What is the shape of an air bubble on a liquid surface?

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Some applications of liquid foams...

...involve bubbles at the liquid-air interface:

- Beer and sparkling wines

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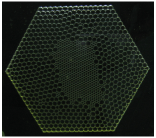
- Household cleaning products

- Firefighting

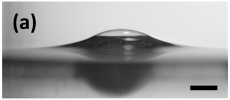
- Bubble rafts

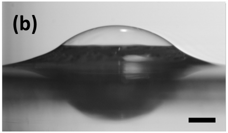


In all of these it is crucial whether bubbles stick together or fall apart.

All bubbles great and small

$r = 1$ mm,
$\theta_b = 25.7^\circ$

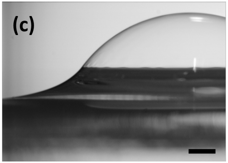




$r = 2.1$ mm,
$\theta_b = 38.5^\circ$

Scale bar: 1 mm

$r = 4.2$ mm,
$\theta_b = 52.5^\circ$



$r = 32.5$ mm,
$\theta_b = 83.9^\circ$

- Small bubbles are mostly immersed in the liquid, with only a small lenticular film emerging.
- Large bubbles are almost perfect hemispheres protruding from the liquid surface.

Aims and strategy

- Find the shape of a bubble lying on a surface of the same liquid.
- This is important for adhesion, boiling, emulsions, wetting. . .
- Whole range of bubble sizes, from small to large.
- As analytically as possible – no specialised numerics.
- Early attempts for small (Nicolson) or 2d (Howell) bubbles only, or large- and small-bubble limits (Aybers).
- Validate our results by comparing with Surface Evolver calculations and experimental data.

Theory: the physics of bubble shape

- Bubble shape is determined by balance of gravity and surface tension, so a key dimensionless quantity is the Bond number:

$$\mathrm{Bo} = \frac{\rho g R^2}{\gamma}$$

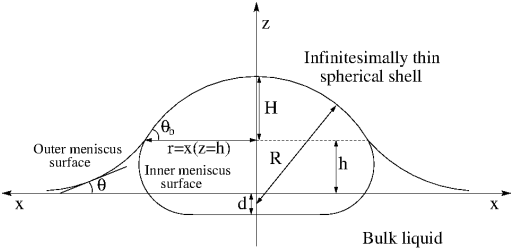
  - Small Bo: surface tension dominates, small bubbles
  - Large Bo: gravity dominates, small menisci
- Relevant lengthscale is the capillary length:

$$\lambda_c = \left(\frac{\gamma}{\rho g}\right)^{1/2}$$

- Use Bo as expansion parameter and λ_c as the unit of length (already identified by Bakker, Blaisdell, Bashforth and Adams, Aybers...).

Theory: model bubble and the Young-Laplace equation

Young-Laplace equation for meniscus around an axisymmetric bubble: curvature = pressure difference across surface



$$\left[1+\left(\frac{dx}{dz}\right)^2\right]^{-3/2}\left[-\frac{d^2x}{dz^2}+\frac{1+\left(\frac{dx}{dz}\right)^2}{x}\right]=\frac{\Delta p}{\gamma}$$

with boundary conditions:

1. $x(z=0)=+\infty$ or $z(x=+\infty)=0$ (outer surface); $dz/dx(x=0)=0$ (inner surface)
2. inner and outer meniscus surfaces must meet tangentially.

No exact analytical solution is known!

Theory: solving the Young-Laplace equation

- Rewrite equation in terms of film inclination θ.
- Use z as the dependent variable and θ as the independent variable.
- Non-dimensionalise all lengths by λ_c and introduce Bo.
- Assume solution to be of the form

$$z' = z_0' + z_1' \mathrm{Bo}^{-1/2} + z_2' \mathrm{Bo}^{-1} + \dots$$

- Systematically solve for z_i' up to desired order of approximation. This is done for the outer and inner meniscus surfaces.
- Get full bubble shape from

$$\Delta x' = \int_{\theta_b}^{\theta} \cot\theta' \frac{dz'}{d\theta'} d\theta'$$

Theory: the non-dimensionalised Young-Laplace equation

- Outer surface:

$$\left(z' + \mathrm{Bo}^{-1/2}\frac{\sin\theta}{\sin\theta_b + \mathrm{Bo}^{-1/2}\int_\theta^{\theta_b}\cot\theta' \frac{dz'}{d\theta'}d\theta'}\right)\frac{dz'}{d\theta} = \sin\theta$$

