Hamiltonian Systems on Polyhedra

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1 Flows on Polyhedra

Let Γ^d be a simple polyhedron with dimension d. We say that a vector field X on Γ^d is tangent to $\partial\Gamma^d$ if X is tangent to every face σ of Γ^d, i.e., $X(p) \in T_p\sigma$ at each point $p \in \sigma$. We denote by $\mathscr{X}(\Gamma^d)$ the vector space of all analytic vector fields X on Γ^d which are tangent to $\partial\Gamma^d$. For any given $X \in \mathscr{X}(\Gamma^d)$ the flow $\phi_X^t : \Gamma^d \to \Gamma^d$ of X is complete and every face of Γ^d is invariant under ϕ_X^t. In particular, the vertices of Γ^d are singularities of the vector field X, and many edges will consist of single orbits flowing from one boundary vertex to the other. Our goal is, for some rather large class of "*regular*" vector fields $X \in \mathscr{X}(\Gamma^d)$, to encapsulate the dynamics of ϕ_X^t along heteroclinic cycles on $\partial\Gamma^d$ in a simple and "computable" dynamical system, that we refer as *the skeleton vector field on the dual cone of* Γ^d.

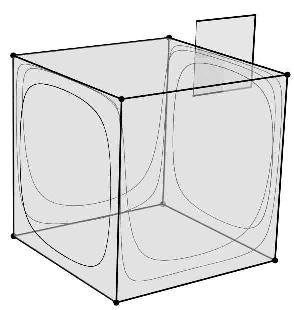

Fig. 1 The dynamics near the edges for a flow ϕ_X^t on the polyhedron $\Gamma^3 = [0,1]^3$.

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Before continuing we give precise definitions of the concepts of polyhedron, dimension, face, vertex, edge and simplicity, while introducing the notation used in the sequel. A subset Γ of some Euclidean space $\mathbb{R}^N$ is called a *polyhedron* if it is a compact convex set which can be represented as a finite intersection of closed half-spaces. Denote by $E(\Gamma)$ the smallest affine subspace of $\mathbb{R}^N$ that contains Γ. The *dimension of a polyhedron* Γ is defined to be the dimension of $E(\Gamma)$. From now on Γ^d will denote a polyhedron of dimension d, which for the sake of simplicity we assume, unless otherwise said, to live in $E(\Gamma^d) = \mathbb{R}^d$. We call *supporting hyperplane of* Γ^d to any affine hyperplane $H \subset \mathbb{R}^d$ such that $H \cap \Gamma^d \neq \emptyset$, and Γ^d is contained in one of the two closed half-spaces determined by H. The intersection of Γ^d with any of its supporting hyperplanes is another polyhedron, called a *face of* Γ^d, or an r-face when its dimension is equal to r. As usual, a *vertex* is any 0-face, and an *edge* is any 1-face of Γ^d. Capital letters A, B, C will denote vertices of Γ^d, while γ will denote a generic edge of Γ^d. By default, the term "face" shall always refer to a $(d-1)$-face, and σ will represent a generic such $(d-1)$-face. We represent by V the set of all vertices, by E the set of all edges, and by F the set of all $(d-1)$-faces of Γ^d.

Definition 1. A family of functions $\{ f_\sigma : \mathbb{R}^d \to \mathbb{R} \}_{\sigma \in F}$ is called a *defining family for* Γ^d if for every face $\sigma \in F$,

1. $f_\sigma : \mathbb{R}^d \to \mathbb{R}$ is an affine function,
2. $f_\sigma(p) = 0$ for all $p \in \sigma$,
3. $f_\sigma(p) \geq 0$ for all $p \in \Gamma^d$, and
4. $\Gamma^d = \bigcap_{\sigma \in F} \{ f_\sigma \geq 0 \}$.

We assume a defining family $\{f_\sigma\}_{\sigma \in F}$ for Γ^d is fixed once and for all.

We call d-simplex to the convex hull of any $d+1$ affinely independent points. These are the simplest polyhedra. A polyhedron Γ^d is called *simple* if each vertex is incident with exactly d faces (edges). This amounts to the supporting hyperplanes $\{f_\sigma = 0\}$ intersecting each other in general position. A d-simplex is of course simple in this sense. A polyhedron Γ^d is simple if and only if every face of its dual polyhedron is a $(d-1)$-simplex.

