\omega a_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\gamma v_0(\hat{x}) D^\beta \Psi_\gamma(y)\, dy + \sum_{i=2}^{5} \mathcal{E}_i(\varepsilon).$$

After rearrangement we obtain

$$V_a^2(\varepsilon) = \varepsilon^n \hat{\rho}_0 \sum_{|\alpha|=|\gamma|=m} k_{\alpha\gamma} D^\alpha u_0(\hat{x}) D^\gamma v_0(\hat{x}) + \sum_{i=2}^{5} \mathcal{E}_i(\varepsilon), \tag{4.34}$$

with $k_{\alpha\beta}$ defined by (4.24). Altogether we arrive at

$$V_a(\varepsilon) = \varepsilon^n \sum_{|\alpha|=|\beta|=m} \big(|\omega|(\hat{\rho}_1 - \hat{\rho}_0) a_{\alpha\beta} + \hat{\rho}_0 k_{\alpha\beta}\big) D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) + \sum_{i=1}^{5} \mathcal{E}_i(\varepsilon).$$

The expression (4.22) of the polarization tensor leads to (4.25). □

The following lemma provides the appropriate estimates for the remainders $\mathcal{E}_i(\varepsilon)$. This is the core of the analysis. The proof is given in Section 7.

Lemma 4.4. *Suppose that f and g are of regularity H^s in a neighborhood of $\hat{x}$ with $s > \max(0, \frac{n}{2} + 1 - m)$. Then the remainders $\mathcal{E}_i(\varepsilon)$ in Lemma 4.3 satisfy $|\mathcal{E}_i(\varepsilon)| = o(\varepsilon^n)$ for each $i = 1, 2, 3, 4, 5$.*

4.6. Topological sensitivity analysis of the cost function

We are now in a position to provide the asymptotic expansion of the cost function.

Theorem 4.5. *For every ε sufficiently small let u_ε be the solution of (2.6). Suppose that the cost function J_ε is such that (4.5) and (4.6) hold true for $\varphi(\varepsilon) = \varepsilon^n$ and $g_\varepsilon = g$ independent of ε, and that f, g are of regularity H^s in a neighborhood of $\hat{x}$ with $s > \max(0, \frac{n}{2} + 1 - m)$. Then we have*

$$\begin{aligned} J_\varepsilon(u_\varepsilon) - J_0(u_0) = \varepsilon^n \Bigg[& \hat{\rho}_0 \sum_{|\alpha|=|\beta|=m} p_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) \\ & + |\omega| \sum_{\substack{|\alpha|\leqslant m \\ |\beta|\leqslant m-1}} q_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) + \delta J_1 + \delta J_2 \Bigg] + o\big(\varepsilon^n\big), \end{aligned} \tag{4.35}$$

with the entries of the polarization tensor $(p_{\alpha\beta})_{|\alpha|=|\beta|=m}$ given by (4.22), and u_0, v_0 solutions of (4.13).

Proof. By Lemmas 4.3 and 4.4 and Definition 5.3, Eq. (4.3) is satisfied with $\varphi(\varepsilon)=\varepsilon^n$ and

$$\delta a := \hat{\rho}_0 \sum_{|\alpha|=|\beta|=m} p_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}) + |\omega| \sum_{\substack{|\alpha|\leqslant m\\ |\beta|\leqslant m-1}} q_{\alpha\beta} D^\alpha u_0(\hat{x}) D^\beta v_0(\hat{x}).$$

The claim follows from Theorem 4.1. □

Remark 4.6. If the source term $f=f_\varepsilon$ depends on ε, then (4.35) is simply modified by the addition of the extra term $\varepsilon^n \delta f$, which stems from (4.4). If now the function g_ε which satisfies (4.5) depends on ε, then (4.28) changes, which results in the additional term $\langle g_\varepsilon - g_0, \eta\rangle$ at the right hand side of (4.29). If $\|g_\varepsilon - g_0\|_{H^{-m}(\Omega)} = o(\varepsilon^{n/2})$, then we still have $|\tilde{v}_\varepsilon - h_\varepsilon|_{H^m(\Omega)} = o(\varepsilon^{n/2})$ in Lemma 7.1, therefore formula (4.35) remains unchanged. But if $\|g_\varepsilon - g_0\|_{H^{-m}(\Omega)}$ is of order $\varepsilon^{n/2}$, then an extra term appears in (4.35). For an example we refer to [6] where such a term has been computed for the Kirchhoff plate problem. Note that the regularity conditions $f, g \in H^s$ apply to f_0 and g_0.

Remark 4.7. Although the condition $\hat{\rho}_1 > 0$ has been used in several places, the topological asymptotic expansion for Neumann holes can be rigorously obtained by taking the value $\hat{\rho}_1 = 0$ in the computation of the polarization tensor, provided that (4.19) still admits a solution (necessarily non-unique) for this value. The proof of this claim is rather technical, and has been done for the Laplace operator and the Kirchhoff plate problem in [4] and [6], respectively. The same idea applies here, therefore we do not reproduce the proof.

5. A class of degenerate problems

5.1. Degenerate expression of the polarization tensor

Definition 5.1 *(Degenerate polarization tensor).* We say that the polarization tensor (4.22) is degenerate when its entries do not remain bounded when the contrast r tends to zero.

In particular, when the polarization tensor is degenerate, the topological sensitivity for Neumann holes is not defined, see Remark 4.7. This situation occurs when the cost functional is discontinuous with respect to the nucleation of a Neumann hole, and it is observed for instance in dimension $n=1$ for the Laplacian (see Section 6.2). We will see that it can also occur in higher dimension, but for higher order operators.

The goal of this section is to give a sufficient condition of degeneracy, as well as to provide an explicit expression of the polarization tensor in this case. To this aim we introduce the family of piecewise constant functions $\zeta_{\alpha\gamma} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\zeta_{\alpha\gamma}(x) = \begin{cases} -\frac{\hat{\rho}_1-\hat{\rho}_0}{\hat{\rho}_1}\delta_{\alpha\gamma} & \text{if } x\in\omega,\\ 0 & \text{if } x \in \mathbb{R}^n\setminus\omega, \end{cases} \tag{5.1}$$

with $\delta_{\alpha\gamma}=1$ if $\alpha=\gamma$, $\delta_{\alpha\gamma}=0$ otherwise. We have for all $\Phi \in W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$:

$$\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \hat{\rho} a_{\alpha\beta} \zeta_{\alpha\gamma} D^\beta \Phi(y)\,dy = -(\hat{\rho}_1-\hat{\rho}_0) \sum_{|\beta|=m} \int_\omega a_{\gamma\beta} D^\beta \Phi(y)\,dy. \tag{5.2}$$

In the definition of the polarization tensor, (4.19) appeared as a critical step. Accordingly the following assumption is made.

Assumption 5.2. For any multi-indices γ with $|\gamma| = m$, there exists a function $\Psi_\gamma \in W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$ satisfying

$$\sum_{|\alpha|=m} a_{\alpha\beta} D^\alpha \Psi_\gamma = \sum_{|\alpha|=m} a_{\alpha\beta}\zeta_{\alpha\gamma} \quad \forall\beta, |\beta| = m. \tag{5.3}$$

It immediately stems from Assumption 5.2, using (5.2), that Ψ_γ solves (4.19). Hence (4.24) results in:

$$\begin{aligned} k_{\alpha\gamma} &= \left(\frac{\hat\rho_1}{\hat\rho_0} - 1\right)\int_\omega \sum_{|\beta|=m} a_{\alpha\beta} D^\beta \Psi_\gamma(y)\,dy = \left(\frac{\hat\rho_1}{\hat\rho_0} - 1\right)\int_\omega \sum_{|\beta|=m} a_{\alpha\beta}\zeta_{\beta\gamma} \\ &= -\left(\frac{\hat\rho_1}{\hat\rho_0} - 1\right)\sum_{|\beta|=m} a_{\alpha\beta}\frac{\hat\rho_1 - \hat\rho_0}{\hat\rho_1}|\omega|\delta_{\beta\gamma} = -a_{\alpha\gamma}\frac{(\hat\rho_1 - \hat\rho_0)^2}{\hat\rho_1\hat\rho_0}|\omega|, \end{aligned} \tag{5.4}$$

with $|\omega|$ the n-dimensional Lebesgue measure of ω. Moreover, plugging (5.4) into (4.22) provides the following closed formula for the polarization tensor.

Proposition 5.3 *(Degenerate polarization tensor). If Assumption 5.2 is fulfilled, then we have*

$$p_{\alpha\gamma} = |\omega|\left(1 - \frac{1}{r}\right)a_{\alpha\gamma}. \tag{5.5}$$

We also note that, in addition to be degenerate in the sense of Definition 5.1, the polarization tensor (5.5) is independent of the shape of ω.