- Inner surface:

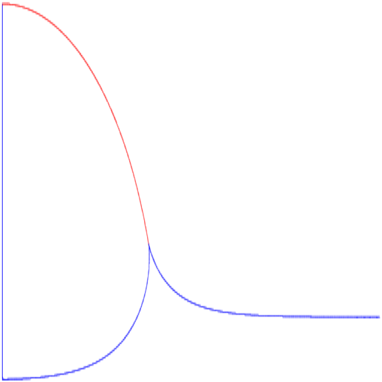
$$\left(4\mathrm{Bo}^{-1/2} + z' - \mathrm{Bo}^{-1/2}\frac{\sin\theta}{\sin\theta_b + \mathrm{Bo}^{-1/2}\int_\theta^{\theta_b}\cot\theta' \frac{dz'}{d\theta'}d\theta'}\right)\frac{dz'}{d\theta} = -\sin\theta$$

Theory: solutions of Young-Laplace equation

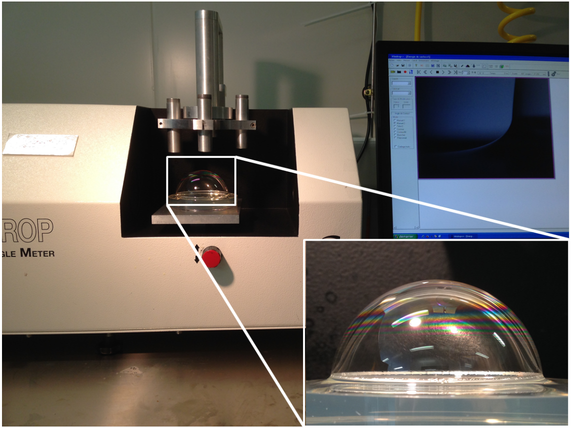
Order	Outer meniscus surface	Inner meniscus surface
z_0'	$\sqrt{2}\,(1-\cos\theta)^{1/2}$	$\sqrt{2}\,(1+\cos\theta)^{1/2}$
z_1'	$-\dfrac{4\left(1-\cos^3\frac{\theta}{2}\right)}{3\sqrt{2}\sin\theta_b\,(1-\cos\theta)^{1/2}}$	$-4+\dfrac{4\left(1-\sin^3\frac{\theta}{2}\right)}{3\sqrt{2}\sin\theta_b\,(1+\cos\theta)^{1/2}}$
z_2'	$\dfrac{1}{z_0'}\displaystyle\int_0^{\theta}\dfrac{\sin\theta'}{z_0'} \times\left[\dfrac{\sin\theta'}{\sin^2\theta_b}\left(\int_{\theta'}^{\theta_b}\dfrac{\cos\theta''}{z_0'}d\theta''\right)+\dfrac{1}{z_0'}\left(z_1'+\dfrac{\sin\theta'}{\sin\theta_b}\right)^2\right]d\theta'$	$\dfrac{1}{z_0'}\displaystyle\int_{\pi}^{\theta}\dfrac{\sin\theta'}{z_0'} \times\left[-\dfrac{\sin\theta'}{\sin^2\theta_b}\left(\int_{\theta'}^{\theta_b}\dfrac{\cos\theta''}{z_0'}d\theta''\right)+\dfrac{1}{z_0'}\left(4+z_1'-\dfrac{\sin\theta'}{\sin\theta_b}\right)^2\right]d\theta'$

Simulation: Surface Evolver (SE)

- Discretises bubble and performs direct numerical minimisation of surface energy for a fixed bubble volume.
- Angle between the two air-liquid interfaces at top of meniscus is $5^\circ \neq 0$ in order to avoid numerical problems. Doubling this leads to differences of at most 9%.

