The College Mathematics Journal

ISSN: 0746-8342 (Print) 1931-1346 (Online) Journal homepage: http://www.tandfonline.com/loi/ucmj20

Dividing the Circle

Pedro J. Freitas & Hugo Tavares

To cite this article: Pedro J. Freitas & Hugo Tavares (2018) Dividing the Circle, The College Mathematics Journal, 49:3, 187-194, DOI: 10.1080/07468342.2018.1440871

To link to this article: https://doi.org/10.1080/07468342.2018.1440871

Published online: 13 Apr 2018.

Dividing the Circle

Pedro J. Freitas and Hugo Tavares

Pedro J. Freitas (pjfreitas@fc.ul.pt, MR ID 342132, ORCID 0000-0003-1206-0257) received his Ph.D. in mathematics from the University of Illinois at Chicago in 1999. He is a member of the Department of History and Philosophy of Sciences at the University of Lisbon. Freitas's current research interests include history of mathematics and elementary mathematics, having written books and papers on the relations between math and other subjects, such as art or magic.

Hugo Tavares (hrtavares@fc.ul.pt, MR ID 823208) received his Ph.D. from the University of Lisbon in 2010 where he now on the mathematics faculty. His research focuses on analysis, more specifically partial differential equations. When his small children allow it, Tavares enjoys reading and running.

As you may have learned in high school, there are geometric constructions for some regular polygons using only a straightedge and compass. The constructions for the triangle, the square, and the hexagon are simple. For the pentagon, the construction is more delicate, but well known. For the octagon, one can simply bisect the angle of the square. Similarly for the decagon.

The reader may have noticed that we skipped two regular polygons, the heptagon (7 sides) and the nonagon or enneagon (9 sides). There is a good reason for the omission: No matter how hard we try, it is impossible to find a straightedge-and-compass construction for either of them. This is a consequence of the following classical result.

Theorem (Gauss–Wantzel). *The division of the circle in n equal parts with straightedge and compass is possible if and only if $n = 2^k p_1 \cdots p_t$ where $p_1, \ldots, p_t$ are distinct Fermat primes.*

Fermat primes are those of the form $2^{2^m} + 1$. Presently, the only known Fermat primes are 3, 5, 17, 257, and 65,537.

Gauss proved, in his early years, that the 17-gon is constructible. He went on to formulate the theorem and Wantzel concluded the proof [**13**]. Visit youtu.be/87uo2TPrsl8 to see David Eisenbud constructing a 17-gon and youtu.be/oYlB5lUGlbw for a discussion of the mathematics involved. See [**10**, chap. 19] for a complete proof of this theorem, where you can also find a reference for the construction of the 257-gon as well as some funny anecdotes about the 65,537-gon.

As mentioned, this result implies that the heptagon and the nonagon are not constructible with straightedge and compass: 7 is not a Fermat prime and $9 = 3 \cdot 3$ with

Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/ucmj.
doi.org/10.1080/07468342.2018.1440871
MSC: 51M15

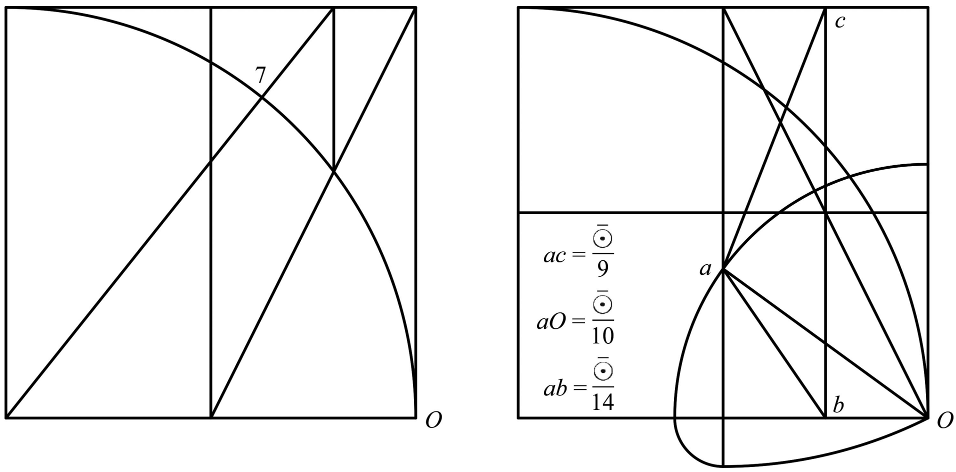

Figure 1. Reproduction of two drawings by Almada Negreiros.

