CONSERVATIVE AND DISSIPATIVE POLYMATRIX REPLICATORS

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ABSTRACT. In this paper we address a class of replicator dynamics, referred as polymatrix replicators, that contains well known classes of evolutionary game dynamics, such as the symmetric and asymmetric (or bimatrix) replicator equations, and some replicator equations for n-person games. Polymatrix replicators form a simple class of algebraic o.d.e.'s on prisms (products of simplexes), which describe the evolution of strategical behaviours within a population stratified in $n \geq 1$ social groups.

In the 80's Raymond Redheffer et al. developed a theory on the class of stably dissipative Lotka-Volterra systems. This theory is built around a reduction algorithm that "infers" the localization of the system's attractor in some affine subspace. It was later proven that the dynamics on the attractor of such systems is always embeddable in a Hamiltonian Lotka-Volterra system. In this paper we extend these results to polymatrix replicators.

1. **Introduction.** Lotka-Volterra (LV) systems were introduced independently by Alfred Lotka [31] and Vito Volterra [50] to model the evolution of biological and chemical ecosystems. The phase space of a Lotka-Volterra system is the non-compact polytope $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i \geq 0, \ i = 1, \ldots, n\}$, where a point in $\mathbb{R}^n$

represents a state of the ecosystem. The LV systems are defined by the following o.d.e.

$$\frac{dx_i}{dt} = x_i\, f_i(x), \quad i = 1, \ldots, n,$$

where usually the so called *fitness functions* $f_i(x)$ are considered to be affine, i.e., of the form

$$f_i(x) = r_i + \sum_{j=1}^{n} a_{ij}\, x_j,$$

where $A = (a_{ij}) \in \mathrm{Mat}_{n \times n}(\mathbb{R})$ is called the system's interaction matrix.

In general, the dynamics of LV systems can be arbitrarily rich, as was first observed by S. Smale [46] who proved that any finite dimensional compact flow can be embedded in a LV system with non linear fitness functions. Later, using a class of embeddings studied by L. Brenig [5], L. Brenig and A. Goriely [6], B. Hernández-Bermejo and V. Fairén [16], it was proven (see [16, Theorems 1 and 2]) that any LV system with polynomial fitness functions can be embedded in a LV system with affine fitness functions. Combining this with Smale's result, we infer that any finite dimensional compact flow can be, up to a small perturbation, embedded in a LV system with affine fitness functions. These facts emphasize the difficulty of studying the general dynamics of LV systems.

In spite of these difficulties, many dynamical consequences have been driven from information on the fitness data $f_i(x)$ for some special classes of LV systems. Two such classes are the *cooperative* and *competitive* LV systems, corresponding to fitness functions satisfying $\frac{\partial f_i}{\partial x_j} \geq 0$ and $\frac{\partial f_i}{\partial x_j} \leq 0$, respectively, for all i, j. Curiously, the fact that Smale's embedding takes place in a competitive LV system influenced the development of the theory of cooperative and competitive LV systems initiated by M. Hirsch [18–20].

In his pioneering work Volterra [50] studies dissipative LV systems as generalizations of the classical predator-prey model. A LV system with interaction matrix $A = (a_{ij})$ is called *dissipative*, resp. *conservative*, if there are constants $d_i > 0$ such that the quadratic form $Q(x) = \sum_{i,j=1}^{n} a_{ij} d_j x_i x_j$ is negative semi-definite, resp. zero. Note that the meaning of the term *dissipative* is not strict because dissipative LV o.d.e.s include conservative LV systems. In addition we remark that conservative LV models are in some sense Hamiltonian systems, a fact that was well known and explored by Volterra.

Given a LV system with interaction matrix $A = (a_{ij})$, we define its interaction graph $G(A)$ to be the undirected graph with vertex set $V = \{1, \ldots, n\}$ that includes an edge connecting i to j whenever $a_{ij} \neq 0$ or $a_{ji} \neq 0$. The LV system and its matrix A are called *stably dissipative* if $\sum_{i,j=1}^{n} \tilde{a}_{ij} x_i x_j \leq 0$ for all $x \in \mathbb{R}^n$ and every small enough perturbation $\tilde{A} = (\tilde{a}_{ij})$ of A such that $G(\tilde{A}) = G(A)$. This notion of stably dissipativeness is due to Redheffer *et al.* whom in a series of papers [36–40] studied this class of models under the name of *stably admissible* systems.

Assuming the system admits an interior equilibrium $q \in \mathrm{int}(\mathbb{R}_+^n)$, Redheffer *et al.* describe a simple reduction algorithm, running on the graph $G(A)$, that 'deduces' the minimal affine subspace of the form $\cap_{i \in I} \{x \in \mathbb{R}_+^n : x_i = q_i\}$ that contains the attractor of every stably dissipative LV system with interaction graph $G(A)$.

Under the scope of this theory, Duarte *et al.* [12] have proven that the dynamics on the attractor of a stably dissipative LV system is always described by a conservative (Hamiltonian) LV system.

The replicator equation, which is now central to Evolutionary Game Theory (EGT), was introduced by P. Taylor and L. Jonker [48]. It models the time evolution of the probability distribution of strategical behaviors within a biological population. Given a *payoff matrix* $A \in \mathrm{Mat}_{n \times n}(\mathbb{R})$, the replicator equation refers to the following o.d.e.

$$x_i' = x_i \left((Ax)_i - x^t A x \right), \quad i = 1, \ldots, n$$

on the simplex $\Delta^{n-1} = \{ x \in \mathbb{R}_+^n : \sum_{j=1}^n x_j = 1 \}$. This equation says that the logarithmic growth of the usage frequency of each behavioural strategy is directly proportional to how well that strategy fares within the population.

Another important class of models in EGT, that includes the *Battle of sexes*, is the *bimatrix replicator* equation. In this model the population is divided in two groups, e.g. males and females, and all interactions involve individuals of different groups. Given two payoff matrices $A \in \mathrm{Mat}_{n \times m}(\mathbb{R})$ and $B \in \mathrm{Mat}_{m \times n}(\mathbb{R})$, for the strategies in each group, the bimatrix replicator refers to the o.d.e.

$$\begin{cases} x_i' & = x_i \left((Ay)_i - x^t A y \right) & i = 1, \ldots, n \\ y_j' & = y_j \left((Bx)_j - y^t B x \right) & j = 1, \ldots, m \end{cases}$$

on the product of simplices $\Delta^{n-1} \times \Delta^{m-1}$. It describes the time evolution of the strategy usage frequencies in each group. These systems were first studied in [43] and [44].

We now introduce the *polymatrix replicator* equation studied in [1]. Consider a population is divided in $p \in \mathbb{N}$ groups, $\alpha = 1, \ldots, p$, each with $n_\alpha \in \mathbb{N}$ behavioral strategies, in a total of $n = \sum_{\alpha=1}^p n_\alpha$ strategies, numbered from 1 to n. The system is described by a single payoff matrix $A \in \mathrm{Mat}_{n \times n}(\mathbb{R})$, which can be decomposed in p^2 blocks $A^{\alpha,\beta} \in \mathrm{Mat}_{n_\alpha \times n_\beta}(\mathbb{R})$ with the payoffs corresponding to interactions between strategies in group α with strategies in group β. Let us abusively write $i \in \alpha$ to express that i is a strategy of the group α. With this notation the polymatrix replicator refers to the following o.d.e.

$$x_i' = x_i \left((A\,x)_i - \sum_{j \in \alpha} x_j\,(A\,x)_j \right), i \in \alpha, \alpha \in \{1, \ldots, p\}.$$

on the product of simplexes $\Delta^{n_1 - 1} \times \ldots \times \Delta^{n_p - 1}$. Notice that interactions between individuals of any two groups (including the same) are allowed. Notice also that this equation implies that competition takes place inside the groups, i.e., the relative success of each strategy is evaluated within the corresponding group.

This class of evolutionary systems includes both the replicator equation (when $p = 1$) and the bimatrix replicator equation (when $p = 2$ and $A^{1,1} = 0 = A^{2,2}$). It also includes the replicator equation for n-person games (when $A^{\alpha,\alpha} = 0$ for all $\alpha = 1, \ldots, p$). This last subclass of polymatrix replicator equations specializes more general replicator equations for n-person games with multi-linear payoffs that were first formulated by Palm [33] and studied by Ritzberger, Weibull [41], Plank [34] among others.

In this paper we define the class of admissible polymatrix replicators (the analogue of stably dissipative for LV systems), and introduce a reduction algorithm similar to the one of Redheffer that 'deduces' the constraints on the localization of the attractor. We also generalize the mentioned theorem in [12] to polymatrix replicator systems.

This paper is organized as follows. In Section 2 we introduce the notion of polymatrix game as well as its associated polymatrix replicator system (o.d.e.), proving some elementary facts about this class of models. In Section 3 we recall some known results of Redheffer *et al.* reduction theory for stably dissipative LV systems. In sections 4 and 5, we define, respectively, the classes of conservative and dissipative polymatrix replicators, and study their properties. In particular, we extend to polymatrix replicators the concept of stably dissipativeness of Redheffer *et al.*. We generalize to this context the mentioned theorem in [12] about the Hamiltonian nature of the limit dynamics of a "stably dissipative" system. Finally, in Section 6 we illustrate our results with a simple example.

2. **Polymatrix Replicators.**

Definition 2.1. A polymatrix game is an ordered pair $(\underline{n}, A)$ where $\underline{n} = (n_1, \ldots, n_p)$ is a list of positive integers, called the game type, and $A \in \mathrm{Mat}_{n \times n}(\mathbb{R})$ a square matrix of dimension $n = n_1 + \ldots + n_p$.

This formal definition has the following interpretation.

Consider a population divided in p groups, labeled by an integer α ranging from 1 to p. Individuals of each group $\alpha = 1, \ldots, p$ have exactly n_α strategies to interact with other members of the population. The strategies of a group α are labeled by positive integers j in the range

$$n_1 + \ldots + n_{\alpha-1} < j \le n_1 + \ldots + n_\alpha .$$

We will write $j \in \alpha$ to mean that j is a strategy of the group α. Hence the strategies of all groups are labeled by the integers $j = 1, \ldots, n$.

The matrix A is the *payoff matrix*. Given strategies $i \in \alpha$ and $j \in \beta$, in the groups α and β respectively, the entry a_{ij} represents an average payoff for an individual using the first strategy in some interaction with an individual using the second. Thus, the payoff matrix A can be decomposed into $n_\alpha \times n_\beta$ block matrices $A^{\alpha,\beta}$, with entries a_{ij}, $i \in \alpha$ and $j \in \beta$, where α and β range from 1 to p.

Definition 2.2. Two polymatrix games $(\underline{n}, A)$ and $(\underline{n}, B)$ with the same type are said to be equivalent, and we write $(\underline{n}, A) \sim (\underline{n}, B)$, when for $\alpha, \beta = 1, \ldots, p$, all the rows of the block matrix $A^{\alpha\beta} - B^{\alpha\beta}$ are equal.

The *state* of the population is described by a point $x = (x^\alpha)_\alpha$ in the *prism*

$$\Gamma_{\underline{n}} := \Delta^{n_1-1} \times \ldots \times \Delta^{n_p-1} \subset \mathbb{R}^n ,$$

where $\Delta^{n_\alpha-1} = \{x \in \mathbb{R}^{n_\alpha} : \sum_{i=1}^{n_\alpha} x_i = 1\}$, $x^\alpha = (x_j)_{j \in \alpha}$ and the entry x_j represents the usage frequency of strategy j within the group α. The prism $\Gamma_{\underline{n}}$ is a $(n-p)$-dimensional simple polytope whose affine support is the $(n-p)$-dimensional space $E^{n-p} \subset \mathbb{R}^n$ defined by the p equations

$$\sum_{i \in \alpha} x_i = 1, \quad 1 \le \alpha \le p .$$

Definition 2.3. A polymatrix game $(\underline{n}, A)$ determines the following o.d.e. on the prism $\Gamma_{\underline{n}}$

$$\frac{dx_i}{dt} = x_i \left((A\,x)_i - \sum_{\beta=1}^{p} (x^\alpha)^T A^{\alpha,\beta} x^\beta \right), \quad \forall\, i \in \alpha, \, 1 \le \alpha \le p , \tag{1}$$

called a polymatrix replicator system.