2 Game Dynamics

Systems as these include many interesting classes from Game Dynamics, for instance the *replicator equation*. Within a population individuals interact using one of n possible strategies. The time evolution of a population distribution $(x_1, \ldots, x_n) \in \Delta^{n-1}$ is ruled by

$$\frac{x_i'}{x_i} = f_i(x_1, \ldots, x_n) - \sum_{k=1}^{n} x_k f_k(x_1, \ldots, x_n) , \tag{1}$$

where Δ^{n-1} stands for the usual $(n-1)$-simplex $\{ (x_1, \ldots, x_n) : x_i \geq 0, \sum_{i=1}^n x_i = 1 \}$. The value $f_i(x_1, \ldots, x_n)$ measures the *absolute fitness* of strategy i for the population distribution $(x_1,\ldots,x_n)\in\Delta^{n-1}$. Likewise, the right-hand-side in (1) expresses the *relative fitness* of strategy i within the same population. In the replicator equation model, strategies in the population thrive or recede proportional to their relative fitnesses. When the functions $f_i(x)$ are linear, say $f_i(x_1,\ldots,x_n)=\sum_{j=1}^n a_{ij}x_j$, the system is determined by a matrix $A=(a_{ij})$ called the payoff matrix. The payoffs a_{ij} are the eigenvalues of the singularities at the vertices, for the associated replicator flow or vector field.

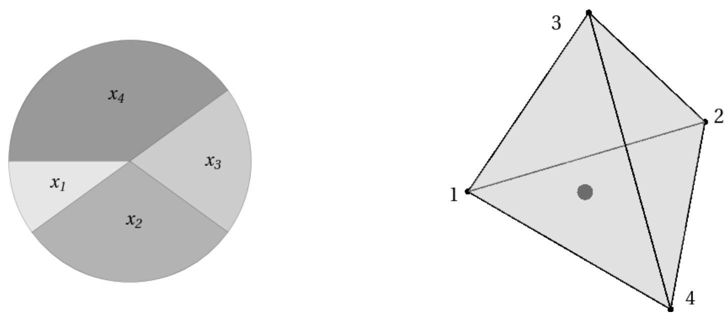

Fig. 2 A point in Δ^3 is a probability vector in $\{1,2,3,4\}$

An important class of equations which reduces to the (linear) replicator equation are the so called Lotka-Volterra equations. They govern the time evolution of a n-species ecosystem $y=(y_1,\ldots,y_n)\in\mathbb{R}^n_+$

$$\frac{y_i'}{y_i}=r_i+\sum_{j=1}^n a_{ij}y_j\,, \tag{2}$$

where y_i measures the size of species i within the ecosystem, a_{ij} is an interaction coefficient between species i and j, while r_i models the interaction of species i with environment. Every Lotka-Volterra system is equivalent to a replicator system in the sense that the underlying vector fields are equivalent. The equivalence is given by the algebraic map defined by

$$x=(x_0,\ldots,x_n)\in\Delta^n\;\longleftrightarrow\;y=(y_1,\ldots,y_n)=\left(\frac{x_1}{x_0},\ldots,\frac{x_n}{x_0}\right)\in\mathbb{R}^n_+\,,$$

which maps the interior of the simplex Δ^n onto the the interior of $\mathbb{R}^n_+$. In the new coordinates $x=(x_0,\ldots,x_n)\in\Delta^n$ the system becomes (up to a time reparametrization)

$$\frac{x_i'}{x_i}=\sum_{j=0}^n\tilde a_{ij}x_j-\sum_{j,k=0}^n\tilde a_{kj}x_kx_j \tag{3}$$

which is a linear replicator with payoff matrix $\widetilde{A}=(\tilde a_{ij})$, where $\tilde a_{ij}=a_{ij}$ when $i,j\geq 1$, $\tilde a_{i0}=r_i$ and $\tilde a_{0i}=0$. This reduction, due to J. Hofbauer [6], consists roughly in letting the n species together with the environment play the roles of $n+1$ strategies.