the Fermat prime 3 repeated in the factorization of 9. Nevertheless, it is possible to find good approximate constructions for both these polygons.

Several artists took an interest in finding ways to divide the circle in n parts and use this in their work, including Dürer [**5**]. In the 20th century, a famous Portuguese modernist artist, José de Almada Negreiros, produced drawings consisting solely of such constructions. We reproduce two of them here.

On the left in Figure 1, the arc $\widehat{O7}$ is one-seventh of the circle; on the right, the lines ab, aO, and ac are the chords for the 14th, 10th, and 9th parts of the circle, respectively. The errors are 0.2% for the 7th part, 1% for the 14th, and an amazing 0.001% for the 9th (the constructible 10th part is exact). See [**4**, **6**] for detailed analysis of these approximate constructions.

General approximate constructions

Generally, the constructions become more and more intricate as we increase the number of sides of the polygon. Moreover, they vary from one polygon to the other. While searching for good approximations, one needs to balance the complexity of the construction and its accuracy. One of the authors once asked a friend who teaches high school descriptive geometry if there is a general approximate construction for the n-gon with such characteristics. It turns out there is. She proceeded to draw the picture in Figure 2 (left) as a construction for the nonagon.

Starting with a circle of diameter s, draw arcs of two circles having radius s centered at endpoints of a diameter. (This figure is known as the *vesica piscis*, the fish's bladder, and appears frequently in the composition of Medieval art, for instance.) Let V be the point where these arcs intersect above the original circle. Divide the diameter into nine equal parts and use the second point from the left to define the ray as shown in Figure 2 (left). The process is completely general—for the n-gon, divide the diameter into n equal parts and use the second point from the left.

This method can be found in some descriptive geometry books [**2**, **8**] and is called the Bion method, see [**1**, **7**]. Tempier proposed a variation [**11**, **12**], using a different ray directed by another point, situated on the diameter two nth parts from the circle center, demonstrated for the nonagon in Figure 2 (right).