5.2. Characterization of a degenerate problem

We shall now give sufficient conditions for Assumption 5.2 to be satisfied. To do so, set $q = \sharp\{\alpha \in \mathbb{N}^n,\ |\alpha| = m\}$ and define the linear map

$$\begin{aligned} \Lambda : \mathbb{R}^q &\to \mathbb{R}^q \\ (U_\alpha)_{|\alpha|=m} &\mapsto (V_\beta)_{|\beta|=m} \end{aligned}$$

such that

$$V_\beta = \sum_{|\alpha|=m} a_{\alpha\beta} U_\alpha.$$

We recall the following general result from [20, Theorem 7.1.20].

Theorem 5.4. *If P is a homogeneous elliptic (i.e. $P(\xi) = 0 \Rightarrow \xi = 0$) polynomial of degree p in $\mathbb{R}^n$, then the differential operator $P(D)$ has a fundamental solution of the form*

$$E = E_0 - Q(x)\log|x|, \tag{5.6}$$

where E_0 is homogeneous of degree $p-n$, $\mathcal{C}^\infty$ and analytic in $\mathbb{R}^n \setminus \{0\}$ and Q is a polynomial which is identically 0 when $n > p$ and is homogeneous of order $p-n$ when $n \leqslant p$. (Cf. [20] for its explicit expression.)

Corollary 5.5. *Let E be the fundamental solution introduced in Theorem 5.4 and $\alpha \in \mathbb{N}^n$ be such that $|\alpha| = k$. For all $R > 1$ there exists $c > 0$ such that, for all $x \in \mathbb{R}^n$ with $|x| > R$,*

$$\begin{aligned} &\text{if } n > p, \quad \left|D^\alpha E(x)\right| \leqslant c|x|^{p-n-k}, \\ &\text{if } n \leqslant p, \quad \left|D^\alpha E(x)\right| \leqslant c|x|^{p-n-k}\log|x|. \end{aligned}$$

Proof. We first concentrate on the term E_0 of the decomposition (5.6). For all $(r, \upsilon) \in \mathbb{R}_+ \times \mathbb{R}^n$ we have

$$E_0(r\upsilon) = r^{p-n}E_0(\upsilon).$$

Differentiating k times with respect to υ in the direction $(\delta\upsilon_1, \dots, \delta\upsilon_k)$ gives

$$d^k E_0(r\upsilon)(r\delta\upsilon_1, \dots, r\delta\upsilon_k) = r^{p-n} d^k E_0(\upsilon)(\delta\upsilon_1, \dots, \delta\upsilon_k),$$

whereby

$$\left\|d^k E_0(r\upsilon)\right\| = r^{p-n-k}\left\|d^k E_0(\upsilon)\right\|.$$

Choosing now $x = r\upsilon$, with $|\upsilon| = 1$ yields

$$\left\|d^k E_0(x)\right\| = |x|^{p-n-k}\left\|d^k E_0(\upsilon)\right\| \leqslant c|x|^{p-n-k},$$

which in turn implies

$$\left|D^\alpha E_0(x)\right| \leqslant c|x|^{p-n-k} \quad \forall|\alpha| = k.$$

This provides the result for $n > p$. We now assume that $n \leqslant p$. Similarly to the previous calculation we obtain

$$\left|D^\alpha Q(x)\right| \leqslant c|x|^{p-n-k} \quad \forall|\alpha| = k. \tag{5.7}$$

Denoting by $\tilde{E}(x) = Q(x)\log|x|$, we have, for some coefficients $c_{\alpha\beta} \geqslant 0$,

$$D^\gamma \tilde{E}(x) = D^\gamma Q(x)\log|x| + \sum_{\substack{|\alpha|+|\beta|=|\gamma| \\ |\beta|\geqslant 1}} c_{\alpha\beta} D^\alpha Q(x) D^\beta\big(\log|x|\big).$$

Using (5.7) we get whenever $|\gamma| = k$ and $|x| > R$:

$$\left|D^\gamma \tilde{E}(x)\right| \leqslant c|x|^{p-n-k}\log|x| + c \sum_{\substack{|\alpha|+|\beta|=k \\ |\beta|\geqslant 1}} |x|^{p-n-|\alpha|}|x|^{-|\beta|} \leqslant c|x|^{p-n-k}\log|x|.$$

Thus we get the results. $\square$

We now state and prove one of the main results of our work, which allows to easily determine whether the polarization tensor associated to an elliptic problem of order $2m$ is degenerate in the sense of Definition 5.1.

Theorem 5.6. *Suppose that* $\operatorname{rank}(\Lambda) = 1$. *Then Assumption 5.2 is fulfilled. In consequence the polarization tensor admits the expression (5.5), hence it is degenerate.*

Proof. Let $V = (V_\beta)_{|\beta|=m} \in \operatorname{im}(\Lambda)$ be such that $V_{\bar\beta} \neq 0$ for some $\bar\beta$. Since $\dim(\operatorname{im}\Lambda) = 1$ we have $\operatorname{im}\Lambda = \operatorname{span}(V)$ and

$$\Lambda\left(\left(D^\alpha\Psi\right)_{|\alpha|=m}\right) = V \quad \Leftrightarrow \quad \sum_{|\alpha|=m} a_{\alpha\bar\beta} D^\alpha \Psi = V_{\bar\beta}. \tag{5.8}$$

For V defined by $V_\beta = \sum_{|\alpha|=m} a_{\alpha\beta}\zeta_{\alpha\gamma}$, γ fixed, a solution to the rightmost equality of (5.8) is given by

$$\Psi_\gamma = E * V_{\bar\beta},$$

where E is the fundamental solution of the operator $\sum_{|\alpha|=m} a_{\alpha\bar\beta} D^\alpha$. Let us show that the polynomial $P(\xi) = \sum_{|\alpha|=m} a_{\alpha\bar\beta}\xi^\alpha$ associated to this operator is elliptic. Thus, assume that $P(\xi) = 0$. For $|\alpha| = m$ we set $U_\alpha = \xi^\alpha$, and we define $U = (U_\alpha)_{|\alpha|=m}$. We have $[\Lambda(U)]_{\bar\beta} = P(\xi) = 0$, and since $\Lambda(U) \in \operatorname{span}(V)$ (i.e., $\Lambda(U) = \lambda V$ for some λ) with $V_{\bar\beta} \neq 0$, we infer $\Lambda(U) = 0$. Therefore, $\sum_{|\alpha|=m} a_{\alpha\beta}\xi^\alpha = 0$ for every β with $|\beta| = m$. Multiplying by ξ^β and summing over $|\beta| = m$, this implies $\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\xi^{\alpha+\beta} = 0$. By the uniform ellipticity assumption (2.4) we derive $\xi = 0$. Therefore, the fundamental solution of the operator $P(D) = \sum_{|\alpha|=m} a_{\alpha\bar\beta} D^\alpha$ satisfies Theorem 5.4. Using Corollary 5.5 with $p = m$, it is easily checked that $E \in W^m(\mathbb{R}^n)$ (defined in Appendix C). Hence $\Psi_\gamma \in W^m(\mathbb{R}^n)$ as well, and the proof is achieved. $\square$

6. Selected applications

6.1. Examples of operators

In this section we review some classical elliptic operators. As the polarization tensor only depends on the principal symbol, we restrict our presentation to homogeneous operators, i.e., we assume that $b_{\alpha\beta,\varepsilon} \equiv 0$. In order to check the uniform ellipticity condition, we set

$$P(\xi) = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\xi^{\alpha+\beta}.$$

6.1.1. In dimension $n = 1$

We have $|\alpha| = m \Rightarrow \alpha = (m)$, hence $q = 1$ and $\operatorname{rank}(\Lambda) = 1$ for every $m \geqslant 1$. This case is always degenerate and the topological asymptotic expansion for ω being the interval $(-1, 1)$ is given by

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = \varepsilon\left[2\hat{\rho}_0\left(1 - \frac{1}{r}\right)\frac{d^m u_0}{dx^m}(\hat{x})\frac{d^m v_0}{dx^m}(\hat{x}) + \delta J_1 + \delta J_2\right] + o(\varepsilon).$$

6.1.2. Laplacian in dimension $n \geqslant 2$

We have $m = 1$, hence $q = n$. Let $(e_i)_{i=1,\dots,n}$ be the canonical basis of $\mathbb{R}^n$. The bilinear form is

$$a_\varepsilon(u, v) = \int_\Omega \rho_\varepsilon \nabla u . \nabla v \, dx = \sum_{i=1}^{n} \int_\Omega \rho_\varepsilon D^{e_i} u D^{e_i} v \, dx,$$

hence $P(\xi) = |\xi|^2$. In the basis formed by the vectors $(e_i)_{i=1,\dots,n}$ the matrix of Λ is the identity matrix of order n, hence $\operatorname{rank}(\Lambda) = n$. This case is not degenerate. The polarization tensor is explicitly known for ellipses and ellipsoids, see e.g. [3,4].