Another important class which falls within the scope of this work is that of asymmetric games, where two groups of individuals within a population, e.g. males and females, interact using different sets of strategies, say n strategies for males and m strategies for females. The phase space of an asymmetric game system is a polyhedron, product of simplices $\Delta^{n-1}\times\Delta^{m-1}$, and the time co-evolution of two population distributions $(x,y)\in\Delta^{n-1}\times\Delta^{m-1}$ is governed by

$$\begin{aligned} \frac{x_i'}{x_i} &= f_i(y_1,\ldots,y_m) - \textstyle\sum_{k=1}^n x_k\, f_k(y_1,\ldots,y_m) \\ \frac{y_j'}{y_j} &= g_j(x_1,\ldots,x_n) - \textstyle\sum_{k=1}^m x_k\, g_k(x_1,\ldots,x_n) \end{aligned} \tag{4}$$

The value $f_i(y)$ measures the absolute fitness of a male strategy i in a female population $y\in\Delta^{m-1}$, while $g_j(x)$ measures the absolute fitness of a female strategy j in a male population $x\in\Delta^{n-1}$. The right-hand-sides in (4) express, respectively, the relative fitnesses of a male strategy i, and of a female strategy j, within the populations of opposite gender. Once more, in this asymmetric game model strategies in the male and female populations thrive or recede proportional to their relative fitnesses. When the functions $f_i(y)$ and $g_j(x)$ are both linear, say $f_i(y_1,\ldots,y_m)=\sum_{j=1}^m a_{ij}\, y_j$ and $g_j(x_1,\ldots,x_n)=\sum_{i=1}^n b_{ji}\, x_i$, the system is determined by a pair of matrices $A=(\,a_{ij}\,)$ of order $n\times m$ and $B=(\,b_{ji}\,)$ of order $m\times n$, called the payoff matrices. Again, the payoffs a_{ij} and b_{ji} are related to the eigenvalues of the singularities at the vertices, for the associated asymmetric game flow or vector field.

3 Skeletons and Dual Cones

Assume for a while $\Gamma^d\subset\mathbb{R}^{d+1}-\{0\}$ and the cone $\widehat{\Gamma}^{d+1}=\{\,tX\,:\,t\geq 0,\, X\in\Gamma^d\,\}$ has dimension $d+1$. In Convex Analysis the dual cone of Γ^d is defined to be

$$(\Gamma^d)^* = \{\, Y\in\mathbb{R}^{d+1}\,:\, Y\cdot X\geq 0,\, \forall X\in\Gamma^d\,\}\ .$$

Here we shall call *dual cone of* Γ^d to the boundary of this set, $\mathscr{C}^*(\Gamma^d)=\partial(\Gamma^d)^*$. We give an alternative description of the dual cone, which is more convenient for our purposes. Denote by Σ^d the dual of the polyhedron Γ^d. We can identify $V^*=V(\Sigma^d)\equiv F$ and $F^*=F(\Sigma^d)\equiv V$. By duality each vertex $A\in V$ stands for a $(d-1)$-face in Σ^d, each face $\sigma\in F$ represents a vertex of Σ^d, and the relation $A\in\sigma$ in Γ^d is equivalent to $\sigma\in A$ in Σ^d. We define

$$\mathscr{C}(\Sigma^d) := \{\, x\in\mathbb{R}^{V^*}\,:\,\exists A\in F^* \ \text{ for all }\ \sigma\in V^*,\, x_\sigma\geq 0 \ \text{ and }\ x_\sigma = 0 \ \text{ if }\ \sigma\notin A\,\}\ ,$$

and for each face ρ of Σ^d we set

$$\Pi_\rho := \{\, x \in \mathbb{R}^{V^*} \,:\, \text{ for all } \sigma \in V^*,\, x_\sigma \geq 0 \text{ and } x_\sigma = 0 \text{ if } \sigma \notin \rho \,\} \,.$$

Then the following properties hold for all faces ρ, ρ' of Σ^d:

1. $\dim \Pi_\rho = \dim_{\Sigma^d}(\rho) + 1$,
2. $\Pi_\rho \subseteq \Pi_{\rho'} \;\;\Leftrightarrow\;\; \rho \subseteq \rho'$ in Σ^d,
3. $\Pi_\rho \cap \Pi_{\rho'} = \Pi_{\rho\cap\rho'}$.