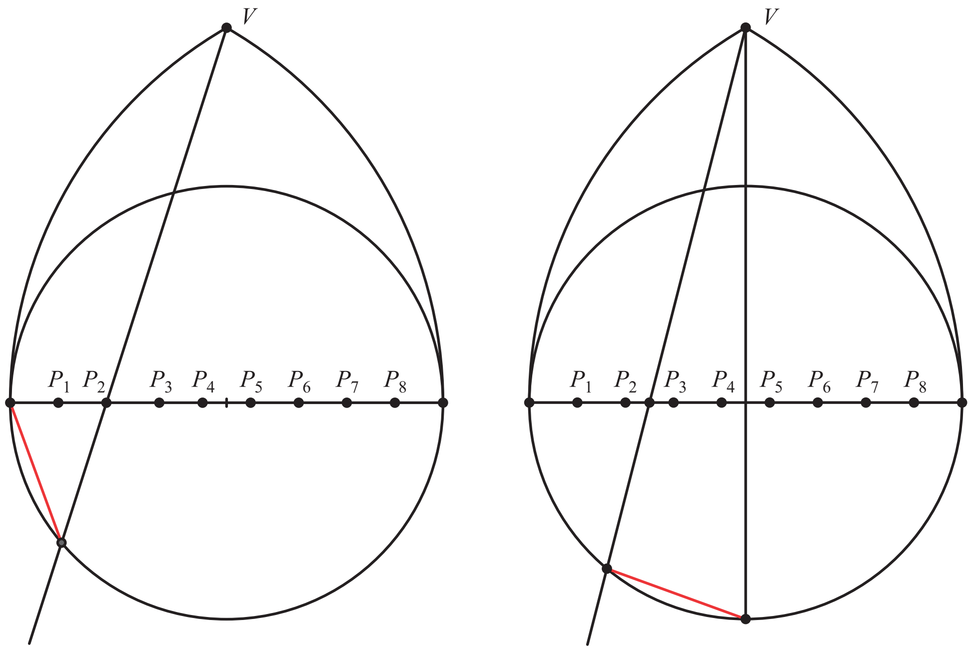

Figure 2. Bion's method for the nonagon, left, and Tempier's, right.

The 19th century papers [**7**, **11**, **12**] include mathematical descriptions of the methods and error tables and point out that the 17-gon constructions are much simpler than Gauss's (even though they are approximate). This material does not seem to be well known in the present mathematical community, with the recent exception [**9**]. We have provided Geogebra applets producing the polygons determined by Bion's method (tube.geogebra.org/student/m630691) and by Tempier's method (tube.geogebra.org/student/m630601).

These constructions are simple and elegant, but it is not at all clear why they give an approximation of the nth part of the circle. What is so special about the point V and the length of two nth parts of the diameter? We will develop a unified analysis of the two methods in order to address these concerns.

Measuring error

To determine the errors produced by these general constructions, we must find the exact values of the angles they determine. Our analysis differs slightly from the original papers [**7**, **11**, **12**] and handles both Bion's and Tempier's methods simultaneously.

Figure 3 shows vertical and horizontal lines meeting at the center of the circle determining, with a ray from the constructions, angles α, β, x, y, and lengths a, b, c, d. We determine the complementary angles x and y that meet at the center of the circle.

Beginning with β, we have

$$\sin\beta = \sin(\pi - \beta) = \frac{b}{c}\,, \quad \cos\beta = \cos(\pi - \beta) = -\frac{a}{c}\,.$$

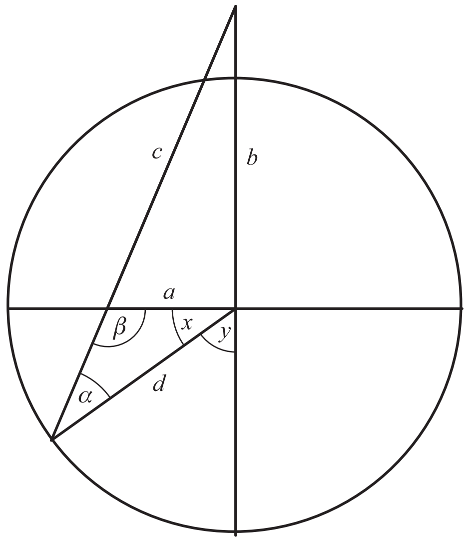

Figure 3. Angles and lengths in the constructions.

Since β is obtuse and given the main restrictions of sine, cosine, and tangent,

$$\beta = \pi - \arcsin\frac{b}{c} = \arccos\left(-\frac{a}{c}\right) = \pi + \arctan\left(-\frac{b}{a}\right). \tag{1}$$

For α, by the law of sines, $(\sin\alpha)/a = (\sin\beta)/d$, which gives

$$\alpha = \arcsin\frac{ab}{cd}. \tag{2}$$

Since α, β, and x are the angles of a triangle, by (1) and (2),

$$x = \pi - \left(\pi - \arcsin\frac{b}{c}\right) - \arcsin\frac{ab}{cd} = \arcsin\frac{b}{c} - \arcsin\frac{ab}{cd}.$$

For y, notice that $\cos(\pi/2 - \alpha) = \sin\alpha = (ab)/(cd)$ so, since $\pi/2 - \alpha \in [0, \pi]$,

$$y = \beta + \alpha - \frac{\pi}{2} = \arccos\left(-\frac{a}{c}\right) - \arccos\frac{ab}{cd}.$$

We can now determine the angles that appear in the approximate methods by calculating a, b, c, and d in a division of the circle into n parts. Assume a unit circle, i.e., $d = 1$, so that the diameter is divided into n parts of length $2/n$.