This equation says that the logarithmic growth rate of each frequency x_i is the difference between its payoff $(Ax)_i = \sum_{j=1}^{n} a_{ij}x_j$ and the average payoff of all strategies in the group α. The flow $\phi_{\underline{n},A}^t$ of this equation leaves the prism $\Gamma_{\underline{n}}$ invariant. Hence, by compactness of $\Gamma_{\underline{n}}$, this flow is complete. The underlying vector field on $\Gamma_{\underline{n}}$ will be denoted by $X_{\underline{n},A}$.

In the case $p = 1$, we have $\Gamma_{\underline{n}} = \Delta^{n-1}$ and (1) is the usual replicator equation associated to the payoff matrix A.

When $p = 2$, and $A^{11} = A^{22} = 0$, $\Gamma_{\underline{n}} = \Delta^{n_1-1} \times \Delta^{n_2-1}$ and (1) becomes the bimatrix replicator equation associated to the pair of payoff matrices (A^{12}, A^{21}).

The polytope $\Gamma_{\underline{n}}$ is parallel to the affine subspace

$$H_{\underline{n}} := \left\{ x \in \mathbb{R}^n : \sum_{j \in \alpha} x_j = 0, \ \text{for} \ \alpha = 1, \ldots, p \right\} . \tag{2}$$

For each $\alpha = 1, \ldots, p$, we denote by $\pi_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ the projection

$$x \mapsto y, \quad y_i := \begin{cases} x_i & \text{if} \quad i \in \alpha \\ 0 & \text{if} \quad i \notin \alpha \end{cases} .$$

We also define $\mathbb{1} := (1, \ldots, 1) \in \mathbb{R}^n$.

Lemma 2.4. *Given a matrix $C \in \mathrm{Mat}_{n \times n}(\mathbb{R})$, the following statements are equivalent:*

(a) $C^{\alpha\beta}$ has equal rows, for all $\alpha, \beta \in \{1, \ldots, p\}$,
(b) $Cx \in H_{\underline{n}}^\perp$, for all $x \in \mathbb{R}^n$.

Moreover, if any of these conditions holds then $X_{\underline{n},C} = 0$ on $\Gamma_{\underline{n}}$.

Proof. Assume (a). Since $H_{\underline{n}}^\perp$ is spanned by the vectors $\pi_\alpha(\mathbb{1})$ with $\alpha = 1, \ldots, p$, we have $v \in H_{\underline{n}}^\perp$ iff $v_i = v_j$ for all $i, j \in \alpha$. Because all rows of C in the group α are equal, we have $(Cx)_i = (Cx)_j$ for all $i, j \in \alpha$. Hence item (b) follows.

Next assume (b). For all $i \in \alpha$, with $\alpha \in \{1, \ldots, p\}$, $Ce_i \in H_{\underline{n}}^\perp$, which implies that $c_{i,k} = c_{j,k}$ for all $j \in \alpha$. This proves (a).

If (a) holds, then for any $\alpha \in \{1, \ldots, p\}$, $i, j \in \alpha$ and $k = 1, \ldots, n$, we have $c_{ik} = c_{jk}$. Hence for any $x \in \Gamma_{\underline{n}}$, and $i, j \in \alpha$ with $\alpha \in \{1, \ldots, p\}$, $(Cx)_i = (Cx)_j$, which implies that $X_{\underline{n},C} = 0$ on $\Gamma_{\underline{n}}$. $\square$

Proposition 1. *Given two polymatrix games $(\underline{n}, A)$ and $(\underline{n}, B)$ with the same type $\underline{n}$, if $(\underline{n}, A) \sim (\underline{n}, B)$ then $X_{\underline{n},A} = X_{\underline{n},B}$ on $\Gamma_{\underline{n}}$.*

Proof. Follows from Lemma 2.4 and the linearity of the correspondence $A \mapsto X_{\underline{n},A}$. $\square$

We have the following obvious characterization of interior equilibria.

Proposition 2. *Given a polymatrix game $(\underline{n}, A)$, a point $q \in \mathrm{int}(\Gamma_{\underline{n}})$ is an equilibrium of $X_{\underline{n},A}$ if and only if $(A\,q)_i = (A\,q)_j$ for all $i, j \in \alpha$ and $\alpha = 1, \ldots, p$.*

In particular the set of interior equilibria of $X_{\underline{n},A}$ is the intersection of some affine subspace with $\mathrm{int}(\Gamma_{\underline{n}})$.

3. **Lotka-Volterra systems.** The standard sector

$$\mathbb{R}_+^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \quad \forall i \in \{1, \ldots, n\}\}.$$

is the phase space of Lotka-Volterra systems.

Definition 3.1. We call *Lotka-Volterra* (LV) any system of differential equations on $\mathbb{R}_+^n$ of the form

$$x_i' = x_i \left(r_i + \sum_{j=1}^n a_{ij} x_j \right), \quad i = 1, \ldots, n . \tag{3}$$

In the canonical interpretation (3) models the time evolution of an ecosystem with n species. Each variable x_i represents the density of species i, the coefficient r_i stands for the intrinsic rate of decay or growth of species i, and each coefficient a_{ij} represents the effect of population j over population i. For instance $a_{ij} > 0$ means that population j benefits population i. The matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ is called the interaction matrix of system (3).

The interior equilibria of (3) are the solutions $q \in \mathbb{R}_+^n$ of the non-homogeneous linear equation $r + A x = 0$. Given $A \in \text{Mat}_{n \times n}(\mathbb{R})$ and $q \in \mathbb{R}^n$ such that $r + A q = 0$, the LV system (3) can be written as

$$\frac{dx}{dt} = X_{A,q}(x) := x * A (x - q) , \tag{4}$$

where $*$ denotes the point-wise multiplication of vectors in $\mathbb{R}^n$.

Definition 3.2. We say that the LV system (4), the matrix A, or the vector field $X_{A,q}$, is *dissipative* iff there is a positive diagonal matrix D such that $Q_{AD}(x) = x^T A D x \leq 0$ for every $x \in \mathbb{R}^n$.

Proposition 3. *If $X_{A,q}$ is dissipative then, for any $D = \text{diag}(d_i)$ as in Definition 3.2, $X_{A,q}$ admits the Lyapunov function*

$$h(x) = \sum_{i=1}^n \frac{x_i - q_i \, \log x_i}{d_i} , \tag{5}$$

which decreases along orbits of $X_{A,q}$.

Proof. The derivative of h along orbits of $X_{A,q}$ is given by

$$\dot{h}(x) = \sum_{i,j=1}^n \frac{a_{ij}}{d_i}(x_i - q_i)(x_j - q_j) = (x - q)^T D^{-1} A (x - q)$$

$$= [D^{-1}(x - q)]^T A D [D^{-1}(x - q)] \leq 0.$$

$\square$

We will denote by $\text{Ker}(A)$ the kernel of a matrix A.

Proposition 4. *If $A \in \text{Mat}_{n \times n}(\mathbb{R})$ is dissipative and D is a positive diagonal matrix such that $Q_{AD} \leq 0$ then $\text{Ker}(A) = D \text{Ker}(A^T)$.*

Proof. Assume first that $Q_A \leq 0$ on $\mathbb{R}^n$ and consider the decomposition $A = M + N$ with $M = (A + A^T)/2$ and $N = (A - A^T)/2$. Clearly $\text{Ker}(M) \cap \text{Ker}(N) \subseteq \text{Ker}(A)$. On the other hand, if $v \in \text{Ker}(A)$ then $v^T M v = v^T A v = 0$. Because $Q_M = Q_A \leq 0$ this implies that $M v = 0$, i.e., $v \in \text{Ker}(M)$. Finally, since $N = A - M$,

$v \in \mathrm{Ker}(N)$. This proves that $\mathrm{Ker}(A) = \mathrm{Ker}(M) \cap \mathrm{Ker}(N)$. Similarly, one proves that $\mathrm{Ker}(A^T) = \mathrm{Ker}(M) \cap \mathrm{Ker}(N)$. Thus $\mathrm{Ker}(A) = \mathrm{Ker}(A^T)$.

In general, if $Q_{AD} \leq 0$, we have $\mathrm{Ker}((AD)^T) = \mathrm{Ker}(DA^T) = \mathrm{Ker}(A^T)$, and $\mathrm{Ker}(AD) = D^{-1}\mathrm{Ker}(A)$. Thus, from the previous case applied to AD we get $D^{-1}\mathrm{Ker}(A) = \mathrm{Ker}(A^T)$. $\square$

Proposition 5. *Any dissipative LV system admits an invariant foliation on* $\mathrm{int}(\mathbb{R}^n_+)$ *with a unique equilibrium point in each leaf.*

Proof. See [13, Proposition 2.1 and Theorem 2.3]. $\square$

On the rest of this section we focus attention on LV systems with interior equilbria $q \in \mathrm{int}(\mathbb{R}^n_+)$. In this case the Lyapunov function h is proper, and hence the forward orbits of (4) are complete. Therefore, the vector field $X_{A,q}$ induces a complete semi-flow $\phi^t_{A,q}$ on $\mathrm{int}(\mathbb{R}^n_+)$.

Definition 3.3. Given a matrix $A = (a_{ij}) \in \mathrm{Mat}_{n \times n}(\mathbb{R})$ of a LV system, we define its associated graph $G(A)$ to have vertex set $\{1, \ldots, n\}$, and to contain an edge connecting vertex i to vertex j iff $a_{ij} \neq 0$ or $a_{ji} \neq 0$.

Given a matrix $A = (a_{ij}) \in \mathrm{Mat}_{n \times n}(\mathbb{R})$ we call *admissible perturbation* of A any other matrix $\tilde{A} = (\tilde{a}_{ij}) \in \mathrm{Mat}_{n \times n}(\mathbb{R})$ such that

$$\tilde{a}_{ij} = 0 \Leftrightarrow a_{ij} = 0.$$

By definition, admissible perturbation are perturbations of A such that $G(A) = G(\tilde{A})$.

Definition 3.4. A matrix $A \in \mathrm{Mat}_{n \times n}(\mathbb{R})$ is said to be *stably dissipative* if any close enough admissible perturbation $\tilde{A}$ of A is dissipative, i.e., if there exists $\varepsilon > 0$ such that for any admissible perturbation $\tilde{A} = (\tilde{a}_{ij})$ of $A = (a_{ij})$,

$$\max_{1 \leq i,j \leq n} |a_{ij} - \tilde{a}_{ij}| < \varepsilon \Rightarrow \tilde{A} \text{ is dissipative.}$$

A LV system (4) is said to be *stably dissipative* if its interaction matrix is stably dissipative.

Lemma 3.5. *Let D be a positive diagonal matrix. If A is a stably dissipative matrix, then AD and $D^{-1}A$ are also stably dissipative.*

Proof. Since A is dissipative there exists a positive diagonal matrix D' such that $Q_{AD'} \leq 0$, which is equivalent to $Q_{(AD)(D^{-1}D')} \leq 0$. Hence AD is dissipative. Analogously, since $Q_{AD'} \leq 0$ we have $Q_{D^{-1}AD'D^{-1}}(x) = Q_{AD'}(D^{-1}x) \leq 0$, which shows that $D^{-1}A$ is dissipative.

Let B be a small enough admissible perturbation of AD. Then there exists an admissible perturbation $\tilde{A}$ of A such that $B = \tilde{A}D$. Since A is stably dissipative the matrix $\tilde{A}$ is dissipative as well. Hence there exists a positive diagonal matrix D'' such that $Q_{\tilde{A}D''} \leq 0$, which is equivalent to $Q_{(\tilde{A}D)(D^{-1}D'')} \leq 0$. This proves that $B = \tilde{A}D$ is dissipative. Therefore AD is stably dissipative.