Because Γ^d is simple, by duality, every r-face of Σ^d is a $(r-1)$-simplex, i.e., it has exactly r vertices. This implies item 1. Properties 2 and 3 are obvious consequences of definitions. Realizing the dual polyhedron Σ^d as a transversal section to the cone $(\Gamma^d)^*$, we can identify $(\Gamma^d)^*$ with $\widehat{\Sigma}^{d+1} = \{\, t\,X \,:\, t \geq 0,\, X \in \Sigma^d \,\}$. Whence, the faces of $\partial(\Gamma^d)^*$ satisfy the exact same properties 1-3 above. In fact, the three models $\mathscr{C}(\Sigma^d)$, $\partial(\Gamma^d)^*$ and $\partial\widehat{\Sigma}^{d+1}$ are piecewise-linear isomorphic. From now on we consider the dual cone of Γ^d to be $\mathscr{C}^*(\Gamma^d) := \mathscr{C}(\Sigma^d)$. Properties 1-3 above can be re-interpreted in terms of Γ^d's faces. Since each r-face ρ of Γ^d corresponds to a $(d-1-r)$-face of Σ^d, we have for all faces ρ, ρ' of Γ^d:

1. $\dim \Pi_\rho = d - \dim_{\Gamma^d}(\rho)$,
2. $\Pi_\rho \subseteq \Pi_{\rho'} \;\;\Leftrightarrow\;\; \rho' \subseteq \rho$ in Γ^d,
3. $\Pi_\rho \cap \Pi_{\rho'} = \Pi_{\rho\vee\rho'}$,

where $\rho \vee \rho'$ stands for smallest face of Γ^d containing $\rho \cup \rho'$. In particular, the dual cone $\mathscr{C}^*(\Gamma^d)$ has a face Π_A for each vertex A of Γ^d, and the intersection $\Pi_A \cap \Pi_B$ of any two meeting faces corresponds to an edge of Γ^d connecting A to B.

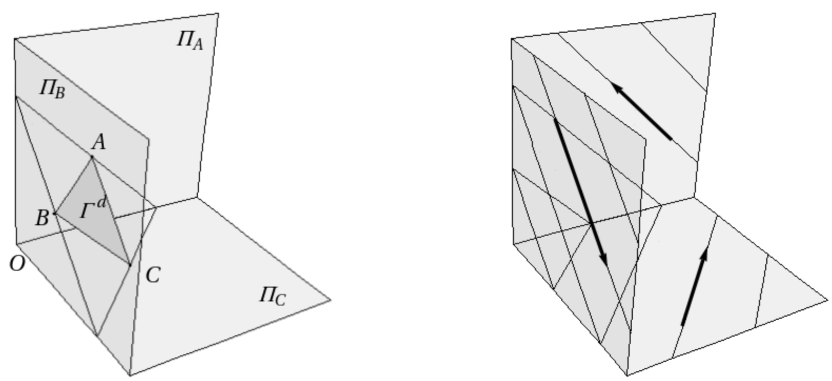

Fig. 3 The dual cone of a triangle polyhedron and a skeleton vector field on it.

A skeleton vector field is a piecewise constant vector field on the dual cone $\mathscr{C}^*(\Gamma^d)$, i.e., one which is constant on each face Π_A, $A \in V$. Any skeleton vector field is given by the finite data $\chi = (\,\chi^A_\sigma\,)_{A\in V,\sigma\in F}$ with $\chi^A_\sigma = 0$ whenever $A \notin \sigma$. We write χ^A for the vector $(\chi^A_\sigma)_{\sigma\in F}$ in the tangent space to Π_A. Orbits of a skeleton vector field χ are defined to be the polygonal curves whose intersection with each face Π_A of $\mathscr{C}^*(\Gamma^d)$ is a line segment parallel to χ^A on Π_A. Notice that orbit continuation is essentially unique, because as an orbit through Π_A reaches the intersection $\Pi_\gamma = \Pi_A \cap \Pi_B$ of two faces Π_A and Π_B of $\mathscr{C}^*(\Gamma^d)$ at some point p interior to Π_γ, there is at most one possible continuation on Π_B, because Π_B is the unique face which meets Π_A at p.

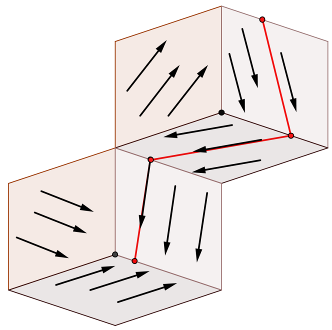

Fig. 4 A finite orbit of a skeleton vector field.

Of course some orbits will end in finite time. This definition gives us an incomplete piecewise linear flow on $\mathscr{C}^*(\Gamma^d)$. Vertices and edges of Γ^d are classified w.r.t. the skeleton vector field χ as figures 5 and 6 indicate.