Bion's method. Here $a = 1 - 4/n = (n-4)/n$ and $b = \sqrt{2^2 - 1^2} = \sqrt{3}$, so that

$$c = \sqrt{\left(\sqrt{3}\right)^2 + \left(\frac{n-4}{n}\right)^2} = \frac{2\sqrt{n^2 - 2n + 4}}{n}.$$

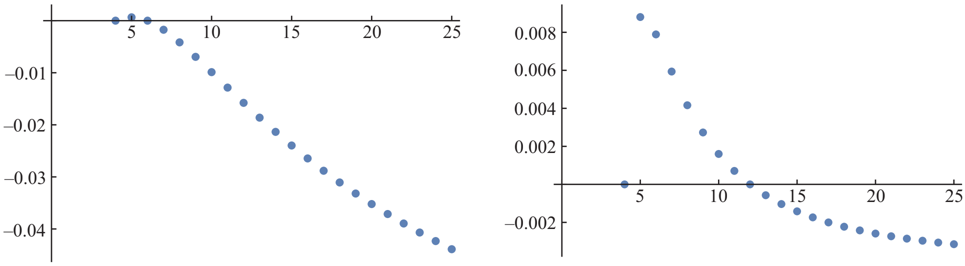

Figure 4. Graph of the relative errors for Bion's method, left, and Tempier's method, right.

Therefore,

$$x = \arcsin \frac{\sqrt{3}n}{2\sqrt{n^2 - 2n + 4}} - \arcsin \frac{\sqrt{3}(n-4)}{2\sqrt{n^2 - 2n + 4}} .$$

Define the following relative error function for $n \geq 4$ as the quotient of the error and the exact angle:

$$n \;\mapsto\; \frac{\frac{2\pi}{n} - x(n)}{\frac{2\pi}{n}} = 1 - \frac{n\,x(n)}{2\pi}.$$

(Actually, one usually uses the absolute value of this expression, but that would provide less information.) Using Mathematica to study this function, we find that it vanishes only at $n = 4$ and 6, has a maximum at $n = 5$, and is strictly decreasing for $n \geq 6$, converging to $1 - 2\sqrt{3}/\pi \approx -0.102658$ as $n \to \infty$. Figure 4 (left) shows the graph for n from 4 to 25. Besides the two cases when Bion's method is exact, the error can range from 0.1% to 10.27%.

Tempier's method. Here $a = 4/n$, $b = \sqrt{3}$, and $c = \sqrt{3n^2 + 16}/n$. In this case, the approximation for $2\pi/n$ is

$$y = \arccos\left(-\frac{4}{\sqrt{3n^2 + 16}}\right) - \arccos \frac{4\sqrt{3}}{\sqrt{3n^2 + 16}} .$$

The analogous relative error function $1 - (n\,y(n))/(2\pi)$ is zero at $n = 4$ and 12, again has a maximum at $n = 5$, and is strictly decreasing for $n \geq 5$, converging to $-(6 + 2\sqrt{3} - 3\pi)/(3\pi) \approx -0.004172$ as $n \to \infty$. See Figure 4 (right) for the corresponding graph for n from 4 to 25. Besides the two cases when Tempier's method is exact, the error is never worse than 0.9%!

Comparing the two methods, one concludes that Bion's is better for $n = 5, 6, 7$, they are more or less equivalent for $n = 8$, and then for $n \geq 9$ Tempier's is much more accurate.

Among the exact constructions are the Bion hexagon and the Tempier dodecagon. Both polygons can be illustrated by Figure 5 with $BF = BC/3$ and $FA = BC/6$. This also implies $BF = 2FA$.