A similar argument proves that $D^{-1}A$ is stably dissipative. $\square$

Definition 3.6. Given a matrix $A \in \mathrm{Mat}_{n \times n}(\mathbb{R})$ and a subset $I \subseteq \{1, \ldots, n\}$, we say that $A_I = (a_{ij})_{(i,j) \in I \times I}$ is the *submatrix $I \times I$* of A.

Lemma 3.7. *Let $A \in \mathrm{Mat}_{n \times n}(\mathbb{R})$ be a stably dissipative matrix. Then, for all $I \subseteq \{1, \ldots, n\}$, the submatrix A_I is stably dissipative.*

Proof. Let $I \subset \{1, \ldots, n\}$ and consider an admissible perturbation $B = (b_{ij})_{i,j \in I}$ of A_I. Define $\tilde{A} = (\tilde{a}_{ij})$ to be the matrix with entries

$$\tilde{a}_{ij} = \begin{cases} b_{ij} & \text{if} \quad (i,j) \in I \times I \\ a_{ij} & \text{if} \quad (i,j) \notin I \times I \end{cases}.$$

Clearly, $\tilde{A}$ is an admissible perturbation of A. Hence there exists a positive diagonal matrix D such that $\tilde{A}D \leq 0$. Letting now D_I be the $I \times I$ submatrix of D, we see that $BD_I = (\tilde{A}D)_I \leq 0$, which concludes the proof. □

Definition 3.8. We call attractor of the LV system (4) the following topological closure

$$\Lambda_{A,q} := \overline{\cup_{x \in \mathbb{R}_+^n} \omega(x)} \,,$$

where $\omega(x)$ is the ω-limit of x by the semi-flow $\{\phi_{A,q}^t : \mathbb{R}_+^n \to \mathbb{R}_+^n\}_{t \geq 0}$.

We need the following classical theorem (see [30, Theorem 2]).

Theorem 3.9 (La Salle). *Given a vector field $f(x)$ on a manifold M, consider the autonomous o.d.e. on M,*

$$x' = f(x). \tag{6}$$

Let $h : M \to \mathbb{R}$ be a smooth function such that

1. *h is a Lyapunov function, i.e., the derivative of h along the flow satisfies $\dot{h}(x) := Dh_x f(x) \leq 0$ for all $x \in M$.*
2. *h is bounded from below.*
3. *h is a proper function, i.e. $\{h \leq a\}$ is compact for all $a \in \mathbb{R}$.*

Then (6) *induces a complete semi-flow on M such that the topological closure of all its ω-limits is contained in the region where the derivative of h along the flow vanishes, i.e.,*

$$\overline{\cup_{x \in M} \omega(x)} \subseteq \{x \in M : \dot{h}(x) = 0\}.$$

The following lemma plays a key role in the theory of stably dissipative systems.

Lemma 3.10. *Given a stably dissipative matrix A, if D is a positive diagonal matrix D such that $Q_{AD} \leq 0$ then for all $i = 1, \ldots, n$ and $w \in \mathbb{R}^n$,*

$$Q_{AD}(w) = 0 \quad \Rightarrow \quad a_{ii} w_i = 0 \,.$$

Proof. See [40]. □

By Theorem 3.9 the attractor $\Lambda_{A,q}$ is contained in the set $\{\dot{h} = 0\}$. By the proof Proposition 3 we have $\dot{h}(x) = Q_{D^{-1}A}(x - q)$. Hence

$$\Lambda_{A,q} \subseteq \left\{x \in \mathbb{R}_+^n : Q_{D^{-1}A}(x - q) = 0\right\},$$

and by Lemma 3.10 it follows that $\Lambda_{A,q} \subseteq \{x : x_i = q_i\}$ for every $i = 1, \ldots, n$ such that $a_{ii} < 0$.

Let us say that a species i *is of type* $\bullet$ to mean that the following inclusion holds $\Lambda_{A,q} \subseteq \{x : x_i = q_i\}$. Similarly, we say that a species i *is of type* $\oplus$, to state that $\Lambda_{A,q} \subseteq \{x : X_{A,q}^i(x) = 0\}$, where $X_{A,q}^i(x)$ stands for the i-th component of the vector $X_{A,q}(x)$. Equivalently, the strategy i is of type $\oplus$ if and only if the sets $\{x_i = \text{const}\}$ are invariant under the flow $\phi_{A,q}^t : \Lambda_{A,q} \hookleftarrow$. With this terminology it can be proven that

Proposition 6. *Given neighbor vertexes j, l in the graph $G(A)$,*

> *(a) If j is of type $\bullet$ or $\oplus$ and all of its neighbors are of type $\bullet$, except for l, then l is of type $\bullet$;*
> *(b) If j is of type $\bullet$ or $\oplus$ and all of its neighbors are of type $\bullet$ or $\oplus$, except for l, then l is of type $\oplus$;*
> *(c) If all neighbors of j are of type $\bullet$ or $\oplus$, then j is of type $\oplus$.*

Proof. See [38]. $\square$

Based on these facts, Redheffer *et al.* introduced a reduction algorithm on the graph $G(A)$ to derive information on the species' types of a stably dissipative LV system (4).

Rule 1. Initially, colour black, $\bullet$, every vertex i such that $a_{ii} < 0$, and colour white, $\circ$, all other vertices.

The reduction procedure consists in applying the following rules, corresponding to valid inference rules:

Rule 2. If j is a $\bullet$ or $\oplus$-vertex and all of its neighbours are $\bullet$, except for one vertex l, then colour l as $\bullet$;

Rule 3. If j is a $\bullet$ or $\oplus$-vertex and all of its neighbours are $\bullet$ or $\oplus$, except for one vertex l, then draw $\oplus$ at the vertex l;

Rule 4. If j is a $\circ$-vertex and all of is neighbours are $\bullet$ or $\oplus$, then draw $\oplus$ at the vertex j.

Redheffer *et al.* define the *reduced graph* of the system, $\mathscr{R}(A)$, as the graph obtained from $G(A)$ by successive applications of the reduction rules 2-4, until they can no longer be applied. An easy consequence of this theory is the following result.

Proposition 7. *Let $A \in \mathrm{Mat}_n(\mathbb{R})$ be a stably dissipative matrix and consider the LV system (4) with an equilibrium $q \in \mathrm{int}(\mathbb{R}^n_+)$.*

> *1. If all vertices of $\mathscr{R}(A)$ are $\bullet$ then q is the unique globally attractive equilibrium.*
> *2. If $\mathscr{R}(A)$ has only $\bullet$ or $\oplus$ vertices then there exists an invariant foliation with a unique globally attractive equilibrium in each leaf.*

Proof. Item (1) is clear because if all vertices are of type $\bullet$ then for every orbit $x(t) = (x_1(t), \ldots, x_n(t))$ of (4), and every $i = 1, \ldots, n$, one has $\lim_{t \to +\infty} x_i(t) = q_i$.

Likewise, if $\mathscr{R}(A)$ has only $\bullet$ or $\oplus$ vertices then every orbit of (4) converges to an equilibrium point, which depends on the initial condition. But by Proposition 5 there exists an invariant foliation $\mathscr{F}$ with a single equilibrium point in each leaf. Hence, the unique equilibrium point in each leaf of $\mathscr{F}$ must be globally attractive. $\square$

Definition 3.11. We say that a dissipative matrix $A \in \mathrm{Mat}_{n \times n}(\mathbb{R})$ is *almost skew-symmetric* iff $a_{ij} = -a_{ji}$ whenever $a_{ii} = 0$ or $a_{jj} = 0$, and the quadratic form Q_A is negative definite on the subspace

$$E = \{ w \in \mathbb{R}^n : w_i = 0 \text{ for all } i \text{ such that } a_{ii} = 0 \} .$$

Definition 3.12. We say that the graph $G(A)$ has a *strong link* ($\bullet\!-\!\bullet$) if there is an edge $\{i, j\}$ between vertexes i, j such that $a_{ii} < 0$ and $a_{jj} < 0$.

Proposition 8 (Zhao-Luo [54])**.** *Given $A \in \mathrm{Mat}_{n \times n}(\mathbb{R})$, A is stably dissipative iff every cycle of $G(A)$ contains at least a strong link and there is a positive diagonal matrix D such that AD is almost skew-symmetric.*

Proof. See [54, Theorem 2.3], or [13, Proposition 3.5]. □

A compactification procedure introduced by J. Hofbauer [22] shows that every Lotka-Volterra system in $\mathbb{R}_+^n$ is orbit equivalent to a replicator system on the n-dimensional simplex Δ^n. We briefly recall this compactification. Let A be a $n \times n$ real matrix and $r \in \mathbb{R}^n$ a constant vector. The Lotka-Volterra equation associated to A and r is defined on $\mathbb{R}_+^n$ as follows

$$\frac{dz_i}{dt} = z_i \left(r_i + (Az)_i \right) \quad 1 \leq i \leq n . \tag{7}$$

For each $j = 1, \ldots, n + 1$, let $\sigma_j := \{ x \in \Delta^n \subset \mathbb{R}^{n+1} : x_j = 0 \}$ and consider the diffeomorphism

$$\phi : \mathbb{R}_+^n \to \Delta^n \backslash \sigma_{n+1}$$

$$(z_1, \ldots, z_n) \mapsto \frac{1}{1 + \sum\limits_{i=1}^{n} z_i} (z_1, \ldots, z_n, 1).$$

A straightforward calculation shows that the push-forward of the vector field (7) is equal to $\frac{1}{x_{n+1}} X_{\tilde{A}}$. where $X_{\tilde{A}}$ is the replicator vector field associated to the payoff matrix

$$\tilde{A} = \begin{pmatrix} a_{11} & \ldots & a_{1n} & r_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \ldots & a_{nn} & r_n \\ 0 & \ldots & 0 & 0 \end{pmatrix} .$$

Since the flows of $\frac{1}{x_{n+1}} X_{\tilde{A}}$ and $X_{\tilde{A}}$ are orbit equivalent, we refer to $X_{\tilde{A}}$ as the compactification of the LV equation (7).

4. Hamiltonian Polymatrix Replicators.

Definition 4.1. We say that any vector $q \in \mathbb{R}^n$ is a *formal equilibrium* of a polymatrix game (n, A) if

(a) $(Aq)_i = (Aq)_j$ for all $i, j \in \alpha$, and all $\alpha = 1, \ldots, p$,
(b) $\sum_{j \in \alpha} q_j = 1$ for all $\alpha = 1, \ldots, p$.

The matrix A induces a quadratic form $Q_A : H_{\underline{n}} \to \mathbb{R}$ defined by $Q_A(w) := w^T A w$, where $H_{\underline{n}}$ is defined in (2).

Definition 4.2. We call diagonal matrix of type $\underline{n}$ any diagonal matrix $D = \text{diag}(d_i)$ such that $d_i = d_j$ for all $i, j \in \alpha$ and $\alpha = 1, \ldots, p$.

Definition 4.3. A polymatrix game $(\underline{n}, A)$ is called *conservative* if it has a formal equilibrium q, and there exists a positive diagonal matrix D of type $\underline{n}$ such that $Q_{AD} = 0$ on $H_{\underline{n}}$.

In [1] we have defined conservative polymatrix game as follows.

Definition 4.4. A polymatrix game $(\underline{n}, A)$ is called *conservative* if

(a) it has a formal equilibrium,
(b) there are matrices $A_0, D \in \text{Mat}_{n \times n}(\mathbb{R})$ such that
   (i) $(\underline{n}, A) \sim (\underline{n}, A_0 D)$,
   (ii) A_0 is skew-symmetric,
   (iii) D is a positive diagonal matrix of type $\underline{n}$.