6.1.3. Bi-Laplacian

For this operator we have $m = 2$ and the bilinear form is

$$a_\varepsilon(u, v) = \int_\Omega \rho_\varepsilon \Delta u \Delta v \, dx.$$

This yields $P(\xi) = |\xi|^4$. Let us first focus for simplicity on the dimension $n = 2$. Ordering the family $(\alpha \in \mathbb{N}^2, |\alpha| = 2)$ as $((2, 0), (0, 2), (1, 1))$ the matrix of Λ in the canonical basis of $\mathbb{R}^3$ is

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is immediately checked that $A \geqslant 0$ and $\operatorname{rank}(\Lambda) = \operatorname{rank}(A) = 1$. This case is thus degenerate. The same thing occurs in any dimension n, with, using a similar ordering, a matrix having as only nonzero coefficients an $n \times n$ upper left block of ones. Therefore we have in any dimension the topological asymptotic expansion:

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = \varepsilon^n\left[\hat{\rho}_0|\omega|\left(1 - \frac{1}{r}\right)\Delta u_0(\hat{x})\Delta v_0(\hat{x}) + \delta J_1 + \delta J_2\right] + o\big(\varepsilon^n\big).$$

6.1.4. Kirchhoff plate model

For this fourth order operator ($m = 2$) in dimension $n = 2$, the bilinear form is

$$a_\varepsilon(u, v) = k \int_\Omega \rho_\varepsilon(\lambda \Delta u \Delta v + 2\mu \nabla\nabla u : \nabla\nabla v) \, dx,$$

where $k = \tau^3/12$, $\tau > 0$ is the thickness of the plate, $\lambda, \mu \geqslant 0$ are the Lamé coefficients. This entails $P(\xi) = k(\lambda + 2\mu)|\xi|^4$, which is uniformly elliptic provided that either $\lambda > 0$ or $\mu > 0$. In the same basis as in the previous case, the matrix of Λ is

$$A = k \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 4\mu \end{pmatrix} \geqslant 0.$$

We find $\det A = 16k^3\mu^2(\lambda + \mu)$, hence $\operatorname{rank}(A) = 3$ provided that $\mu > 0$. The problem is thus non-degenerate. The polarization tensor for a circular inclusion has been obtained in [6].

6.2. Numerical illustrations

6.2.1. One-dimensional problem

We consider the compliance functional associated to a second order one-dimensional equation, where, for $0 < a < 1/2$, the source term f has the form:

$$f(x) = \begin{cases} 2, & 0 < x < a, \\ 0, & a \leqslant x < 1. \end{cases}$$

The cost functional associated to the unperturbed problem reads

$$J_0(u_0) = \int_0^1 f u_0 = 2\int_0^a u_0,$$

with u_0 solution to

$$\begin{cases} -u_0''(x) = 2, & 0 < x < a, \\ -u_0''(x) = 0, & a \leqslant x < 1, \\ u_0(0) = u_0(1) = 0. \end{cases}$$

The expression of u_0 can be easily shown to be

$$u_0(x) = \begin{cases} -x^2 - a^2x + 2ax, & 0 < x < a, \\ -a^2(x-1), & a \leqslant x < 1. \end{cases}$$

The compliance associated to the perturbed problem reads

$$J_\varepsilon(u_\varepsilon) = \int_0^1 f u_\varepsilon = 2\int_0^a u_\varepsilon,$$

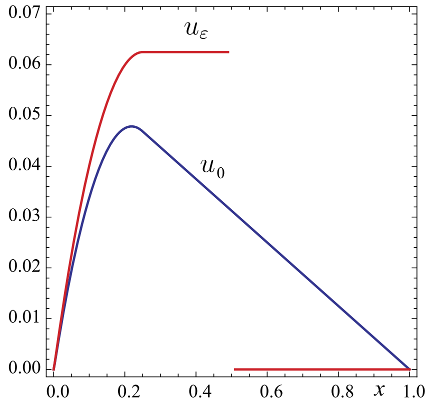


Fig. 1. One-dimensional case: Solutions $u_0(x)$ and $u_\varepsilon(x)$ for $a = 1/4$.

with u_ε solution to

$$\begin{cases} -u_\varepsilon''(x) = 2, & 0 < x < a, \\ -u_\varepsilon''(x) = 0, & a \leqslant x < \dfrac{1}{2} - \varepsilon, \\ -u_\varepsilon''(x) = 0, & \dfrac{1}{2} + \varepsilon < x < 1, \\ u_\varepsilon'\left(\dfrac{1}{2} - \varepsilon\right) = u_\varepsilon'\left(\dfrac{1}{2} + \varepsilon\right) = 0, & \\ u_\varepsilon(0) = u_\varepsilon(1) = 0. & \end{cases}$$

This means that the domain is topologically perturbed by the introduction of a hole of size 2ε, with homogeneous Neumann boundary condition. The explicit solution is given by

$$u_\varepsilon(x) = \begin{cases} -x^2 + 2ax, & 0 < x < a, \\ a^2, & a \leqslant x < \frac{1}{2} - \varepsilon, \\ 0, & \frac{1}{2} + \varepsilon < x < 1. \end{cases}$$

The solutions u_0 and u_ε are represented in Fig. 1 for $a = 1/4$. As we have already mentioned, this is a degenerate case. In fact, in this simple example the difference between $J_\varepsilon(u_\varepsilon)$ and $J_0(u_0)$ is explicitly given by a jump independent of ε, namely

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = a^4.$$

This means that the cost functional $J_\varepsilon(u_\varepsilon)$ is not continuous with respect ε. Hence, the topological derivative for a Neumann hole is not defined.

6.2.2. Bi-Laplacian operator

Let us consider three balls $B_1, B_a, B_\varepsilon \in \mathbb{R}^2$ with centers at the origin and radii 1, a and ε, respectively, such that $\varepsilon < a < 1$. We consider the compliance functional associated to the bi-Laplacian operator with source term

$$f(x) = \begin{cases} 8 & \text{if } x \in B_1 \setminus \overline{B_a}, \\ 0 & \text{if } x \in B_a. \end{cases}$$

The cost functional associated to the unperturbed problem reads

$$J_0(u_0) = \int_{B_1} f u_0 = 8 \int_{B_1 \setminus \overline{B_a}} u_0,$$

with u_0 solution to

$$\begin{cases} \Delta^2 u_0 = 8 & \text{in } B_1 \setminus \overline{B_a}, \\ \Delta^2 u_0 = 0 & \text{in } B_a, \\ \left.\begin{array}{l} u_0 = 0 \\ \partial_n u_0 = 0 \end{array}\right\} & \text{on } \partial B_1. \end{cases}$$

For the perturbed problem with a homogeneous Neumann condition on the boundary of a hole B_ε the cost functional is

$$J_\varepsilon(u_\varepsilon) = \int_{B_1} f u_\varepsilon = 8 \int_{B_1 \setminus \overline{B_a}} u_\varepsilon,$$

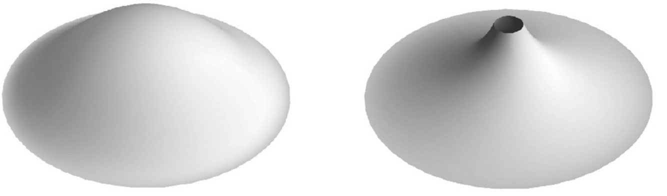
with u_ε solution to

$$\begin{cases} \Delta^2 u_\varepsilon = 8 & \text{in } B_1 \setminus \overline{B_a}, \\ \Delta^2 u_\varepsilon = 0 & \text{in } B_a \setminus \overline{B_\varepsilon}, \\ \left.\begin{array}{l} u_\varepsilon = 0 \\ \partial_n u_\varepsilon = 0 \end{array}\right\} & \text{on } \partial B_1, \\ \left.\begin{array}{l} \Delta u_\varepsilon = 0 \\ \partial_n \Delta u_\varepsilon = 0 \end{array}\right\} & \text{on } \partial B_\varepsilon. \end{cases}$$

Using a polar coordinate system (r, θ), we find analytical expressions for both u_0 and u_ε by separation of variables, as plotted in Fig. 2. Due to the axis-symmetry of the problems, their solutions can be written in terms of r only, as shown in Fig. 3. In this example the difference between $J_\varepsilon(u_\varepsilon)$ and $J_0(u_0)$ is again given by a jump independent of ε, namely

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = \pi \left(a^4 - 4a^2 \log a - 1\right)^2.$$

The cost functional $J_\varepsilon(u_\varepsilon)$ is not continuous with respect to ε at $\varepsilon = 0$, which confirms that this case is degenerate.