It is straightforward to prove this exactness using more elementary methods: Take a parallel to BV through point A and let G be its intersection with VF. Using the similarity of triangles BFV and GFA, one can show $\angle BAG = \pi/3$ (which implies $\angle GAF = \pi/6$), so that G is on the circle. This is what we need, as $\pi/3$ and $\pi/6$ are the center angles for the hexagon and dodecagon, respectively.

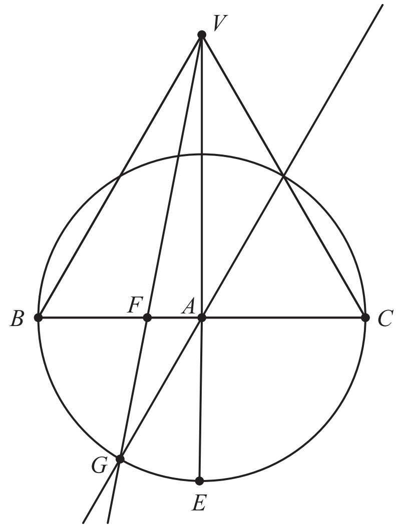

Figure 5. One figure for the Bion hexagon and the Tempier dodecagon.

The rectification of the quadrant

We now turn to the question about the point V. Why was this point chosen for these constructions?

We noticed that, in [2], there is a construction for rectification of arcs smaller than $\pi/2$ that uses a point very close to V. The point R in Figure 6 is marked on the vertical line at distance 3/4 of the radius from D (we chose R for rational). This construction is used to produce segment $A'B$ with length approximately equal to that of the arc $\widehat{AB}$. Without delving too much on the accuracy of this construction, we calculate the length when the arc is a full quadrant such as $\widehat{QB}$.

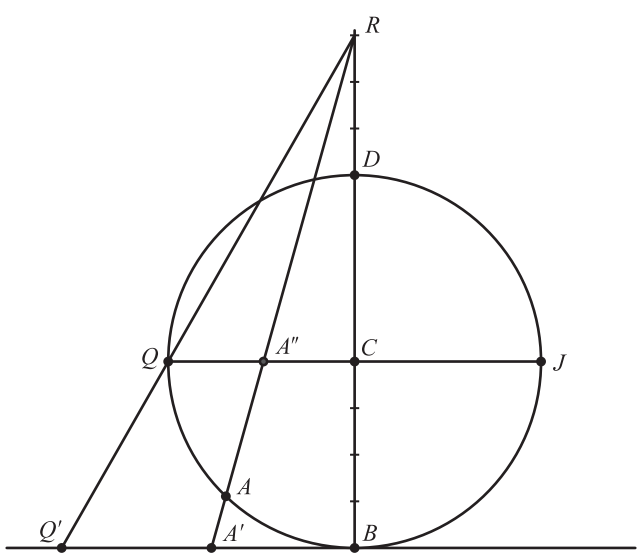

Figure 6. Rectification of arcs.

By simple proportions (and still using a unit circle), we have $Q'B/RB = QC/RC$ which leads to $Q'B = (11/4)/(7/4) = (22/7)/2$. That is, the position of R gives a rectification of the quadrant equivalent to the famous approximation of π as $22/7 \approx 3.142857$.

Using V from the constructions to rectify the quadrants leads to the implicit approximation of π as $2 + 2\sqrt{3}/3 \approx 3.154701$.

One can also work the other way around: What is the position of a point P on BD such that the rectification of the full quadrant is exact? Working out the proportion once more, we find the distance from P to C would be $2/(\pi - 2) \approx 1.751938$. This is a nonconstructible length, since it is transcendental; R is a very good approximation. Table 1 summarizes these computations. If we use this point P in the Tempier process and calculate the corresponding relative error, the limit is 0 as $n \to \infty$.

Table 1. Comparing points for the constructions.

point	distance to C	2 (rectified quadrant)
V	$\sqrt{3} \approx 1.732051$	$2 + 2\sqrt{3}/3 \approx 3.154701$
R	$7/4 = 1.75$	$22/7 \approx 3.142857$
P	$2/(\pi - 2) \approx 1.751938$	$\pi \approx 3.141593$

We conclude that points V, R, and P are all reasonable base points for rectifying arcs. In particular, it probably explains why V was chosen: Of the two constructible lengths, it is probably the easier to mark.