However, we will prove in Proposition 11 that these two definitions are equivalent.

Let $\{e_1, \ldots, e_n\}$ denote the canonical basis in $\mathbb{R}^n$, and $V_{\underline{n}}$ be the set of vertices of $\Gamma_{\underline{n}}$. Each vertex $v \in V_{\underline{n}}$ can be written as $v = e_{i_1} + \cdots + e_{i_p}$, with $i_\alpha \in \alpha$, $\alpha = 1, \ldots, p$, and it determines the set

$$\mathscr{V}_v := \{ (i, i_\alpha) : i \in \alpha, \ i \neq i_\alpha, \ \alpha = 1, \ldots, p \}$$

of cardinal $n - p = \dim(H_{\underline{n}})$. Notice that $(i, j) \in \mathscr{V}_v$ iff $i \neq j$ are in the same group and $v_j = 1$. Hence there is a natural identification $\mathscr{V}_v \equiv \{ i \in \{1, \ldots, n\} : v_i = 0 \}$. For every vertex v, the family $\mathscr{B}_v := \{ e_i - e_j : (i, j) \in \mathscr{V}_v \}$ is a basis of $H_{\underline{n}}$.

Lemma 4.5. *For any vertex v of $\Gamma_{\underline{n}}$ and $x, q \in \Gamma_{\underline{n}}$,*

$$x - q = \sum_{(i,j) \in \mathscr{V}_v} (x_i - q_i)(e_i - e_j) .$$

Proof. Let v be a vertex of $\Gamma_{\underline{n}}$. Notice that for all $\alpha = 1, \ldots, p$,

$$-(x_{i_\alpha} - q_{i_\alpha}) = \sum_{\substack{i \neq i_\alpha \\ i \in \alpha}} (x_i - q_i) .$$

$$\sum_{(i,j) \in \mathscr{V}_v} (x_i - q_i)(e_i - e_j) = \sum_{\alpha=1}^{p} \sum_{\substack{i \neq i_\alpha \\ i \in \alpha}} (x_i - q_i)(e_i - e_{i_\alpha})$$

$$= \sum_{\alpha=1}^{p} \sum_{\substack{i \neq i_\alpha \\ i \in \alpha}} (x_i - q_i)e_i - \sum_{\alpha=1}^{p} \sum_{\substack{i \neq i_\alpha \\ i \in \alpha}} (x_i - q_i)e_{i_\alpha}$$

$$= \sum_{\alpha=1}^{p} \sum_{\substack{i \neq i_\alpha \\ i \in \alpha}} (x_i - q_i)e_i + \sum_{\alpha=1}^{p} (x_{i_\alpha} - q_{i_\alpha})e_{i_\alpha}$$

$$= \sum_{\alpha=1}^{p} \sum_{i \in \alpha} (x_i - q_i)e_i = x - q$$

$\square$

Given ordered pairs of strategies in the same group $(i, j), (k, l)$, i.e., $i, j \in \alpha$ and $k, l \in \beta$ for some $\alpha, \beta \in \{1, \ldots, p\}$, define

$$A_{(i,j),(k,l)} := a_{ik} + a_{jl} - a_{il} - a_{jk} .$$

Proposition 9. *The coefficients $A_{(i,j),(k,l)}$ do not depend on the representative A of the polymatrix game $(\underline{n}, A)$.*

Proof. Consider the matrix $B = A - C$, where the blocks $C^{\alpha\beta} = (c_{ij})_{i \in \alpha, j \in \beta}$ of C have equal rows for all $\alpha, \beta = 1, \ldots, p$. Let $(i, j) \in \alpha$ and $(k, l) \in \beta$ with $\alpha, \beta \in \{1, \ldots, p\}$. Then

$$\begin{aligned} B_{(i,j),(k,l)} &= b_{ik} + b_{jl} - b_{il} - b_{jk} \\ &= a_{ik} - c_k + a_{jl} - c_l - a_{il} + c_l - a_{jk} + c_k \\ &= A_{(i,j),(k,l)} , \end{aligned}$$

where c_k is the constant entry on the kth-column of $C^{\alpha\beta}$. $\square$

Definition 4.6. Given $v \in V_{\underline{n}}$, we define $A_v \in \mathrm{Mat}_{d \times d}(\mathbb{R})$, $d = n - p$, to be the matrix with entries $A_{(i,j),(k,l)}$, indexed in $\mathscr{V}_v \times \mathscr{V}_v$, and $G(A_v)$ to be its associated graph (see Definition 3.3).

Proposition   10.   *The   matrix A_v represents   the   quadratic   form $Q_A : H_{\underline{n}} \to \mathbb{R}$ in the basis $\mathscr{B}_v$.*

  More precisely, if q is a formal equilibrium of the polymatrix game $(\underline{n}, A)$ then the quadratic form $Q_A : H_{\underline{n}} \to \mathbb{R}$ is given by

$$Q_A(x - q) = \sum_{(i,j),(k,l) \in \mathscr{V}_v} A_{(i,j),(k,l)} \, (x_i - q_i) \, (x_k - q_k) \,. \tag{8}$$

Proof. Using lemma 4.5, we have

$$
\begin{aligned}
Q_A(x - q) &= \left(\sum_{(i,j) \in \mathscr{V}_v} (x_i - q_i)(e_i - e_j) \right)^T A \left(\sum_{(k,l) \in \mathscr{V}_v} (x_k - q_k)(e_k - e_l) \right) \\
&= \sum_{(i,j),(k,l) \in \mathscr{V}_v} (e_i - e_j)^T A (e_k - e_l)(x_i - q_i)(x_k - q_k) \\
&= \sum_{(i,j),(k,l) \in \mathscr{V}_v} A_{(i,j),(k,l)}(x_i - q_i)(x_k - q_k) \,,
\end{aligned}
$$

$\square$

Remark 1. All matrices A_v, with $v \in V_{\underline{n}}$, have the same rank because they represent, in different basis, the same (non-symmetric) bilinear form $B_A : H_{\underline{n}} \times H_{\underline{n}} \to \mathbb{R}$, $B_A(v, w) := v^T A w$.

Proposition 11. *Definitions 4.3 and 4.4 are equivalent.*

Proof. Given a matrix C with blocks $C^{\alpha\beta} = (c_{ij})_{i \in \alpha, j \in \beta}$ having equal rows for all $\alpha, \beta = 1, \ldots, p$, it is clear that $C_{(i,j),(k,l)} = 0$ for all pairs of strategies (i,j), (k,l) in the same group. Hence, by Proposition 10, Q_C vanishes on $H_{\underline{n}}$.

  If $(\underline{n}, A)$ is conservative in the sense of Definition 4.4 then there are matrices: A_0 skew-symmetric, and D positive diagonal of type $\underline{n}$, such that $(\underline{n}, A) \sim (\underline{n}, A_0 D)$. It follows that $(\underline{n}, A D^{-1}) \sim (\underline{n}, A_0)$ and as observed above the matrix $C = A D^{-1} - A_0$ satisfies $Q_C = 0$ on $H_{\underline{n}}$. Finally, since A_0 is skew-symmetric, we have $Q_{AD^{-1}} = 0$ on $H_{\underline{n}}$. In other words, $(\underline{n}, A)$ is conservative in the sense of Definition 4.3.

  Conversely, assume that A is conservative in the sense of Definition 4.3. Then for some positive diagonal matrix D of type $\underline{n}$, $Q_{AD^{-1}}$ vanishes on $H_{\underline{n}}$.

  Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of $\mathbb{R}^n$ where the vectors $v_\alpha = \frac{1}{\sqrt{n_\alpha}} \pi_\alpha(\mathbb{1})$, with $\alpha \in \{1, \ldots, p\}$, form a orthonormal basis of $H_{\underline{n}}^\perp$, and the family $\{v_{p+1}, \ldots, v_n\}$ is any orthonormal basis of $H_{\underline{n}}$.

  Let $m_{ij} = \langle AD^{-1}v_i, v_j \rangle$, for all $i, j = 1, \ldots, n$, so that $M = (m_{ij})_{i,j}$ represents the linear endomorphism $AD^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ w.r.t. the basis $\{v_1, \ldots, v_n\}$. Since $Q_{AD^{-1}} = 0$ on $H_{\underline{n}}$, the $(n - p) \times (n - p)$ sub-matrix M' of M, formed by the last $n - p$ rows and columns of M, is skew-symmetric.

  Let $M_0 \in \mathrm{Mat}_{n \times n}(\mathbb{R})$ be a skew-symmetric matrix that shares with M its last $n - p$ rows. Let $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ be the linear endomorphism represented by the matrix M_0 w.r.t. the basis $\{v_1, \ldots, v_n\}$, and identify A_0 with the matrix that represents it w.r.t. the canonical basis. Because M_0 is skew-symmetric, and $\{v_1, \ldots, v_n\}$ orthonormal, A_0 is skew-symmetric too.

Then $C = A D^{-1} - A_0$ is represented by the matrix $M - M_0$ w.r.t. the basis $\{v_1, \ldots, v_n\}$. Since the last $n - p$ rows of $M - M_0$ are zero, the range of C : $\mathbb{R}^n \to \mathbb{R}^n$ is contained in $H_{\underline{n}}^{\perp}$. Hence, by Lemma 2.4, $(\underline{n}, AD^{-1}) \sim (\underline{n}, A_0)$, which implies $(\underline{n}, A) \sim (\underline{n}, A_0 D)$. Since A_0 is skew-symmetric, this proves that $(\underline{n}, A)$ is conservative in the sense of Definition 4.4. $\qquad\square$

Remark 2. For all $w \in H_{\underline{n}}$, $Q_{D^{-1}A}(w) = Q_{AD}(D^{-1}w)$. Hence, because $D\, H_{\underline{n}} = H_{\underline{n}}$ for any diagonal matrix D of type $\underline{n}$

1. $Q_{A D}(w) = 0 \quad \forall\, w \in H_{\underline{n}} \quad \Leftrightarrow \quad Q_{D^{-1}A}(w) = 0 \quad \forall\, w \in H_{\underline{n}}$.
2. $Q_{A D}(w) \leq 0 \quad \forall\, w \in H_{\underline{n}} \quad \Leftrightarrow \quad Q_{D^{-1}A}(w) \leq 0 \quad \forall\, w \in H_{\underline{n}}$.

Lemma 4.7. *Given* $A \in \mathrm{Mat}_{n \times n}(\mathbb{R})$, *if* q *is a formal equilibrium of* $X_{\underline{n},A}$, *and* $D = \mathrm{diag}(d_i)$ *is a positive diagonal matrix of type* $\underline{n}$, *then the derivative of*

$$h(x) = -\sum_{i=1}^{n} \frac{q_i}{d_i} \log x_i \tag{9}$$

along the flow of $X_{\underline{n},A}$ *satisfies*

$$\dot{h}(x) = Q_{D^{-1}A}(x - q)\ .$$

Proof.

$$
\begin{aligned}
\dot{h} &= -\sum_{\alpha=1}^{p}\sum_{i\in\alpha} \frac{q_i}{d_i}\frac{\dot{x}_i}{x_i} = -\sum_{\alpha=1}^{p}\sum_{i\in\alpha}\frac{q_i}{d_i}\left((Ax)_i - \sum_{\beta=1}^{p}(x^\alpha)^t A^{\alpha,\beta} x^\beta\right)\\
&= -q^T D^{-1} A x + x^T D^{-1} A x = (x-q)^T D^{-1} A x\\
&= (x-q)^T D^{-1} A x - \underbrace{(x-q)^T D^{-1} A q}_{=0}\\
&= (x-q)^T D^{-1} A (x-q) = Q_{D^{-1}A}(x-q)\ .
\end{aligned}
$$

To explain the vanishing term notice that for all $\alpha \in \{1, \ldots, p\}$ and $i, j \in \alpha$, $(A\,q)_i = (A\,q)_j$, $d_i = d_j$ and $\sum_{k\in\alpha}(x_k - q_k) = 0$. $\qquad\square$

Proposition 12. *If* $(\underline{n}, A)$ *is conservative, and* q *and* $D = \mathrm{diag}(d_i)$ *are as in Definition 4.3, then* (9) *is a first integral for the flow of* $X_{\underline{n},A}$, *i.e.,* $\dot{h} = 0$ *along the flow of* $X_{\underline{n},A}$.