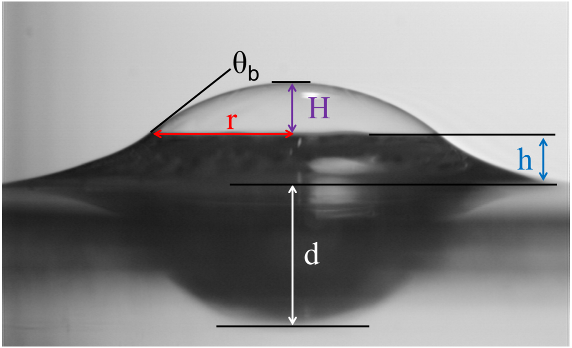


Experimental set-up



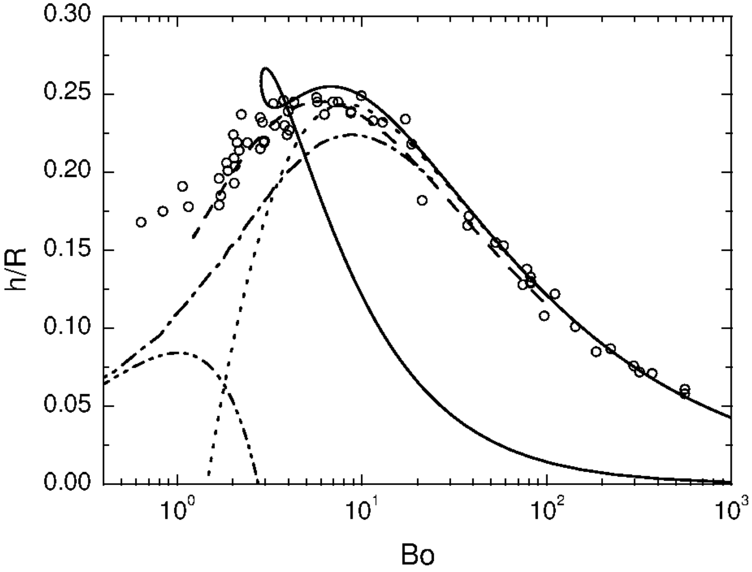
- Contact angle meter (GBX Scientific Instruments, France).
- Commercially available soap solution (Pustefix, Germany), surface tension $\gamma = 28.2 \pm 0.3$ mJ m^{-2}, $\lambda_c \approx 1.7$ mm.
- Air-filled pendant soap bubbles generated with pipette, then placed on liquid surface. Base radii in range 0.5–40 mm.

Experimentally-measured quantities

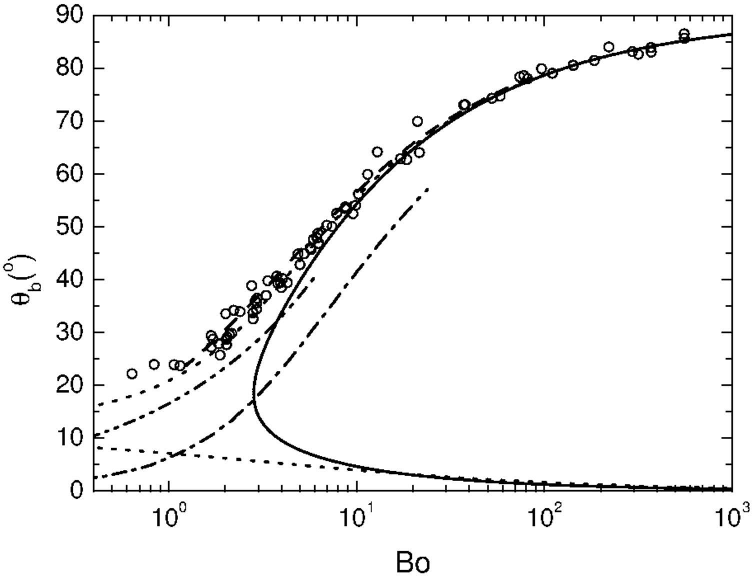


- $\theta_b = 2\tan^{-1}(H/r)$: bubble contact angle.
- r: radius of top of meniscus or base of bubble.
- H: bubble height measured from top of meniscus.
- h: meniscus height.
- d: bubble depth.