(a) solution u_0 (b) solution u_ε

Fig. 2. Bi-Laplacian case: solutions u_0 and $u_\varepsilon \leqslant 0.3$ for $a = 1/2$.

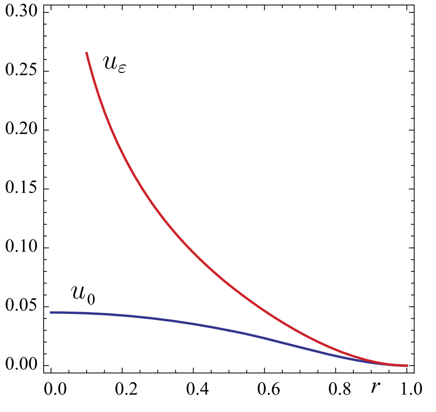


Fig. 3. Bi-Laplacian case: profile of the solutions $u_0(r)$ and $u_\varepsilon(r \geqslant 0.1)$ for $a = 1/2$.

7. Estimation of the remainders

This section is devoted to the proof of Lemma 4.4. We will use the letter c to denote a generic positive constant independent of ε.

7.1. Preliminary estimates

Recall that we have defined

$$\tilde{v}_\varepsilon = v_\varepsilon - v_0 \in H^m_0(\Omega).$$

We introduce the difference

$$e_\varepsilon = \tilde{v}_\varepsilon - h_\varepsilon \in H^m(\Omega)/\mathcal{P}_{m-1},$$

where $h_\varepsilon \in W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$ solves (4.16). Moreover, we set

$$\tilde{\rho}_\varepsilon(x) = \begin{cases} \rho_\varepsilon(x) & \text{if } x \in \Omega, \\ \hat{\rho}_0 & \text{if } x \in \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

Lemma 7.1. *Let $a > 0$ be such that the open ball of center $\hat{x}$ and radius a, denoted by B_a, is contained in Ω, and choose an arbitrary $\delta' \in (0, 1/2)$. For ε small enough we have that*

$$|h_\varepsilon|_{H^m(\mathbb{R}^n\setminus B_a)} \leqslant c\varepsilon^{n-\delta'}, \qquad |e_\varepsilon|_{H^m(\Omega)} \leqslant c\varepsilon^{n/2+\delta''}, \tag{7.1}$$

$$\|\tilde{v}_\varepsilon\|_{H^m(\Omega)} \leqslant c\varepsilon^{n/2}, \qquad \|\tilde{v}_\varepsilon\|_{H^m(\Omega\setminus B_a)} \leqslant c\varepsilon^{n/2+\delta''}, \qquad \|\tilde{v}_\varepsilon\|_{H^{m-1}(\Omega)} \leqslant c\varepsilon^{n/2+\delta''} \tag{7.2}$$

for some $\delta'' > 0$.

Proof. The proof is divided into five steps.

• *First step: estimation of* $|h_\varepsilon|_{H^m(\mathbb{R}^n)}$. By elliptic regularity applied to (4.31) (see Lemmas 3.4 and 3.5) we get $\|H_\varepsilon\|_{W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}} \leqslant c$, hence in particular

$$|H_\varepsilon|_{H^m(\mathbb{R}^n)} \leqslant c. \tag{7.3}$$

A change of variable results in

$$|h_\varepsilon|_{H^m(\mathbb{R}^n)} \leqslant c\varepsilon^{n/2}. \tag{7.4}$$

• *Second step: estimation of* $|h_\varepsilon|_{H^m(\mathbb{R}^n\setminus B_a)}$. We derive from (4.31) that, for all $\phi_\varepsilon \in W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$,

$$\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \hat{\rho}_0 a_{\alpha\beta} D^\alpha H_\varepsilon(y) D^\beta \phi_\varepsilon(y)\,dy \tag{7.5}$$

$$\begin{aligned}
&= \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \hat{\rho} a_{\alpha\beta} D^\alpha H_\varepsilon(y) D^\beta \phi_\varepsilon(y)\,dy + \sum_{|\alpha|=|\beta|=m} \int_\omega (\hat{\rho}_0 - \hat{\rho}_1) a_{\alpha\beta} D^\alpha H_\varepsilon(y) D^\beta \phi_\varepsilon(y)\,dy \\
&= (\hat{\rho}_0 - \hat{\rho}_1) \sum_{|\alpha|=|\beta|=m} \int_\omega a_{\alpha\beta} D^\alpha v_0(\hat{x}+\varepsilon y) D^\beta \phi_\varepsilon(y)\,dy \\
&\quad + (\hat{\rho}_0 - \hat{\rho}_1) \sum_{|\alpha|=|\beta|=m} \int_\omega a_{\alpha\beta} D^\alpha H_\varepsilon(y) D^\beta \phi_\varepsilon(y)\,dy
\end{aligned}$$

$$= (\hat{\rho}_0 - \hat{\rho}_1) \sum_{|\alpha|=|\beta|=m} \int_\omega a_{\alpha\beta} \big(D^\alpha v_0(\hat{x}+\varepsilon y) + D^\alpha H_\varepsilon(y)\big) D^\beta \phi_\varepsilon(y)\,dy. \tag{7.6}$$

We define the distribution $\mathcal{T}_\varepsilon \in \mathcal{D}'(\mathbb{R}^n)$ by

$$\langle \mathcal{T}_\varepsilon, \eta\rangle = (\hat{\rho}_0 - \hat{\rho}_1) \sum_{|\alpha|=|\beta|=m} \int_\omega a_{\alpha\beta} \big(D^\alpha v_0(\hat{x}+\varepsilon y) + D^\alpha H_\varepsilon(y)\big) D^\beta \eta(y)\,dy.$$

We therefore have, in the sense of distributions,

$$(-1)^m \sum_{|\alpha|=|\beta|=m} \hat{\rho}_0 a_{\alpha\beta} D^{\alpha+\beta} H_\varepsilon = \mathcal{T}_\varepsilon. \tag{7.7}$$

We call E the fundamental solution of the differential operator $(-1)^m \sum_{|\alpha|=|\beta|=m} \hat{\rho}_0 a_{\alpha\beta} D^{\alpha+\beta}$, whereby a solution of (7.7) is given by $H_\varepsilon^\bullet = \mathcal{T}_\varepsilon * E$. By elliptic regularity, since clearly

$\mathcal{T}_\varepsilon \in H^{-m}(\mathbb{R}^n)$, we have $H_\varepsilon^\bullet \in H^m_{loc}(\mathbb{R}^n)$. In addition, if $\operatorname{dist}(x,\omega) > 0$ we have the expressions

$$H_\varepsilon^\bullet(x) = (\hat\rho_0 - \hat\rho_1) \sum_{|\alpha|=|\beta|=m} \int_\omega a_{\alpha\beta}\big(D^\alpha v_0(\hat x + \varepsilon y) + D^\alpha H_\varepsilon(y)\big) D^\beta E(x-y)\,dy$$

and

$$D^\gamma H_\varepsilon^\bullet(x) = (\hat\rho_0 - \hat\rho_1) \sum_{|\alpha|=|\beta|=m} \int_\omega a_{\alpha\beta}\big(D^\alpha v_0(\hat x + \varepsilon y) + D^\alpha H_\varepsilon(y)\big) D^{\gamma+\beta} E(x-y)\,dy. \tag{7.8}$$

The Cauchy–Schwarz inequality applied to (7.8) implies

$$\big|D^\gamma H_\varepsilon^\bullet(x)\big| \leqslant c \sum_{|\beta|=m} \bigg(\int_\omega \big|D^{\gamma+\beta} E(x-y)\big|^2\,dy\bigg)^{1/2}.$$

By Corollary 5.5 with $p = 2m$ we infer that, for any $R > 2$, $\delta' \in (0,1/2)$ and $|\gamma| = m-k$ such that $\omega \subset B(0,R/2)$,

$$\big|D^\gamma H_\varepsilon^\bullet(x)\big| \leqslant c|x|^{k-n+\delta'} \quad \forall |x| > R. \tag{7.9}$$

This implies in particular that $H_\varepsilon^\bullet \in W^m(\mathbb{R}^n)$, and by uniqueness that $H_\varepsilon^\bullet$ is a representative for H_ε. Recalling that $h_\varepsilon(x) = \varepsilon^m H_\varepsilon(\varepsilon^{-1}(x-\hat x))$, we select the representative $h_\varepsilon^\bullet(x) = \varepsilon^m H_\varepsilon^\bullet(\varepsilon^{-1}(x-\hat x))$, hence

$$D^\gamma h_\varepsilon^\bullet(x) = \varepsilon^k D^\gamma H_\varepsilon^\bullet\big(\varepsilon^{-1}(x-\hat x)\big), \quad |\gamma| = m-k. \tag{7.10}$$

From (7.9) we derive

$$\begin{aligned}\big|D^\gamma h_\varepsilon(x)\big| &\leqslant c\varepsilon^k \big|\varepsilon^{-1}(x-\hat x)\big|^{k-n+\delta'} \\ &= c\varepsilon^{n-\delta'}|x-\hat x|^{k-n+\delta'} \quad \forall |x| > \varepsilon R,\ |\gamma| = m-k.\end{aligned} \tag{7.11}$$

Therefore, choosing an arbitrary $a > 0$, we obtain

$$\big\|h_\varepsilon^\bullet\big\|_{W^m(\mathbb{R}^n\setminus B_a)} \leqslant c\varepsilon^{n-\delta'} \tag{7.12}$$

for any ε small enough. In particular this yields $|h_\varepsilon|_{H^m(\mathbb{R}^n\setminus B_a)} \leqslant c\varepsilon^{n-\delta'}$.