Moreover, $X_{\underline{n},A}$ *is Hamiltonian w.r.t. a stratified Poisson structure on the prism* $\Gamma_{\underline{n}}$, *having* h *as its Hamiltonian function.*

Proof. The first part follows from Lemma 4.7 and Remark 2. The second follows from [1, theorem 3.20]. $\qquad\square$

5. Dissipative Polymatrix Replicators.

Definition 5.1. A polymatrix game $(\underline{n}, A)$ is called *dissipative* if it has a formal equilibrium q, and there exists a positive diagonal matrix D of type $\underline{n}$ such that $Q_{A D} \leq 0$ on $H_{\underline{n}}$.

Proposition 13. *If* $(\underline{n}, A)$ *is dissipative, and* q *and* D *are as in Definition 5.1, then*

$$h(x) = -\sum_{i=1}^{n} \frac{q_i}{d_i} \log x_i$$

is a Lyapunov decreasing function for the flow of $X_{\underline{n},A}$, i.e., $\frac{dh}{dt} \leq 0$ along the flow of $X_{\underline{n},A}$.

Proof. Follows from Lemma 4.7, and Remark 2. $\qquad\square$

Definition 5.2. A polymatrix game $(\underline{n}, A)$ is called *admissible* if $(\underline{n}, A)$ is dissipative and for some vertex $v \in \Gamma_{\underline{n}}$ the matrix A_v is stably dissipative (see Definition 3.4). We denote by $V^*_{\underline{n},A}$ the subset of vertices $v \in V_{\underline{n}}$ such that A_v is stably dissipative.

Proposition 14. *Let q be a formal equilibrium of the polymatrix game $(\underline{n}, A)$. Given $v \in V_{\underline{n}}$ and $(i,j) \in \mathscr{V}_v$, then we have the following quotient rule*

$$\frac{d}{dt}\left(\frac{x_i}{x_j}\right) = \frac{x_i}{x_j} \sum_{(k,l)\in\mathscr{V}_v} A_{(i,j),(k,l)}\left(x_k - q_k\right). \tag{10}$$

Proof. Let v be a vertex of $\Gamma_{\underline{n}}$, $(i,j) \in \mathscr{V}_v$, and q be a formal equilibrium. Using Lemma 4.5, we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{x_i}{x_j}\right) &= \frac{x_i}{x_j}\left((Ax)_i - (Ax)_j\right) \\
&= \frac{x_i}{x_j}\left((A(x-q))_i - (A(x-q))_j\right) \\
&= \frac{x_i}{x_j} \sum_{(k,l)\in\mathscr{V}_v} (e_i - e_j)^T A(e_k - e_l)(x_k - q_k) \\
&= \frac{x_i}{x_j} \sum_{(k,l)\in\mathscr{V}_v} A_{(i,j),(k,l)}(x_k - q_k).
\end{aligned}$$

$\qquad\square$

Proposition 15. *If the dissipative polymatrix replicator associated to $(\underline{n}, A)$ has an equilibrium $q \in \mathrm{int}\left(\Gamma_{\underline{n}}\right)$, then for any state $x_0 \in \mathrm{int}\left(\Gamma_{\underline{n}}\right)$ and any pair of strategies i, j in the same group, the solution $x(t)$ of (1) with initial condition $x(0) = x_0$ satisfies*

$$\frac{1}{c} \leq \frac{x_i(t)}{x_j(t)} \leq c, \quad for\ all \quad t \geq 0,$$

where $c = c(x)$ is a constant depending on x.

Proof. Notice that the Lyapunov function h in Proposition 13 is a proper function because $q \in \mathrm{int}(\Gamma_{\underline{n}})$. Given $x_0 \in \mathrm{int}\left(\Gamma_{\underline{n}}\right)$, $h(x_0) = a$ for some constant $a > 0$. By Proposition 13 the compact set $K = \{x \in \mathrm{int}(\Gamma_{\underline{n}}) : h(x) \leq a\}$ is forward invariant by the flow of $X_{\underline{n},A}$. In particular, the solution of the polymatrix replicator with initial condition $x(0) = x_0$ lies in K. Hence the quotient $\frac{x_i}{x_j}$ has a minimum and a maximum in K. $\qquad\square$

Proposition 16. *Given a dissipative polymatrix game $(\underline{n}, A)$, if $X_{\underline{n},A}$ admits an equilibrium $q \in \mathrm{int}(\Gamma_{\underline{n}})$ then there exists a $X_{\underline{n},A}$-invariant foliation $\mathscr{F}$ on $\mathrm{int}(\Gamma_{\underline{n}})$ such that every leaf of $\mathscr{F}$ contains exactly one equilibrium point.*

Proof. Fix some vertex $v \in V_{\underline{n}}$. Recall that the entries of A_v are indexed in the set $\mathscr{V}_v \equiv \{i \in \{1,\ldots,n\} : v_i = 0\}$. Given a vector $w = (w_i)_{i\in\mathscr{V}_v} \in \mathbb{R}^{n-p}$, we denote by $\bar{w}$ the unique vector $\bar{w} \in H_{\underline{n}}$ such that $\bar{w}_i = w_i$ for all $i \in \mathscr{V}_v$.

Let $\mathscr{E} \subset \mathbb{R}^n$ be the affine subspace of all points $x \in \mathbb{R}^n$ such that for all $\alpha = 1,\ldots,p$ and all $i,j \in \alpha$, $(Ax)_i = (Ax)_j$ and $\sum_{j\in\alpha} x_j = 1$. By definition $\mathscr{E} \cap \mathrm{int}(\mathbb{R}^n)$

is the set of interior equilibria of $X_{\underline{n},A}$. We claim that $\mathscr{E} = \{q + \bar{w} \colon w \in \mathrm{Ker}(A_v)\}$. To see this it is enough to remark that $w \in \mathrm{Ker}(A_v)$ if and only if

$$(A\bar{w})_i - (A\bar{w})_j = (e_i - e_j)^T A\bar{w} = 0 \quad \forall \, (i,j) \in \mathscr{V}_v.$$

Given $b \in \mathrm{Ker}(A_v^T)$, consider the function $g_b \colon \mathrm{int}(\mathbb{R}_+^n) \to \mathbb{R}$ defined by $g_b(x) := \sum_{j=1}^n \bar{b}_j \log x_j$. The restriction of g_b to $\Gamma_{\underline{n}}$ is invariant by the flow of $X_{\underline{n},A}$. Note we can write

$$g_b(x) = \sum_{l=1}^n \bar{b}_l \log x_l = \sum_{(i,j)\in\mathscr{V}_v} b_i \log\left(\frac{x_i}{x_j}\right),$$

and differentiating g_b along the flow of $X_{\underline{n},A}$, by Proposition 14 we get

$$\dot{g}_b(x) = b^T A_v (x_k - q_k)_{k\in\mathscr{V}_v} = 0 \quad \text{for all } x \in \Gamma_{\underline{n}}.$$

Fix a basis $\{b_1, \ldots, b_k\}$ of $\mathrm{Ker}(A_v^T)$, and define $g \colon \mathrm{int}(\mathbb{R}_+^n) \to \mathbb{R}^k$ by $g(x) := (g_{b_1}(x), \ldots, g_{b_k}(x))$. This map is a submersion. For that consider the matrix $B \in \mathrm{Mat}_{k\times n}(\mathbb{R})$ whose rows are the vectors $\bar{b}_j$, $j = 1, \ldots, k$. We can write $g(x) = B \log x$, where $\log x = (\log x_1, \ldots, \log x_n)$. Hence $Dg_x = B D_x^{-1}$, where $D_x = \mathrm{diag}(x_1, \ldots, x_n)$, and because B has maximal rank, $\mathrm{rank}(B) = k$, the map g is a submersion. Hence g determines the foliation $\mathscr{F}$ whose leaves are the pre-images $g^{-1}(c) = \{g \equiv c\}$ with $c \in \mathbb{R}^k$.

Let us now explain why each leaf of $\mathscr{F}$ contains exactly one point in $\mathscr{E}$. Consider the vector subspace parallel to $\mathscr{E}$, $E_0 := \{\bar{w} \colon w \in \mathrm{Ker}(A_v)\}$. Because $(\underline{n}, A)$ is dissipative, $A_v \in \mathrm{Mat}_{d\times d}(\mathbb{R})$, $d = n-p$, is also dissipative, and by Proposition 4, $\mathrm{Ker}(A_v)$ and $\mathrm{Ker}(A_v^T)$ have the same rank. Therefore $\dim(E_0) = k$. Let $\{c_1, \ldots, c_{n-k}\}$ be a basis of $E_0^\perp \subset \mathbb{R}^n$ and consider the matrix $C \in \mathrm{Mat}_{(n-k)\times n}(\mathbb{R})$ whose rows are the vectors c_j, $j = 1, \ldots, n - k$. The matrix C provides the following description $\mathscr{E} = \{x \in \mathbb{R}^n \colon C(x - q) = 0\}$. Consider the matrix $U = \begin{bmatrix} B \\ \hline C \end{bmatrix} \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, which is nonsingular because by Proposition 4, $\mathrm{Ker}(A_v) = D\,\mathrm{Ker}(A_v^T)$, for some positive diagonal matrix D.

The intersection $g^{-1}(c) \cap \mathscr{E}$ is described by the non-linear system

$$x \in g^{-1}(c) \cap \mathscr{E} \quad \Leftrightarrow \quad \begin{cases} B \log x = c \\ C(x - q) = 0 \end{cases}.$$

Considering $u = \log x$, this system becomes

$$\begin{cases} B\,u = c \\ C(e^u - q) = 0 \end{cases}.$$

It is now enough to see that

$$\begin{cases} B\,u = c \\ C(e^u - q) = 0 \end{cases} \quad \text{and} \quad \begin{cases} B\,u' = c \\ C(e^{u'} - q) = 0 \end{cases}$$

imply $u = u'$. By the mean value theorem, for every $i \in \{1, \ldots, n\}$ there is some $\tilde{u}_i \in [u_i, u_i']$ such that

$$e^{u_i} - e^{u_i'} = e^{\tilde{u}_i}(u_i - u_i'),$$

which in vector notation is to say that

$$e^u - e^{u'} = D_{e^{\tilde{u}}}(u - u') = e^{\tilde{u}} * (u - u').$$

Hence

$$
\begin{cases} B\,(u - u') = 0 \\ C(e^u - e^{u'}) = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} B\,(u - u') = 0 \\ C\,D_{e^{\tilde{u}}}(u - u') = 0 \end{cases}
$$

$$
\Leftrightarrow \quad \left[\frac{B}{C\,D_{e^{\tilde{u}}}}\right](u - u') = 0
$$

$$
\Leftrightarrow \quad U\left[\begin{array}{c|c} I & 0 \\ \hline 0 & D_{e^{\tilde{u}}} \end{array}\right](u - u') = 0 \ .
$$

Therefore, because $\left[\begin{array}{c|c} I & 0 \\ \hline 0 & D_{e^{\tilde{u}}} \end{array}\right]$ is non-singular, we must have $u = u'$.

Restricting $\mathscr{F}$ to $\mathrm{int}(\Gamma_{\underline{n}})$ we obtain a $X_{\underline{n},A}$-invariant foliation on $\mathrm{int}(\Gamma_{\underline{n}})$. Notice that the restriction $g_{|\mathrm{int}(\Gamma_{\underline{n}})} : \mathrm{int}(\Gamma_{\underline{n}}) \to \mathbb{R}^k$ is invariant by the flow of $X_{\underline{n},A}$ because all its components are.