So, when we look for the value of the angle in the Tempier or Bion methods, we can look at the length of the corresponding (approximately) rectified segment on the tangent line. Now, if you divide the diameter into n equal parts, this corresponds to a division of the full rectified half-circle into n equal parts as well. Each part will then have length π/n, so one has to take two of these in order to get a length of $2\pi/n$, as we wish. And this explains why we must take two segments on the diameter in order to define the angle $2\pi/n$.

In practice, approximations are acceptable in certain circumstances; some errors are so small that the very thickness of the trace renders them negligible. Our computations confirm that the Bion and Tempier methods are good options for a regular n-gon, being acceptable even in the cases when an exact construction is known.

This study arose in relation to the first author's ongoing work on Almada Negreiros's geometric drawings, mentioned at the beginning. Many of the artist's works include approximate geometrical constructions for the division of the circle in equal parts (see modernismo.pt for more of his work). See [3] for a general presentation of the mathematics in Almada's work. The book [4] includes 29 of Almada's drawings made from a mathematical viewpoint along with analysis of then. Expanding on these studies, the website gulbenkian.pt/almada-comecar/en gives mathematical and historical explanations of the geometrical constructions in Almada's monumental mural *Começar*.

Acknowledgment. Figure 4 was made in Mathematica, the rest were created with Geogebra. The first author was partially supported by FCT/Portugal through project UID/MAT/04721/2013, the second by FCT/Portugal through project UID/MAT/04459/2013.

Summary. Some but not all regular polygons can be constructed using only straightedge and compass; the Gauss–Wantzel theorem states precisely which. The known constructions differ

from one polygon to the other. There are, however, general processes for determining the side of an arbitrary n-gon approximately, but sometimes with great precision. We describe two such methods, named for Bion and Tempier, analyze their errors, and explain why these approximate constructions work. We also highlight some related geometric artwork of Almada Negreiros.

References

[1] Bion, N. (1752). *Traité de la Construction et des Principaux usages des Instruments de Mathématique*, 4th ed. Paris: Jombert.

[2] Braga, T. (1997). *Desenho Linear Geométrico*, 14th ed. São Paulo: Ícone.

[3] Costa, S. P., Freitas, P. J. (2015). Almada Negreiros and the geometric canon. *J. Math. Arts* 9: 27–36. doi.org/10.1080/17513472.2015.1012699.

[4] Costa, S. P., Freitas, P. J. (2015). *Livro de Problemas de Almada Negreiros*. Lisbon: Portuguese Mathematical Society.

[5] Dürer, A. (1538). *Underweysung der Messung, mit dem Zirckel und Richtscheyt*. Nüremberg, Germany: Andreae.

[6] Freitas, P. J. (2015). Almada Negreiros and the nonagon. *Recreat. Math. Mag.* 3: 39–50.

[7] Housel, M. (1853). Division pratique de la circonférence en parties égales. *Nouv. Ann. Math.* 12: 77–80.

[8] Marques Leitão, C. A. (1909). *Desenho*, book 2. Lisbon: Fernandes.

[9] Milnikel, R. S. (2015). A new angle on an old construction: Approximating inscribed n-gons. *Math. Mag.* 88: 260–269. doi.org/10.4169/math.mag.88.4.260.

[10] Stewart, I. (2015). *Galois Theory*, 4th ed. Boca Raton, FL: CRC.

[11] Tempier, M. (1853). Division pratique de la circonférence en parties égales. *Nouv. Ann. Math.* 12: 345–347.

[12] Tempier, M. (1854). Division pratique de la circonférence en parties égales. *Nouv. Ann. Math.* 13: 295.

[13] Wantzel, P. M. L. (1837). Recherches sur les moyens de reconnaître si un problème de géométrie peut se résoudre avec la règle et le compas. *J. Math. Pures Appl.* 2: 366–372.