- *Third step: estimation of* $\|h_\varepsilon\|_{H^{m-1}(\Omega)}$. From (7.9) we obtain, for $|\gamma| = m-k$,

$$\big\|D^\gamma H_\varepsilon^\bullet\big\|_{L^2(C(R,\varepsilon^{-1}R_0))} \leqslant c\varepsilon^{\frac{n}{2}-k-\delta'} + c \tag{7.13}$$

and

$$\big\|D^\gamma H_\varepsilon^\bullet\big\|_{L^2(C(R,2R))} \leqslant c,$$

where $C(a,b)$ stands for the ring of radii a and b. The above inequality together with (7.3) yields, thanks to the Poincaré inequality, that

$$\|H_\varepsilon^\bullet\|_{H^m(B_R)} \leqslant c. \tag{7.14}$$

Combining (7.13) and (7.14) we arrive at

$$\|D^\gamma H_\varepsilon^\bullet\|_{L^2(B_{\varepsilon^{-1}R_0})} \leqslant c + c\varepsilon^{\frac{n}{2}-k-\delta'}.$$

A change of variables provides

$$\|D^\gamma h_\varepsilon^\bullet\|_{L^2(\Omega)} \leqslant c\varepsilon^{k+\frac{n}{2}} + c\varepsilon^{n-\delta'}.$$

For $k \geqslant 1$, as $\delta' \in (0, n/2)$, the right hand side of the above inequality is of order $O(\varepsilon^{\frac{n}{2}+\delta''})$ for some $\delta'' > 0$. It follows that

$$\|h_\varepsilon^\bullet\|_{H^{m-1}(\Omega)} \leqslant c\varepsilon^{\frac{n}{2}+\delta''}. \tag{7.15}$$

- *Fourth step: estimation of* $|e_\varepsilon|_{H^m(\Omega)}$. We set

$$\langle \hat{A}_\varepsilon u, v\rangle_{H^{-m}(\Omega),H_0^m(\Omega)} = \sum_{|\alpha|=|\beta|=m} \int_\Omega \hat{\rho}_\varepsilon a_{\alpha\beta} D^\alpha u D^\beta v\, dx$$

so that, in view of (4.16) applied to the selected representative $h_\varepsilon^\bullet$ and for a test function $\eta \in H_0^m(\Omega)$ extended by 0, we have

$$\hat{A}_\varepsilon h_\varepsilon^\bullet = -(\hat{A}_\varepsilon - \hat{A}_0)v_0.$$

Recalling that

$$\left(A_\varepsilon + B_\varepsilon^*\right)v_\varepsilon = \left(A_0 + B_0^*\right)v_0 = -g,$$

we find

$$\left(A_\varepsilon + B_\varepsilon^*\right)\tilde{v}_\varepsilon = -\left(A_\varepsilon - A_0 + B_\varepsilon^* - B_0^*\right)v_0.$$

This entails, for $e_\varepsilon^\bullet = \tilde{v}_\varepsilon - h_\varepsilon^\bullet$,

$$\left(A_\varepsilon + B_\varepsilon^*\right)e_\varepsilon^\bullet = S_\varepsilon := -(A_\varepsilon - \hat{A}_\varepsilon)h_\varepsilon^\bullet - B_\varepsilon^* h_\varepsilon^\bullet - \left(A_\varepsilon - A_0 - \hat{A}_\varepsilon + \hat{A}_0 + B_\varepsilon^* - B_0^*\right)v_0. \tag{7.16}$$

In addition it holds $e_\varepsilon^\bullet = -h_\varepsilon^\bullet$ on $\partial\Omega$. By Proposition 3.3 and classical arguments of elliptic regularity and trace theory we infer that

$$\|e_\varepsilon^\bullet\|_{H^m(\Omega)} \leqslant c\left(\|S_\varepsilon\|_{H^{-m}(\Omega)} + \|h_\varepsilon^\bullet\|_{W^m(\mathbb{R}^n\setminus B_a)}\right).$$

Yet for every $\eta \in H_0^m(\Omega)$ we have

$$\begin{aligned}
&\langle S_\varepsilon, \eta\rangle_{H^{-m}(\Omega),H_0^m(\Omega)} \\
&\quad = - \sum_{|\alpha|=|\beta|=m} \int_\Omega (\rho_\varepsilon - \hat{\rho}_\varepsilon) a_{\alpha\beta} D^\alpha h_\varepsilon^\bullet D^\beta \eta \, dx - \sum_{\substack{|\alpha|\leqslant m \\ |\beta|\leqslant m-1}} \int_\Omega b_{\alpha\beta,\varepsilon} D^\alpha \eta D^\beta h_\varepsilon^\bullet \, dx \\
&\qquad - \sum_{|\alpha|=|\beta|=m} \int_\Omega (\rho_\varepsilon - \rho_0 - \hat{\rho}_\varepsilon + \hat{\rho}_0) a_{\alpha\beta} D^\alpha v_0 D^\beta \eta \, dx - \sum_{\substack{|\alpha|\leqslant m \\ |\beta|\leqslant m-1}} \int_{\omega_\varepsilon} q_{\alpha\beta} D^\alpha \eta D^\beta h_\varepsilon^\bullet \, dx.
\end{aligned}$$

Using (7.12), (7.15), and the fact that $\rho_\varepsilon - \rho_0 = \hat{\rho}_\varepsilon - \hat{\rho}_0$ for every ε small enough, we get

$$\langle S_\varepsilon, \eta\rangle_{H^{-m}(\Omega),H_0^m(\Omega)} \leqslant c\varepsilon^{\frac{n}{2}+\delta''} \|\eta\|_{H^m(\Omega)}.$$

Using once more (7.12) we arrive at

$$\left\|e_\varepsilon^\bullet\right\|_{H^m(\Omega)} \leqslant c\varepsilon^{\frac{n}{2}+\delta''}. \tag{7.17}$$

• *Fifth step: estimates on* $\tilde{v}_\varepsilon$. From (7.1) and $\tilde{v}_\varepsilon = h_\varepsilon^\bullet + e_\varepsilon^\bullet$ we derive $|\tilde{v}_\varepsilon|_{H^m(\Omega\setminus B_a)} \leqslant c\varepsilon^{\frac{n}{2}+\delta''}$. The Poincaré inequality entails $\|\tilde{v}_\varepsilon\|_{H^m(\Omega\setminus B_a)} \leqslant c\varepsilon^{\frac{n}{2}+\delta''}$. Likewise, (7.4) yields $\|\tilde{v}_\varepsilon\|_{H^m(\Omega)} \leqslant c\varepsilon^{\frac{n}{2}}$. We also derive from (7.15) and (7.17)

$$\|\tilde{v}_\varepsilon\|_{H^{m-1}(\Omega)} \leqslant c\varepsilon^{\frac{n}{2}+\delta''}. \tag{7.18}$$

All the estimates are now proven. □

We are now in a position to estimate the remainders $\mathcal{E}_i(\varepsilon)$, $i = 1, 2, 3, 4, 5$ of Lemma 4.3.