Since all points in $\mathrm{int}(\Gamma_{\underline{n}}) \cap \mathscr{E}$ are equilibria, each leaf of the restricted foliation contains exactly one equilibrium point. □

Definition 5.3. We call attractor of the polymatrix replicator (1) the following topological closure

$$
\Lambda_{\underline{n},A} := \overline{\cup_{x \in \Gamma_{\underline{n}}} \omega(x)} \ ,
$$

where $\omega(x)$ is the ω-limit of x by the flow $\{\varphi_{\underline{n},A}^t : \Gamma_{\underline{n}} \to \Gamma_{\underline{n}}\}_{t \in \mathbb{R}}$.

Proposition 17. *Given a dissipative polymatrix replicator associated to $(\underline{n}, A)$ with an equilibrium $q \in \mathrm{int}\left(\Gamma_{\underline{n}}\right)$ and a diagonal matrix D as in Definition 5.1, we have that*

$$
\Lambda_{\underline{n},A} \subseteq \left\{ x \in \Gamma_{\underline{n}} \ : \ Q_{D^{-1}A}(x - q) = 0 \right\} \ .
$$

Proof. By Theorem 3.9 the attractor $\Lambda_{\underline{n},A}$ is contained in the region where $\dot{h} = 0$. The conclusion follows then by Lemma 4.7. □

Given an admissible polymatrix replicator associated to $(\underline{n}, A)$ with an equilibrium $q \in \mathrm{int}\left(\Gamma_{\underline{n}}\right)$, we say that a strategy i *is of type* • to mean that the following inclusion holds $\Lambda_{\underline{n},A} \subseteq \{x \in \Gamma_{\underline{n}} : x_i = q_i\}$. Similarly, we say that a strategy i is *of type* $\oplus$ to state that $\Lambda_{\underline{n},A} \subseteq \{x \in \Gamma_{\underline{n}} : X_{\underline{n},A}^i(x) = 0\}$, where $X_{\underline{n},A}^i(x)$ stands for the i-th component of the vector $X_{\underline{n},A}(x)$. Given two strategies i and j in the same group, we say that *i and j are related* when the orbits on the attractor $\Lambda_{\underline{n},A}$ preserve the foliation $\{ \frac{x_i}{x_j} = \mathrm{const.} \}$.

For any $v \in V_{\underline{n}}$ we will denote by a_{ij}^v the entries of the matrix A_v.

With this terminology we have

Proposition 18. *Given an admissible polymatrix game $(\underline{n}, A)$ with an equilibrium $q \in \mathrm{int}\left(\Gamma_{\underline{n}}\right)$ the following statements hold:*

(1) *For any graph $G(A_v)$ with $v \in V_{\underline{n},A}^*$:*
  (a) *if i is a strategy such that $v_i = 0$ and $a_{ii}^v < 0$, then i is of type •;*
  (b) *if j is a strategy of type • or $\oplus$ and all neighbours of j but (possibly) l in $G(A_v)$ are of type •, then l is also of type •;*
  (c) *if j is a strategy of type • or $\oplus$ and all neighbours of j but (possibly) l in $G(A_v)$ are of type • or $\oplus$, then l is also of type $\oplus$;*
(2) *For any graph $G(A_v)$ with $v \in V_{\underline{n}}$:*
  (d) *if all neighbours of a strategy j in $G(A_v)$ are of type • or $\oplus$, then j is related to the unique strategy j', in the same group as j, such that $v_{j'} = 1$.*

Proof. The proof involves the manipulation of algebraic relations holding on the attractor. To simplify the terminology we will say that some algebraic relation holds to mean that it holds on the attractor.

Choose a positive diagonal matrix D of type $\underline{n}$ such that $Q_{AD} \leq 0$ on $H_{\underline{n}}$, and set $\tilde{A} := D^{-1}A$. By Lemma 3.5, for any $v \in V_{\underline{n}}$, the matrices A_v and $\tilde{A}_v$ have the same dissipative and stably dissipative character. Hence $V_{\underline{n},A}^* = V_{\underline{n},\tilde{A}}^*$.

Given $v \in V_{\underline{n},A}^*$, for any solution $x(t)$ of the polymatrix replicator in the attractor, we have that $Q_{\tilde{A}_v}(x(t) - q) = 0$. Hence, as $\tilde{A}_v$ is stably dissipative and $a_{ii}^v < 0$, by Lemma 3.10 follows that $x_i(t) = q_i$ on the attractor, which proves (a).

Given $v \in V_{\underline{n},A}^*$ we have that $\tilde{A}_v$ is stably dissipative. By Proposition 17, we obtain

$$\sum_{(k,l)\in\mathscr{V}_v} \tilde{A}_{(j,j'),(k,l)}(x_k - q_k) = 0$$

on the attractor.

Observe that if j is of type $\bullet$, then $x_j = q_j$, and if j is of type $\oplus$, then $a_{jj}^v = A_{(j,j'),(j,j')} = 0$, where j' is the unique strategy in the same group as j such that $v_{j'} = 1$.

Let j, l be neighbour vertices in the graph $G(A_v)$.

Let us prove (b). If j is of type $\bullet$ or $\oplus$ and all of its neighbours are of type $\bullet$, except for l, then

$$\tilde{A}_{(j,j'),(l,l')}(x_l - q_l) = 0 \,,$$

from which follows that $x_l = q_l$ because $A_{(j,j'),(l,l')} = d_j \tilde{A}_{(j,j'),(l,l')} \neq 0$, where l' is the unique strategy in the same group as l such that $v_{l'} = 1$. This proves (b).

Let us prove (c). If j is of type $\bullet$ or $\oplus$ and all of its neighbours are of type $\bullet$ or $\oplus$, except for l, then

$$A_{(j,j'),(l,l')}(x_l - q_l) = c \,,$$

for some constant c. Hence because $A_{(j,j'),(l,l')} \neq 0$, x_l is constant which proves (c).

Let us prove (d). Suppose all neighbours of a strategy j are of type $\bullet$ or $\oplus$. By the polymatrix quotient rule (see Proposition 14),

$$\frac{d}{dt}\left(\frac{x_j}{x_{j'}}\right) = \frac{x_j}{x_{j'}} \sum_{(k,l)\in\mathscr{V}_v} A_{(j,j'),(k,l)}(x_k - q_k) \,.$$

Since all neighbours of j are of type $\bullet$ or $\oplus$ we obtain

$$\frac{d}{dt}\left(\frac{x_j}{x_{j'}}\right) = \frac{x_j}{x_{j'}} C \,,$$

for some constant C. Hence

$$\frac{x_j}{x_{j'}} = B_0 \, e^{Ct} \,,$$

where $B_0 = \frac{x_j(0)}{x_{j'}(0)}$. By Proposition 15 we have that the constant C must be 0. Hence there exists a constant $B_0 > 0$ such that $\frac{x_j}{x_{j'}} = B_0$, which proves (d). $\qquad\square$

Proposition 19. *If in a group α all strategies are of type $\bullet$ (respectively of type $\bullet$ or $\oplus$) except possibly for one strategy i, then i is also of type $\bullet$ (respectively of type $\oplus$).*

Proof. Suppose that in a group α all strategies are of type $\bullet$ or $\oplus$ except for one strategy i. We have that $x_k = c_k$, for some constant c_k, for each $k \neq i$. Thus,

$$x_i = 1 - \sum_{\substack{j \in \alpha \\ j \neq i}} x_j = 1 - \sum_{j=\bullet} x_j - \sum_{k=\oplus} x_k = 1 - \sum_{j=\bullet} q_j - \sum_{k=\oplus} c_k \,.$$

Hence i is of type $\oplus$.

If in a group α all strategies are of type $\bullet$, the proof is analogous. $\square$

Proposition 20. *Assume that in a group α with n strategies, $n - k$ of them, with $0 \leq k < n$, are of type $\bullet$ or $\oplus$, and denote by S the set of the remaining k strategies. If the graph with vertex set S, obtained by drawing an edge between every pair of related strategies in S, is connected, then all strategies in S are of type $\oplus$.*

Proof. Since all strategies in $\alpha \setminus S$ are of type $\bullet$ or $\oplus$, for the strategies in S we have that

$$\sum_{i \in S} x_i = 1 - C \,, \tag{11}$$

where $C = \sum_{j \in \alpha \setminus S} x_j$.

Let G_S be the graph with vertex set S obtained drawing an edge between every pair of related strategies in S. Since G_S is connected we have that it contains a tree. Considering the $k-1$ relations between the strategies in S given by that tree, we have $k-1$ linearly independent equations of the form $x_i = C_{ij} x_j$ for pairs of strategies i and j in S, where C_{ij} is a constant. Together with (11) we obtain k linear independent equations for the k strategies in S, which implies that $x_i = $ constant, for every $i \in S$. This concludes the proof. $\square$

Based on these facts we introduce a reduction algorithm on the set of graphs $\{ G(A_v) : v \in V_{\underline{n}} \}$ to derive information on the strategies of an admissible polymatrix game $(\underline{n}, A)$.

In each step, we also register the information obtained about each strategy in what we call the "information set", where all strategies of the polymatrix are represented.

The algorithm is about labelling (or colouring) strategies with the "colours" $\bullet$ and $\oplus$. The algorithm acts upon all graphs $G(A_v)$ with $v \in V_{\underline{n}}$ as well as on the information set. It is implicit that after each rule application, the new labels (or colours) are transferred between the graphs $G(A_v)$ and the information set, that is, if in a graph $G(A_v)$ a strategy i has been coloured $i = \bullet$, then in all other graphs containing the strategy i, we colour it $i = \bullet$, as well on the information set.

Some rules just can be applied to graphs $G(A_v)$ such that $v \in V_{\underline{n},A}^*$, while others can be applied to all graphs.

Rule 1. Initially, for each graph $G(A_v)$ such that $v \in V_{\underline{n},A}^*$ colour in black ($\bullet$) any strategy i such that $a_{ii}^v < 0$. Colour in white ($\circ$) all other strategies.

The reduction procedure consists in applying the following rules, corresponding to valid inferences rules. For each graph $G(A_v)$ such that $v \in V_{\underline{n},A}^*$:

Rule 2. If i has colour $\bullet$ or $\oplus$ and all neighbours of i but j in $G(A_v)$ are $\bullet$, then colour $j = \bullet$.

Rule 3. If i has colour $\bullet$ or $\oplus$ and all neighbours of i but j in $G(A_v)$ are $\bullet$ or $\oplus$, then colour $j = \oplus$.

For each graph $G(A_v)$ such that $v \in V_{\underline{n}}$:

Rule 4. If i has colour $\circ$ and all neighbours of i in $G(A_v)$ are $\bullet$ or $\oplus$, then we put a link between strategies j and j' in the "information set", where j' is the unique strategy such that $v_{j'} = 1$ and j' is in the same group as j.

The following rules can be applied to the set of all strategies of the polymatrix game.

Rule 5. If in a group all strategies have colour $\bullet$ (respectively, $\bullet, \oplus$) except for one strategy i, then colour $i = \bullet$ (respectively, $i = \oplus$).

Rule 6. If in a group some strategies have colour $\bullet$ or $\oplus$, and the remaining strategies are related forming a connected graph, then colour with $\oplus$ all that remaining strategies.

We define the *reduced information set* $\mathscr{R}(\underline{n}, A)$ as the $\{\bullet, \oplus, \circ\}$-coloring on the set of strategies $\{1, \ldots, n\}$, which is obtained by successive applications to the graphs $G(A_v)$, $v \in V_{\underline{n}}$, of the reduction rules 1-6, until they can no longer be applied.