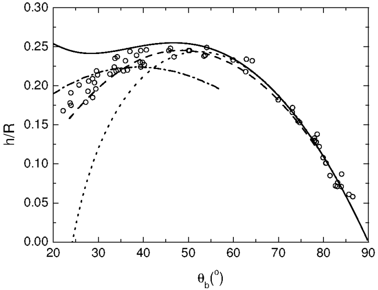
Results: meniscus height, experiment + theory + SE



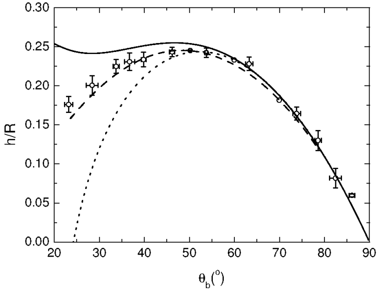
Results: contact angle, experiment + theory + SE



Results: meniscus height and contact angle, experiment + theory + SE

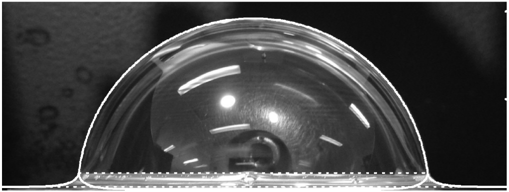


Raw data

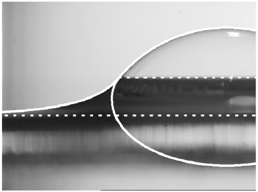


Binned data with error bars

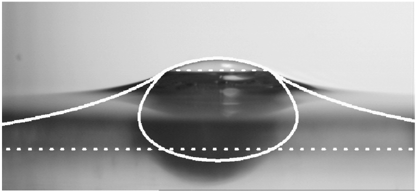
Results: full bubble profiles, experiment + theory



Large bubble (large Bo), $\theta_b = 83.16^\circ$

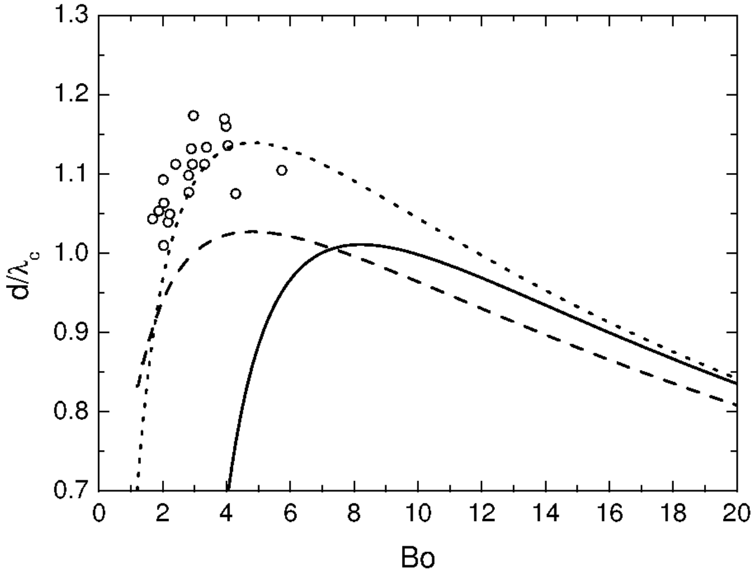


Intermed bubble (moderate Bo), $\theta_b = 50.06^\circ$



Small bubble (small Bo), $\theta_b = 25.74^\circ$

Results: bubble depth, experiment + theory + SE



Summary and conclusions

- Large bubbles ($\mathrm{Bo} \gg 1$) are almost perfect hemispheres, small bubbles ($\mathrm{Bo} \lesssim 1$) are mostly immersed in the liquid.
- The Young-Laplace equation for an axisymmetric bubble can be integrated semi-analytically if $\mathrm{Bo} \gg 1$.
- Calculated bubble shapes are in good agreement with both experimental results and Surface Evolver calculations, for $\mathrm{Bo} \gtrsim 4$.
- We can predict the dependence of meniscus height on bubble contact angle: this is non-monotonic.
- We can describe how the bubble bottom changes from deep and curved (small bubbles) to shallow and flat (large bubbles).
- It is essential to consider the full 3d character of the bubbles.
- Next, do bubble on solid surfaces!

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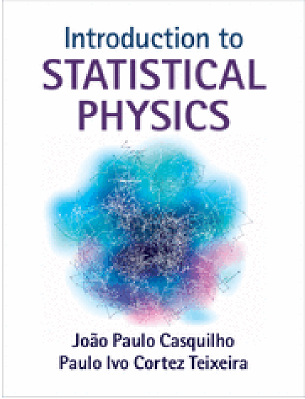
Shameless publicity plug



Introduction to Statistical Physics
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