7.2. First remainder

Due to the assumed regularity of f and g, it follows that $D^\alpha u_0$ and $D^\alpha v_0$ are $\mathcal{C}^1$ in a neighborhood of $\hat{x}$. By the mean value inequality we arrive at

$$\left|\mathcal{E}_1(\varepsilon)\right| \leqslant c\varepsilon^{n+1}.$$

7.3. Second remainder

The Cauchy–Schwarz inequality entails

$$\mathcal{E}_2(\varepsilon) \leqslant c\varepsilon\sqrt{|\omega_\varepsilon|}|\tilde{v}_\varepsilon|_{H^m(\Omega)}.$$

Using Lemma 7.1 we straightforwardly get

$$\left|\mathcal{E}_2(\varepsilon)\right| \leqslant c\varepsilon^{n+1}.$$

7.4. Third remainder

The Cauchy–Schwarz inequality yields

$$\left|\mathcal{E}_3(\varepsilon)\right| \leqslant c\|1\|_{L^2(\omega_\varepsilon)}|\tilde{v}_\varepsilon|_{H^{m-1}(\Omega)}.$$

From Lemma 7.1 we infer

$$\left|\mathcal{E}_3(\varepsilon)\right| \leqslant c\varepsilon^{n+\delta''}.$$

7.5. Fourth remainder

The Cauchy–Schwarz inequality yields

$$\left|\mathcal{E}_4(\varepsilon)\right| \leqslant c\|1\|_{L^2(\omega_\varepsilon)}|e_\varepsilon|_{H^m(\Omega)}.$$

Then Lemma 7.1 entails

$$\left|\mathcal{E}_4(\varepsilon)\right| \leqslant c\varepsilon^{n+\delta''}.$$

7.6. Fifth remainder

We begin by observing that, subtracting (4.32) from (4.31), one gets for any $\phi \in W^m(\mathbb{R}^n)$

$$\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \hat{\rho}(y)a_{\alpha\beta}D^\alpha(H_\varepsilon - H)(y)D^\beta\phi\,dy$$
$$= -\sum_{|\alpha|=|\beta|=m} \int_{\omega} (\hat{\rho}_1 - \hat{\rho}_0)a_{\alpha\beta}\big(D^\alpha v_0(\hat{x}+\varepsilon y) - D^\alpha v_0(\hat{x})\big)D^\beta\phi(y)\,dy.$$

Applying the Cauchy–Schwarz inequality to the right hand side and using the $\mathcal{C}^1$ regularity of v_0 we get

$$\left|\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \hat{\rho}(y)a_{\alpha\beta}D^\alpha(H_\varepsilon - H)(y)D^\beta\phi\,dy\right| \leqslant c\varepsilon|\phi|_{H^m(\mathbb{R}^n)}.$$

By elliptic regularity (see Lemma 3.4) we infer that

$$|H_\varepsilon - H|_{H^m(\mathbb{R}^n)} \leqslant c\varepsilon.$$

This implies by the Cauchy–Schwarz inequality that

$$\left|\mathcal{E}_5(\varepsilon)\right| \leqslant c\varepsilon^{n+1}.$$

8. Generalization to elliptic systems

We explain here how the previous results can be generalized to the differential systems. We restrict ourselves to homogeneous differential operators merely for notational simplicity. We concentrate on the main changes, that is, the expression of the polarization tensor.

8.1. General case

We consider now a vector field $u_\varepsilon = (u_\varepsilon^1, \dots, u_\varepsilon^N) \in H_0^m(\Omega)^N$ solution of

$$\sum_{ij} \sum_{\alpha,\beta} \int_\Omega \rho_\varepsilon a_{\alpha\beta}^{ij} D^\alpha u_\varepsilon^i D^\beta \eta^j \, dx = \sum_i \langle f^i, \eta^i \rangle \quad \forall \eta \in H_0^m(\Omega)^N.$$

By convention Latin indices, written as superscripts, range over the set $\{1, \dots, N\}$, whereas Greek multi-indices are of length n. The system coefficients $(a_{\alpha\beta}^{ij})$ are supposed to satisfy the following properties.

- Symmetry: it holds for every α, β, i, j

$$a_{\alpha\beta}^{ij} = a_{\beta\alpha}^{ji}. \tag{8.1}$$

- Positivity: for any family of real numbers (y_α^i) it holds

$$\sum_{ij} \sum_{\alpha\beta} a_{\alpha\beta}^{ij} y_\alpha^i y_\beta^j \geqslant 0. \tag{8.2}$$

- Uniform ellipticity: there exists $\kappa > 0$ such that

$$\sum_{ij} \sum_{\alpha\beta} a_{\alpha\beta}^{ij} \xi^{\alpha+\beta} z_i \bar{z}_j \geqslant \kappa |\xi|^{2m} \sum_i |z_i|^2 \quad \forall (\xi, z) \in \mathbb{R}^n \times \mathbb{C}^N. \tag{8.3}$$

Based on these assumptions the asymptotic analysis can be easily generalized, which is left to the reader. This leads to define the function $H = (H^1, \dots, H^N)$, instead of (4.32), as the solution of

$$\begin{aligned} &\sum_{ij} \sum_{\alpha\beta} \int_{\mathbb{R}^n} \hat{\rho}(y) a_{\alpha\beta}^{ij} D^\alpha H^i(y) D^\beta \Phi^j(y) \, dy \\ &\quad = -(\hat{\rho}_1 - \hat{\rho}_0) \sum_{ij} \sum_{\alpha\beta} \int_\omega a_{\alpha\beta}^{ij} D^\alpha v_0^i(\hat{x}) D^\beta \Phi^j(y) \, dy \end{aligned} \tag{8.4}$$

for every family of functions $\Phi^1, \dots, \Phi^N \in W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$. By linearity, we have

$$H^i = \sum_l \sum_\gamma D^\gamma v_0^l(\hat{x}) \Psi_\gamma^{il}, \tag{8.5}$$

with $\Psi_\gamma^{il} \in W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$ solving, for each γ, l,

$$\sum_{ij}\sum_{\alpha\beta}\int_{\mathbb{R}^n} \hat\rho(y) a_{\alpha\beta}^{ij} D^\alpha \Psi_\gamma^{il}(y) D^\beta \Phi^j(y)\,dy = -(\hat\rho_1-\hat\rho_0)\sum_j\sum_\beta\int_\omega a_{\gamma\beta}^{lj} D^\beta\Phi^j(y)\,dy \tag{8.6}$$

for all $\Phi^1,\dots,\Phi^N \in W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$. The polarization tensor $(p_{\alpha\beta}^{ij})$ is defined by

$$p_{\alpha\beta}^{ij} = |\omega|(r-1)a_{\alpha\beta}^{ij} + k_{\alpha\beta}^{ij}, \tag{8.7}$$

with

$$k_{\alpha\gamma}^{il} = (r-1)\sum_j\sum_\beta a_{\alpha\beta}^{ij}\int_\omega D^\beta \Psi_\gamma^{jl}(y)\,dy. \tag{8.8}$$

Theorem 8.1. *Suppose that the cost function J_ε is such that (4.5) and (4.6) hold true for $\varphi(\varepsilon)=\varepsilon^n$ and $g_\varepsilon = g$ independent of ε, and that f, g are of regularity H^s in a neighborhood of $\hat x$, $s > \max(0, \frac{n}{2}+1-m)$. Then we have*

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = \varepsilon^n\left[\hat\rho_0\sum_{ij}\sum_{\alpha\beta} p_{\alpha\beta}^{ij} D^\alpha u_0^i(\hat x) D^\beta v_0^j(\hat x) + \delta J_1 + \delta J_2\right] + o\big(\varepsilon^n\big).$$

8.2. Degenerate case

We introduce the family of piecewise constant functions $\zeta_{\alpha\gamma}^{il} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\zeta_{\alpha\gamma}^{il}(x) = \begin{cases} -\frac{\hat\rho_1-\hat\rho_0}{\hat\rho_1}\delta_{\alpha\gamma}^{il} & \text{if } x\in\omega,\\ 0 & \text{if } x \in \mathbb{R}^n\setminus\omega, \end{cases} \tag{8.9}$$

with $\delta_{\alpha\gamma}^{il} = 1$ if $\alpha=\gamma$ and $i=l$, $\delta_{\alpha\gamma}^{il}=0$ otherwise. Assumption 5.2 is modified as follows.

Assumption 8.2. For any γ, l, there exist functions $\Psi_\gamma^{il} \in W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$ satisfying

$$\sum_i\sum_\alpha a_{\alpha\beta}^{ij} D^\alpha \Psi_\gamma^{il} = \sum_i\sum_\alpha a_{\alpha\beta}^{ij}\zeta_{\alpha\gamma}^{il} \quad \forall\beta, j. \tag{8.10}$$

We arrive at the following expression of the polarization tensor.