Proposition 21. *Let $(\underline{n}, A)$ be an admissible polymatrix game, and consider the associated polymatrix replicator* (1) *with an interior equilibrium $q \in \mathrm{int}(\Gamma_{\underline{n}})$.*

1. *If all vertices of $\mathscr{R}(\underline{n}, A)$ are $\bullet$ then q is the unique globally attractive equilibrium.*
2. *If $\mathscr{R}(\underline{n}, A)$ has only $\bullet$ or $\oplus$ vertices then there exists an invariant foliation with a unique globally attractive equilibrium in each leaf.*

Proof. Item (1) is clear because if all strategies are of type $\bullet$ then for every orbit $x(t) = (x_1(t), \ldots, x_n(t))$ of (1), and every $i = 1, \ldots, n$, one has $\lim_{t \to +\infty} x_i(t) = q_i$.

Likewise, if $\mathscr{R}(\underline{n}, A)$ has only $\bullet$ or $\oplus$ vertices then every orbit of (1) converges to an equilibrium point, which depends on the initial condition. But by Proposition 16 there exists an invariant foliation $\mathscr{F}$ with a single equilibrium point in each leaf. Hence, the unique equilibrium point in each leaf of $\mathscr{F}$ must be globally attractive. $\square$

The following definition corresponds to a one-step reduction of the attractor dynamics.

Definition 5.4. Given a polymatrix game $(\underline{n}, A)$, a strategy $l \in \alpha$, for some group α, and a point $q \in \mathrm{int}(\Gamma_{\underline{n}})$, we call (q, l)-*reduction of* $(\underline{n}, A)$ a new polymatrix game $(\underline{n}(l), A(l))$ obtained removing the strategy l from the group α, where $\underline{n}(l) := (n_1, \ldots, n_{\alpha-1}, n_\alpha - 1, n_{\alpha+1}, \ldots, n_p)$, and the matrix $A(l) = (a_{ij}(l))$ indexed in $\{1, \ldots, l-1, l+1, \ldots, n\}$ has the following entries:

$$a_{ij}(l) := \begin{cases} a_{ij} - a_{lj} & \text{if } j \notin \alpha \\ (a_{ij} - a_{lj})(1 - q_l) + (a_{il} - a_{ll})q_l & \text{if } j \in \alpha \setminus \{l\} . \end{cases} \quad (12)$$

The map $\psi_l : \Gamma_{\underline{n}} \cap \{x_l = q_l\} \to \Gamma_{\underline{n}(l)}$, $\psi_l(x) = \check{x}^l = (x_j)_{j \neq l}$, defines a natural identification.

Proposition 22. *Let $(\underline{n}, A)$ be a polymatrix game with an equilibrium $q \in \mathrm{int}(\Gamma_{\underline{n}})$. Given a strategy $l \in \alpha$, for some group α, the (q, l)-reduction $(\underline{n}(l), A(l))$ of $(\underline{n}, A)$ is such that if $x \in \Gamma_{\underline{n}} \cap \{x_l = q_l\}$ and $X_{\underline{n}, A}(x)$ is tangent to $\{x_l = q_l\}$, that is $X_{\underline{n}, A}^l(x) = 0$, then for all $j \neq l$,*

$$X_{\underline{n}, A}^j(x) = X_{\underline{n}(l), A(l)}^j(\check{x}^l) .$$

Proof. Suppose that for some $\alpha \in \{1, \ldots, p\}$ there exists $l \in \alpha$ such that $x \in \Gamma_{\underline{n}} \cap \{x_l = q_l\}$ and $X_{\underline{n},A}^l(x) = 0$.

Since $\sum_{\substack{j \in \alpha \\ j \neq l}} x_j = 1 - q_l$, considering the change of variables

$$y_j = \begin{cases} \frac{x_j}{1-q_l} & \text{if } j \in \alpha \setminus \{l\} \\ x_j & \text{if } j \notin \alpha \end{cases} \,, \tag{13}$$

we have that $\sum_{j \in \alpha \setminus \{l\}} y_j = 1$.

By Proposition 1, we can assume $A = (a_{ij})$ has all entries equal to zero in row l, i.e., $a_{lj} = 0$ for all j. Thus we obtain

$$\frac{dx_l}{dt} = x_l \left(-\sum_{\beta=1}^{p} (x^\alpha)^t A^{\alpha\beta} x^\beta \right) \,.$$

Hence, making $x_l = q_l$, the replicator equation (1) becomes

(i) if $i \in \alpha \setminus \{l\}$,

$$\frac{dx_i}{dt} = x_i \left(\sum_{\substack{j=1 \\ j \neq l}}^{n} a_{ij} x_j + a_{il} q_l - \sum_{\substack{k \in \alpha \\ k \neq l}} \sum_{j=1}^{n} a_{kj} x_k x_j \right) \tag{14}$$

(ii) if $i \in \beta \neq \alpha$, the equation is essentially the same, with $x_l = q_l$.

Observe that $\sum_{\beta=1}^{p} (x^\alpha)^t A^{\alpha\beta} x^\beta = 0$ because we are assuming that $x \in \Gamma_{\underline{n}} \cap \{x_l = q_l\}$ and $X_{\underline{n},A}^l(x) = 0$.

Hence we can add

$$-\frac{q_l}{1-q_l} \sum_{\beta=1}^{p} (x^\alpha)^t A^{\alpha\beta} x^\beta$$

to each equation for $\frac{dx_i}{dt}$, with $i \in \alpha \setminus \{l\}$, without changing the vector field $X_{\underline{n},A}$ at the points $x \in \Gamma_{\underline{n}} \cap \{x_l = q_l\}$ where $X_{\underline{n},A}(x)$ is tangent to $\{x_l = q_l\}$. So equation (14) becomes

$$\frac{dx_i}{dt} = x_i \left(\sum_{\substack{j=1 \\ j \neq l}}^{n} a_{ij} x_j + a_{il} q_l - \frac{1}{1-q_l} \sum_{\substack{k \in \alpha \\ k \neq l}} \sum_{j=1}^{n} a_{kj} x_k x_j \right) \tag{15}$$

Now, using the change of variables (13), equation (15) becomes

$$\frac{dy_i}{dt} = y_i \left(f_i - \sum_{\substack{k \in \alpha \\ k \neq l}} y_k f_k \right) \quad (i \in \alpha) \,, \tag{16}$$

where $f_i = \sum_{j \in \alpha \setminus \{l\}} a_{ij}(1 - q_l) y_j + a_{il} q_l + \sum_{j \notin \alpha} a_{ij} y_j$.

Let $\check{\alpha} \equiv \alpha \setminus \{l\}$. Setting $a_{il} q_l = a_{il} q_l (\sum_{j \in \check{\alpha}} y_j)$,

$$\frac{dy_i}{dt} = y_i \left(g_i - \sum_{k \in \beta} y_k g_k \right) \,, \quad i \in \beta, \ \beta \in \{1, \ldots, p\} \,, \tag{17}$$

where $g_i = \sum_{j \in \check{\alpha}} (a_{ij}(1-q_l) + a_{il}q_l)y_j + \sum_{j \notin \check{\alpha}} a_{ij}y_j$, defines a new polymatrix game in dimension $n-1$. In fact, (17) is the replicator equation of the polymatrix game $(\underline{n}(l), A(l))$, where, since we have assumed that $a_{lj} = 0$ for all j, (12) becomes

$$a_{ij}(l) = \begin{cases} a_{ij} & \text{if } j \notin \check{\alpha} \\ a_{ij}(1-q_l) + a_{il}q_l & \text{if } j \in \check{\alpha} \, . \end{cases}$$

$\square$

Remark 3. Under the assumptions of Proposition 22, when $n_\alpha = 2$, considering for instance that the group α consists of strategies $l-1$ and l, $x_l = q_l$ implies that $x_{l-1} = 1 - q_l = q_{l-1}$. Hence we can further reduce the polymatrix game $(\underline{n}(l), A(l))$ to a new polymatrix game with type $(n_1, \ldots, n_{\alpha-1}, n_{\alpha+1}, \ldots, n_p)$ and payoff matrix indexed in $\{1, \ldots, \ell-2, l+1, \ldots, n\}$.

Corollary 1. *Let $(\underline{n}, A)$ be a polymatrix game with an equilibrium $q \in \text{int}\left(\Gamma_{\underline{n}}\right)$. Given a set $Q \subset \{1, \ldots, n\}$ of strategies such that*

$$\Lambda_{\underline{n},A} \subseteq \bigcap_{l \in Q} \{x_l = q_l\} \, ,$$

then there exists a new polymatrix game $(\underline{m}, B)$, where $m_\alpha = |\alpha \setminus Q|$ for every $\alpha = 1, \ldots, p$, and an identification $\psi : \Gamma_{\underline{n}} \cap \bigcap_{l \in Q} \{x_l = q_l\} \to \Gamma_{\underline{m}}$ such that $X_{\underline{n},A} = X_{\underline{m},B} \circ \psi$ on the attractor $\Lambda_{\underline{n},A}$.

In other words, the attractor $\Lambda_{\underline{n},A}$ lives on a lower dimension polymatrix replicator of type $\underline{m}$.

Proof. Apply Proposition 22 repeatedly. $\square$

Lemma 5.5. *Given a polymatrix game $(\underline{n}, A)$ and a diagonal matrix D of type $\underline{n}$, we have*

$$(A\,D)_v = A_v\,D_v \, ,$$

where A_v is given in Definition 4.6 and D_v is the submatrix of D indexed in $\mathcal{V}_v = \{\, i \in \{1, \ldots, n\} : v_i = 0 \,\}$.

Proof. Given indices $i, k \in \mathcal{V}_v$, take j, resp. l, in the group of i, resp. k, such that $v_j = v_l = 1$.

Since D is of type $\underline{n}$ we have $d_k = d_l$. By Definition 4.6,

$$\begin{aligned} ((A\,D)_v)_{ik} = (A\,D)_{(i,j),(k,l)} &= a_{ik}d_k + a_{jl}d_l - a_{il}d_l - a_{jk}d_k \\ &= (a_{ik} + a_{jl} - a_{il} - a_{jk})\,d_k \\ &= A_{(i,j),(k,l)}\,d_k = (A_v\,D_v)_{ik} \, . \end{aligned}$$

$\square$

Lemma 5.6. *Let $(\underline{n}, A)$ be an admissible polymatrix game and D a diagonal matrix as in Definition 5.1. Given $v \in V_{\underline{n},A}^*$ such that $v_l = 0$ and $a_{ll}^v < 0$ for some $l \in \alpha$ with $\alpha \in \{1, \ldots, p\}$, there exists a positive diagonal matrix $\check{D}$ of type $\underline{n}(l)$ such that $(A(l)\check{D})_{\check{v}}$ is the submatrix of $(AD)_v$ obtained eliminating row and column l. Moreover*

(a) *$(\underline{n}(l), A(l))$ is admissible, and;*
(b) *$\check{v} \in V_{\underline{n}(l),A(l)}^*$.*

Proof. By Proposition 1, we can assume $A = (a_{ij})$ has all entries equal to zero in row l, i.e., $a_{lj} = 0$ for all j.

Since $(\underline{n}, A)$ is admissible and $v \in V_{\underline{n},A}^*$, $(AD)_v$ is stably dissipative.

Consider the set $I = \{i \in \{1, \ldots, n\} : v_i = 0 \text{ and } a_{ii}^v = 0\}$. By Proposition 8, the submatrix $B_v = (a_{ij}^v d_j)_{i,j \in I}$ of $(AD)_v = A_v D_v$ is skew-symmetric.

Let $\Gamma_{\underline{n}(l)}$ be the polytope corresponding to the new polymatrix replicator in lower dimension, given by Proposition 22 and defined by matrix $A(l) = (a_{ij}(l))_{i,j \neq l}$.

Observing that $v_i = 0$ for all strategies i of the matrix $(AD)_v$, we can choose the vertex $\check{v}$ in the polytope $\Gamma_{\underline{n}(l)}$ determined by the exact same strategies as v. Notice that $v_l = 0$ for the removed strategy l.