Proposition 8.3 *(Degenerate polarization tensor). If Assumption 8.2 is fulfilled, then we have*

$$p_{\alpha\gamma}^{il} = |\omega|\left(1-\frac{1}{r}\right)a_{\alpha\gamma}^{il}. \tag{8.11}$$

9. Conclusion

In this work we have derived the general form of the topological asymptotic expansion for a wide range of linear elliptic operators of order $2m$. We have also identified a class of degenerate problems, for which the closed formulation of the polarization tensor has been obtained. We have given a simple algebraic criterion to recognize the degenerate cases, and we have shown that a typical example of degenerate operator is the bi-Laplacian. As a consequence, the physical models whose state equations obey a PDE involving the bi-Laplacian will exhibit peculiar nucleation properties. By nucleation it is here meant changes of the physical properties of the body by removing and adding infinitesimal quantities of different materials with a view to the minimization of a cost function, usually taken as the energy of the model. As an example, heterogeneities in an elastic continuum can be modeled as small strain gradient perturbations, which in the scalar setting would mean an energy comprising a term of the form $\epsilon|\Delta u|^2$, with u standing for the displacement and ϵ a small parameter, and a state equation thereby involving Δ^2 (see [15] where such a problem is treated in the framework of homogenization). Another example is provided by time-dependent phase-change models involving Cahn–Hilliard type equations; a recent application in geology which could also fit our setting has been numerically studied in [21]. Finally we mention that the dislocation problem [26] which involves the bi-Laplacian (as a simplified model for the incompatibility operator appearing in elasticity of dislocated elastic bodies) will be further analyzed with a view to the results developed in the present paper.

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Appendix A. Proof of Lemma 3.1

We use the standard notation $\widehat{u}$ for the Fourier–Plancherel transform of $u \in L^2(\mathbb{R}^n)$, $\bar{z}$ for the complex conjugate of z and $|z|$ for the modulus of z.

By density, we can assume that u belongs to $\mathcal{D}(\Omega)$, the set of compactly supported and infinitely differentiable functions defined in Ω. Next we extend u by zero outside Ω. Due to the positivity assumption, the function $\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^\alpha u D^\beta u$ is nonnegative. Thus, with $\underline{\rho} = \min(\operatorname{essinf}_\Omega \rho_0, \hat{\rho}_1)$, we have

$$\langle A_\varepsilon u, u\rangle_{H^{-m}(\Omega), H_0^m(\Omega)} \geqslant \underline{\rho} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \int_\Omega D^\alpha u D^\beta u \, dx.$$

Passing to the Fourier transform, we have by the Parseval equality

$$\langle A_\varepsilon u, u\rangle_{H^{-m}(\Omega), H_0^m(\Omega)} \geqslant \underline{\rho} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \int_{\mathbb{R}^n} \widehat{D^\alpha u}\, \overline{\widehat{D^\beta u}} \, d\xi = \underline{\rho} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \int_{\mathbb{R}^n} \xi^{\alpha+\beta} |\widehat{u}|^2 \, d\xi.$$

The uniform ellipticity assumption yields

$$\langle A_\varepsilon u, u\rangle_{H^{-m}(\Omega),H_0^m(\Omega)} \geqslant \underline{\rho}\kappa \int\limits_{\mathbb{R}^n} |\xi|^{2m} |\widehat{u}|^2 \, d\xi.$$

The expansion of $|\xi|^{2m} = (\sum_{i=1}^n \xi_i^2)^m$ results in an expression of the form

$$|\xi|^{2m} = \sum_{|\alpha|=m} c_\alpha \xi^{2\alpha}$$

for some coefficients $c_\alpha \geqslant \underline{c} > 0$. This entails

$$\langle A_\varepsilon u, u\rangle_{H^{-m}(\Omega),H_0^m(\Omega)} \geqslant \underline{\rho}\kappa \sum_{|\alpha|=m} c_\alpha \int\limits_{\mathbb{R}^n} \left|\xi^\alpha \widehat{u}\right|^2 d\xi \geqslant \underline{\rho}\kappa\underline{c} \sum_{|\alpha|=m} \int\limits_{\mathbb{R}^n} \left|\xi^\alpha \widehat{u}\right|^2 d\xi.$$

Using again the Parseval equality leads to

$$\langle A_\varepsilon u, u\rangle_{H^{-m}(\Omega),H_0^m(\Omega)} \geqslant \underline{\rho}\kappa\underline{c} \sum_{|\alpha|=m} \int\limits_{\Omega} \left|D^\alpha u\right|^2 dx = \underline{\rho}\kappa\underline{c} |u|^2_{H^m(\Omega)}.$$

Appendix B. Collectively compact operators

Let X be a Banach space and $\mathcal{K}$ be a subset of $\mathcal{L}(X)$, where $\mathcal{L}(X)$ is the set of bounded linear operators from X into itself. We say that $\mathcal{K}$ is collectively compact if the set $\{Kx,\ x \in X,\ \|x\| \leqslant 1,\ K \in \mathcal{K}\}$ is relatively compact. The following result is a corollary of Theorem 1.6 of [9]. A proof can be found in [5].

Theorem B.1. *Let $\mathcal{K}$ be a collectively compact set of bounded linear operators of X. Assume further that $\mathcal{K}$ is pointwise sequentially compact, i.e., for every sequence (K_n) of $\mathcal{K}$ there exists a subsequence (K_{n_p}) and $K \in \mathcal{K}$ such that $K_{n_p}x \to Kx$ for all $x \in X$. If $I - K$ is invertible for all $K \in \mathcal{K}$, then*

$$\sup_{K\in\mathcal{K}} \left\|(I-K)^{-1}\right\| < \infty. \tag{B.1}$$

Appendix C. Weighted and quotient Sobolev spaces

In this appendix we define the functional spaces which provide existence theorems in $\mathbb{R}^n$. The main result is found in Corollary C.5 which is the restatement of Lemma 3.5. Before arriving at this result several preliminary lemmas must be proved.

Let B_a be the open ball centered at the origin and of radius a. We will denote by $r = |x|$ the radial coordinate.

Lemma C.1. *Let $a > 0$, $B'_a = \mathbb{R}^n \setminus \overline{B_a}$ and $q \in (-\infty, 1]$. If $2q + n \neq 0$, then it holds for all $u \in \mathcal{D}(B'_a)$*

$$\left\|r^q u\right\|_{L^2(B'_a)} \leqslant \frac{2}{|2q+n|} \left\|r^{q+1} \nabla u\right\|_{L^2(B'_a)}.$$

Proof. Let $\upsilon \in S_n$, the unit sphere of $\mathbb{R}^n$. Integrating by parts yields

$$\int_a^\infty r^{2q+n-1} u(r\upsilon)^2\,dr = -\frac{2}{2q+n}\int_a^\infty r^{2q+n} u(r\upsilon)\nabla u(r\upsilon).\upsilon\,dr$$
$$\leqslant \frac{2}{2q+n}\int_a^\infty r^{2q+n}\big|u(r\upsilon)\big|\big|\nabla u(r\upsilon)\big|\,dr.$$

We obtain by the Cauchy–Schwarz inequality

$$\int_a^\infty r^{2q+n-1} u(r\upsilon)^2\,dr \leqslant \frac{2}{2q+n}\left(\int_a^\infty r^{2q+n-1}\big|u(r\upsilon)\big|^2 dr\right)^{1/2}\left(\int_a^\infty r^{2q+n+1}\big|\nabla u(r\upsilon)\big|^2\,dr\right)^{1/2}.$$

This implies

$$\int_a^\infty r^{2q+n-1} u(r\upsilon)^2\,dr \leqslant \left(\frac{2}{2q+n}\right)^2\int_a^\infty r^{2q+n+1}\big|\nabla u(r\upsilon)\big|^2\,dr.$$

Next, we have

$$\begin{aligned}\|r^q u\|^2_{L^2(B'_a)} &= \int_{S_n}\int_a^\infty r^{2q+n-1}u(r\upsilon)^2\,dr\,d\upsilon \\ &\leqslant \left(\frac{2}{2q+n}\right)^2\int_{S_n}\int_a^\infty r^{2q+n+1}\big|\nabla u(r\upsilon)\big|^2\,dr\,d\upsilon \\ &= \left(\frac{2}{2q+n}\right)^2\big\|r^{q+1}\nabla u\big\|^2_{L^2(B'_a)},\end{aligned}$$

which leads to the desired result. □

Let $\delta \in (0, 1/2)$ be fixed. For every $k \in \mathbb{N}$ we introduce the weight functions as follows:

$$w_k(x) = \left(1+|x|^2\right)^{\frac{p_k}{2}} \tag{C.1}$$

with

$$p_k = \begin{cases} 0 & \text{if } k = 0, \\ -k-\delta & \text{if } k \geqslant 1 \text{ and } n = 1, \\ 1-\frac{n}{2}-k-\delta & \text{if } k \geqslant 1 \text{ and } n \geqslant 2. \end{cases}$$

Lemma C.2. *Let $a>0$, $B'_a=\mathbb{R}^n\setminus\overline{B_a}$, $m\in\mathbb{N}$. For all $k=0,\dots,m$ and every $u\in\mathcal{D}(B'_a)$, we have*

$$\sup_{|\alpha|=m-k}\left\|w_k D^\alpha u\right\|_{L^2(B'_a)}\leqslant c_k\sup_{|\alpha|=m}\left\|D^\alpha u\right\|_{L^2(B'_a)},$$

where c_k is a positive constant.

Proof. The result is obvious for $k=0$, thus we assume that $k\geqslant 1$. We treat first the case $n\geqslant 2$. By induction from Lemma C.1, we infer for $|\beta|=m-k$,

$$\left\|r^{1-\frac{n}{2}-k-\delta}D^\beta u\right\|_{L^2(B'_a)}\leqslant c\sup_{|\alpha|=m}\left\|r^{1-\frac{n}{2}-\delta}D^\alpha u\right\|_{L^2(B'_a)}\leqslant ca^{1-\frac{n}{2}-\delta}\sup_{|\alpha|=m}\left\|D^\alpha u\right\|_{L^2(B'_a)}.$$

The desired estimate follows straightforwardly. Suppose now that $n=1$. Again by induction from Lemma C.1, we obtain for $|\beta|=m-k$

$$\left\|r^{-k-\delta}D^\beta u\right\|_{L^2(B'_a)}\leqslant c\sup_{|\alpha|=m}\left\|r^{-\delta}D^\alpha u\right\|_{L^2(B'_a)}\leqslant ca^{-\delta}\sup_{|\alpha|=m}\left\|D^\alpha u\right\|_{L^2(B'_a)},$$

leading to the result. □

For any open subset A of $\mathbb{R}^n$ we define the space

$$W^m(A)=\left\{u\in\mathcal{D}'(A)\;\middle|\;\forall k=0,\dots,m,\ |\alpha|=m-k\Rightarrow w_k D^\alpha u\in L^2(A)\right\},\tag{C.2}$$

where the weights are given by (C.1). It is endowed with the norm

$$\|u\|_{W^m(A)}=\left[\sum_{k=0}^{m}\sum_{|\alpha|=m-k}\left\|w_k D^\alpha u\right\|^2_{L^2(A)}\right]^{1/2}.$$

This norm is associated with an inner product $\langle .,.\rangle_{W^m(A)}$, for which it is easily shown that $W^m(A)$ is a Hilbert space.

We define $W^m_0(A)$ as the closure of $\mathcal{D}(A)$ in $W^m(A)$. Let $u\in W^m(B'_a)$ and η be a smooth function such that $\eta=1$ in B_{2a} and $\eta=0$ in B'_{3a}. Then $\eta u\in H^m(B'_a)$ and $(1-\eta)u\in W^m(\mathbb{R}^n)$. This allows us to define the normal trace of u on ∂B_a of order j, $j\leqslant m-1$, denoted by $\partial^j_n u$. Also, one may prove by standard arguments (see e.g. [1]) that $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^m(\mathbb{R}^n)$. This implies the following result.

Lemma C.3. *We have*

$$W^m_0\left(B'_a\right)=\left\{u\in W^m\left(B'_a\right)\;\middle|\;\partial^j_n u=0\ \forall j=0,\dots,m-1\right\}.$$

Proposition C.4. *Let $\mathcal{H}$ be a closed subspace of $W^m(\mathbb{R}^n)$ and $\|.\|_{\mathcal{H}}$ be a norm on $\mathcal{H}$ such that, for some constants $c_1,c_2>$, it holds*

$$c_1|u|_{H^m(\mathbb{R}^n)}\leqslant\|u\|_{\mathcal{H}}\leqslant c_2\|u\|_{W^m(\mathbb{R}^n)}\qquad\forall u\in\mathcal{H}.$$

Then, on the space $\mathcal{H}$, the norms $\|.\|_{\mathcal{H}}$ and $\|.\|_{W^m(\mathbb{R}^n)}$ are equivalent.

Proof. We must show that there exists a constant c_3 such that

$$\|u\|_{W^m(\mathbb{R}^n)} \leqslant c_3 \|u\|_{\mathcal{H}}.$$

By contradiction, we assume that there exists a sequence $(u_p) \in \mathcal{H}$ such that, for every p,

$$\|u_p\|_{W^m(\mathbb{R}^n)} = 1, \qquad \|u_p\|_{\mathcal{H}} < \frac{1}{p}. \tag{C.3}$$

Let η be defined as above. The Leibniz formula provides, for all $v \in W^m(B'_a)$,

$$|\eta v|_{H^m(B_{3a})} \leqslant c|v|_{H^m(B_{3a})} + c\|v\|_{H^{m-1}(B_{3a})}, \tag{C.4}$$

$$\left|(1-\eta)v\right|_{H^m(B'_{2a})} \leqslant c|v|_{H^m(B'_{2a})} + c\|v\|_{H^{m-1}(B_{3a})}. \tag{C.5}$$

The embedding of $H^m(B_{3a})$ into $H^{m-1}(B_{3a})$ is compact and the sequence (u_p) is bounded in $H^m(B_{3a})$. We still denote by (u_p) a subsequence such that $u_p \to w$ in $H^{m-1}(B_{3a})$. Using Lemma C.2, (C.5) and the assumptions we get

$$\left\|(1-\eta)(u_p - u_q)\right\|_{W^m(B'_{2a})} \leqslant c\|u_p - u_q\|_{\mathcal{H}} + c\|u_p - u_q\|_{H^{m-1}(B_{3a})}.$$

Moreover, the Poincaré inequality in $H^m_0(B_{3a})$ together with (C.4) and the assumptions yield

$$\left\|\eta(u_p - u_q)\right\|_{H^m(B_{3a})} \leqslant c\|u_p - u_q\|_{\mathcal{H}} + c\|u_p - u_q\|_{H^{m-1}(B_{3a})}.$$

Therefore, (ηu_p) and $((1-\eta)u_p)$ are Cauchy sequences in $H^m(B_{3a})$ and $W^m(B'_{2a})$, respectively. Thus, there exist $(v_1, v_2) \in H^m(B_{3a}) \times W^m(B'_{2a})$ such that $\eta u_p \to v_1$ in $H^m(B_{3a})$ and $(1-\eta)u_p \to v_2$ in $W^m(B'_{2a})$. After summation, we infer $u_p \to v := v_1 + v_2$ in $W^m(\mathbb{R}^n)$. By assumption, this limit holds also in $\mathcal{H}$. From (C.3) we obtain a contradiction. □

Let $\mathcal{P}_{m-1}$ be the space of polynomials of degree not greater than $m-1$. It is easily checked that $\mathcal{P}_{m-1}$ is a subspace of $W^m(\mathbb{R}^n)$. The quotient space $W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$ is endowed with the norm

$$u \mapsto \|u\|_{W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}} = \inf_{p \in \mathcal{P}_{m-1}} \|u + p\|_{W^m(\mathbb{R}^n)}, \tag{C.6}$$

where u is an arbitrary representative of its class. Proposition C.4 implies that the seminorm $|u|_{H^m(\mathbb{R}^n)}$ is an equivalent norm to $\|u\|_{W^m(\mathbb{R}^n)}$ on $W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$. This will be made clear in the following corollary, which is a restatement of Lemma 3.5, and whose proof is now given.

Corollary C.5. *There exists $c > 0$ such that, for all $u \in W^m(\mathbb{R}^n)$,*

$$\|u\|_{W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}} \leqslant c|u|_{H^m(\mathbb{R}^n)}.$$

Proof. By standard arguments of the calculus of variations the infimum in (C.6) is attained at a unique point. The Euler–Lagrange equation applied to the problem with squared norm reads for the minimizer $v := u + p$

$$\langle v, \tilde{p} \rangle_{W^m(\mathbb{R}^n)} = 0 \quad \forall \tilde{p} \in \mathcal{P}_{m-1},$$

which is equivalent to

$$\left\langle v, x^\alpha \right\rangle_{W^m(\mathbb{R}^n)} = 0 \quad \forall |\alpha| \leqslant m - 1. \tag{C.7}$$

Therefore, $W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$ can be identified with the subspace of the functions $v \in W^m(\mathbb{R}^n)$ satisfying (C.7). In addition, the seminorm $|.|_{H^m(\mathbb{R}^n)}$ is a norm on this space. We then apply Proposition C.4 with $\mathcal{H} = W^m(\mathbb{R}^n)/\mathcal{P}_{m-1}$ and $\|u\|_{\mathcal{H}} := |u|_{H^m(\mathbb{R}^n)}$. □

Remark C.6. It appears from inspection of the proof that Corollary C.5 remains true if $\mathbb{R}^n$ is replaced by any connected open set containing the origin.

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