As in the proof of Proposition 22 the matrix $A(l)$ is defined by

$$a_{ij}(l) = \begin{cases} a_{ij} & \text{if } j \notin \check{\alpha} \\ a_{ij}(1 - q_l) + a_{il}q_l & \text{if } j \in \check{\alpha} \, . \end{cases}$$

Hence

$$a_{ij}^{\check{v}}(l) = \begin{cases} a_{ij}^v & \text{if } j \notin \check{\alpha} \\ (1 - q_l)a_{ij}^v & \text{if } j \in \check{\alpha} \, , \end{cases}$$

where $a_{ij}^{\check{v}}(l) \equiv (a_{ij}(l))^{\check{v}}$ are the entries of matrix $A(l)_{\check{v}}$.

Considering the positive diagonal matrix

$$\check{D} = \text{diag}\left(I_1, \ldots, \frac{1}{1 - q_l} I_\alpha, \ldots, I_p \right) \, ,$$

we have that $(A(l)\check{D})_{\check{v}}$ is the submatrix B_v of $(AD)_v$ obtained by removing the row and column corresponding to strategy l. By Lemma 5.5, $(A(l)\check{D})_{\check{v}} = A(l)_{\check{v}} \check{D}_{\check{v}}$. Hence, by Lemma 3.7, $A(l)_{\check{v}} \check{D}_{\check{v}}$ is stably dissipative, and consequently, by Lemma 3.5, $A(l)_{\check{v}}$ is also stably dissipative. $\square$

Proposition 22 and Lemma 5.6 allows us to generalize [12, Theorem 4.5] about the Hamiltonian nature of the limit dynamics in *admissible* polymatrix replicators.

Theorem 5.7. *Consider a polymatrix replicator* (1) *on* $\Gamma_{\underline{n}}$, *and assume that the system is admissible and has an equilibrium* $q \in \text{int}\left(\Gamma_{\underline{n}}\right)$. *Then the limit dynamics of* (1) *on the attractor* $\Lambda_{\underline{n},A}$ *is described by a Hamiltonian polymatrix replicator in some lower dimensional prism* $\Gamma_{\underline{n}'}$.

Proof. By definition there exists a vertex $v \in \Gamma_{\underline{n}}$ such that $A_v = (a_{ij}^v)$ is stably dissipative. Applying Proposition 22 and Lemma 5.6 we obtain a new polymatrix replicator in lower dimension that is admissible.

We can iterate this process until the corresponding vertex $\check{v}$ in the polytope is such that, $a_{ii}^{\check{v}} = 0$ for all i with $\check{v}_i = 0$.

Let us denote the resulting polymatrix game by $(\underline{r}, A')$. By Proposition 8, for some positive diagonal matrix D' of type $\underline{r}$, $(A' D')_{\check{v}}$ is skew-symmetric. Hence $Q_{A' D'} = 0$ on $H_{\underline{r}}$, and by Definition 4.3 the polymatrix game $(\underline{r}, A')$ is conservative. Notice that this polymatrix game has essentially the same formal equilibrium up to coordinate rescalings. Thus by Proposition 12 the vector field $X_{\underline{r},A'}$ is Hamiltonian. $\square$

6. **An Example.** Consider the polymatrix replicator system associated to the
polymatrix game $\mathcal{G} = ((3,2), A)$, where

$$
A = \begin{bmatrix}
-1 & 8 & -7 & 3 & -3 \\
-10 & -1 & 11 & 3 & -3 \\
11 & -7 & -4 & -6 & 6 \\
-3 & -3 & 6 & 0 & 0 \\
3 & 3 & -6 & 0 & 0
\end{bmatrix}.
$$

We denote by $X_{\mathcal{G}}$ the vector field associated to this polymatrix replicator defined
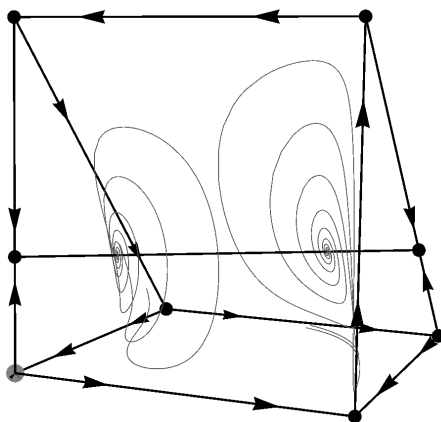on the polytope $\Gamma_{(3,2)} = \Delta^2 \times \Delta^1$.

FIGURE 1. Four orbits in two different leafs of the polymatrix game $\mathcal{G}$.

In this example we want to illustrate the reduction algorithm on the set of graphs
$\{ G(A_v) : v \in V_{(3,2)} \}$ to derive information on the strategies of the polymatrix game
$\mathcal{G}$ as described in section 5. We will see that this polymatrix game is admissible
and verify the validity of the conclusion of Theorem 5.7 for this example.

$v_1 = (1,4)$	$v_2 = (1,5)$	$v_3 = (2,4)$	$v_4 = (2,5)$	$v_5 = (3,4)$	$v_6 = (3,5)$

TABLE 1. Vertex labels.

In this game the strategies are divided in two groups, $\{1,2,3\}$ and $\{4,5\}$. The
vertices of the phase space $\Gamma_{(3,2)}$ will be designated by pairs in $\{1,2,3\} \times \{4,5\}$,
where the label (i,j) stands for the point $e_i + e_j \in \Gamma_{(3,2)}$. To simplify the notation
we designate the prism vertices by the letters $v_1, \ldots, v_6$ according to table 1.
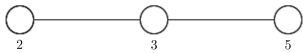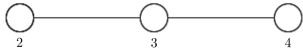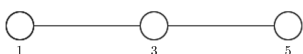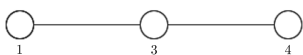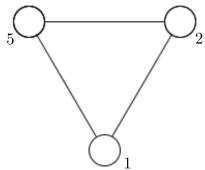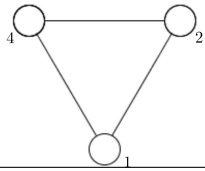
Vertex	A_v	$G(A_v)$
$v_1 \in V_{\underline{n},A}^*$	$\begin{bmatrix} 0 & 27 & 0 \\ -27 & -9 & 18 \\ 0 & -18 & 0 \end{bmatrix}$	
$v_2 \in V_{\underline{n},A}^*$	$\begin{bmatrix} 0 & 27 & 0 \\ -27 & -9 & -18 \\ 0 & 18 & 0 \end{bmatrix}$	
$v_3 \in V_{\underline{n},A}^*$	$\begin{bmatrix} 0 & -27 & 0 \\ 27 & -9 & 18 \\ 0 & -18 & 0 \end{bmatrix}$	
$v_4 \in V_{\underline{n},A}^*$	$\begin{bmatrix} 0 & -27 & 0 \\ 27 & -9 & -18 \\ 0 & 18 & 0 \end{bmatrix}$	
$v_5 \notin V_{\underline{n},A}^*$	$\begin{bmatrix} -9 & 18 & -18 \\ -36 & -9 & -18 \\ 18 & 18 & 0 \end{bmatrix}$	
$v_6 \notin V_{\underline{n},A}^*$	$\begin{bmatrix} -9 & 18 & 18 \\ -36 & -9 & 18 \\ -18 & -18 & 0 \end{bmatrix}$	

TABLE 2. Matrix A_v and its graph $G(A_v)$ for each vertex v.

The point $q \in \text{int}\left(\Gamma_{(3,2)}\right)$ given by

$$q = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right),$$

is an equilibrium of our polymatrix replicator $X_{\mathcal{G}}$. In particular it is also a formal equilibrium of $\mathcal{G}$ (see Definition 4.1).

The quadratic form $Q_A : H_{(3,2)} \to \mathbb{R}$ induced by matrix A is

$$Q_A(x) = -9\,x_3^2,$$

where $x = (x_1, x_2, x_3, x_4, x_5) \in H_{(3,2)}$. By Definition 5.1, $\mathcal{G}$ is dissipative.

In table 2 we present for each vertex v in the prism the corresponding matrix A_v and graph $G(A_v)$.

Considering vertex $v_1 = (1, 4)$ for instance, by Proposition 8, we have that matrix A_{v_1} is stably dissipative. Hence, by Definition 5.2, $\mathcal{G}$ is admissible and $v_1 \in V_{\underline{n},A}^*$.

Table 3 represents the steps of the reduction procedure applied to $\mathcal{G}$. Let us describe it step by step:

Step	Rule	Vertex	Strategy	Group 1			Group 2	
1	1	v_1, v_2, v_3, v_4	3	○ 1	○ 2	● 3	○ 4	○ 5
2	4	v_4 (or v_5)	4, 5	○ 1	○ 2	● 3	○—○ 4 5	
3	6	–	4, 5	○ 1	○ 2	● 3	⊕ 4	⊕ 5
4	3	v_1, v_2	1, 2	⊕ 1	⊕ 2	● 3	⊕ 4	⊕ 5

TABLE 3. Information set of all strategies (by group) of $\mathcal{G}$, where for each step, we mention the rule, the vertex (or vertices) and the strategy (or strategies) to which we apply the rule.

(Step 1) Initially, considering the vertices v_1, v_2, v_3 and v_4 we apply *rule 1* to the corresponding graphs $G(A_{v_1}), G(A_{v_2}), G(A_{v_3})$ and $G(A_{v_4})$, and we colour in black (●) strategy 3. We obtain the graphs depicted in column "Step 1" in table 4;

(Step 2) In this step we can consider vertex v_4 (or v_5) to apply *rule 4*. Hence, we put a link between strategies 4 and 5 in group 2;

(Step 3) In this step we apply *rule 6* to strategies 4 and 5, and we colour with ⊕ that strategies. We obtain the graphs depicted in column "Step 3" in table 4;

(Step 4) Finally, we apply *rule 3* to vertices v_2 and v_3 in the corresponding graphs of the column "Step 3" in table 4, and we colour with ⊕ the strategy 2. Analogously we apply *rule 3* to vertices v_1 and v_3 in the corresponding graphs of the column "Step 3" in table 4, and we colour with ⊕ the strategy 1. We obtain the graphs depicted in column "Step 4" table 4.

Since $\mathcal{G}$ is admissible and has an equilibrium $q \in \text{int}\left(\Gamma_{(3,2)}\right)$, by Theorem 5.7 we have that its limit dynamics on the attractor $\Lambda_{\mathcal{G}}$ is described by a Hamiltonian polymatrix replicator in a lower dimensional prism. Considering the strategy 3 in group 1, by Definition 5.4 we obtain the $(q, 3)$-reduction $((2, 2), A(3))$ where $\tilde{A} := A(3)$ is the matrix

$$\tilde{A} = \begin{bmatrix} -9 & 9 & 9 & -9 \\ -9 & 9 & 9 & -9 \\ -6 & 6 & 6 & -6 \\ -6 & 6 & 6 & -6 \end{bmatrix}.$$

Consider now the polymatrix replicator associated to the game $\tilde{\mathcal{G}} = \left((2, 2), \tilde{A}\right)$, which is equivalent to the trivial game $((2, 2), 0)$. Hence its replicator dynamics on the polytope $\Gamma_{(2,2)} = \Delta^1 \times \Delta^1$ is trivial, in the sense that all points are equilibria. In particular the associated vector field $X_{\tilde{\mathcal{G}}} = 0$ is Hamiltonian.

TABLE 4. The graphs obtained in each step of the reduction algorithm for $\mathcal{G}$.

Since the reduced information set $\mathscr{R}(\mathcal{G})$ is of type $\{\bullet, \oplus\}$, by Proposition 21 the flow of $X_{\mathcal{G}}$ admits an invariant foliation with a single globally attractive equilibrium on each leaf (see Figure 1). Therefore, the attractor $\Lambda_{\mathcal{G}}$ is just a line segment of equilibria, which embeds in the Hamitonian flow of $X_{\bar{\mathcal{G}}} = 0$, as asserted by Proposition 22.

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