Definition 2. Given a vertex $A \in V$, we say that A is a

1. $\chi-$attractor $\Leftrightarrow$ $-\chi^A \in \Pi_A$
2. $\chi-$repellor $\Leftrightarrow$ $\chi^A \in \Pi_A$
3. χ-saddle $\Leftrightarrow$ $\chi^A \notin \Pi_A$ and $-\chi^A \notin \Pi_A$

Because we will be looking for recurrent behavior, χ-saddle vertices are the interesting ones, for if a vertex A is a $\chi-$repellor, resp. a $\chi-$attractor, then Π_A is forward, resp. backward, invariant by the flow of χ^A.

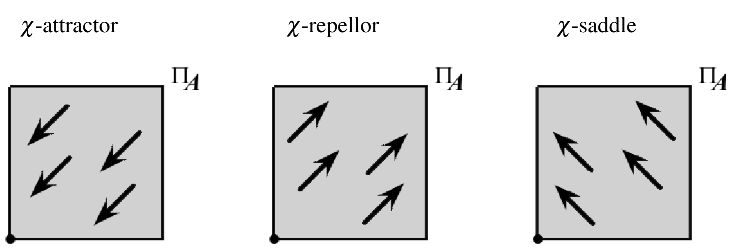

Fig. 5 The classification of vertices for a skeleton vector field.

Let γ be an edge connecting two vertices $A,B \in V$. Take $\sigma,\rho \in F$ to be the unique faces such that $\gamma \cap \sigma = \{A\}$ and $\gamma \cap \rho = \{B\}$.

Definition 3. We say that γ is

1. χ-attracting $\Leftrightarrow$ $\chi^A_\sigma < 0$ and $\chi^B_\rho < 0$
2. χ-repelling $\Leftrightarrow$ $\chi^A_\sigma > 0$ and $\chi^B_\rho > 0$
3. χ-neutral $\Leftrightarrow$ $\chi^A_\sigma = 0$ and $\chi^B_\rho = 0$
4. χ-flowing $\Leftrightarrow$ $\chi^A_\sigma \chi^B_\rho < 0$.

All other edges are said to be χ-undefined.

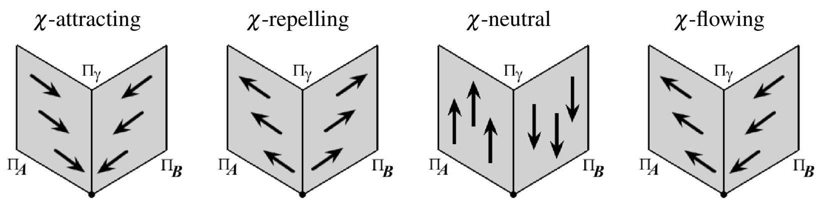

Fig. 6 The classification of edges for a skeleton vector field.

We shall not consider skeleton vector fields with χ-undefined edges. When all vertices are χ-saddles and all edges are either χ-neutral or χ-flowing then some recurrence occurs. This will be the case of the Hamiltonian systems introduced below. Note the flowing edges are naturally oriented, from a source vertex, we denote by $s(\gamma)$, to a target vertex, denoted by $t(\gamma)$. Let $G_\chi(\Gamma^d)$ be the oriented graph consisting of all vertices, and all oriented edges of χ-flowing type of Γ^d. The dynamics of a skeleton vector field can be described in terms of piecewise linear return maps. Fixing an edge γ of $G_\chi(\Gamma^d)$ we can define the return map $R^\chi_\gamma : \Pi_\gamma \to \Pi_\gamma$. These return maps satisfy:

(1) the domain of R^χ_γ splits into a finite or countable number of open convex cones Π_ξ, each associated with a cycle ξ of $G_\chi(\Gamma^d)$ starting and ending with γ, and not passing through γ in between.
(2) the restriction of R^χ_γ to each cone Π_ξ is a linear map, and
(3) the linear branches of R^χ_γ, as well as their domains, are computable.

The return maps R^χ_γ and their domains Π_ξ can be expressed in terms of matrices in $\mathbb{R}^{F\times F}$ whose coefficients are functions of the data χ^A_σ. Given an edge $\gamma \in G_\chi(\Gamma^d)$, let $A = s(\gamma)$ be the source of γ, and $\sigma_0 \in F$ be the unique face such that $\sigma_0 \cap \gamma = \{A\}$. We associate the following $F \times F$ matrix to the edge γ,

$$M_\gamma = \left(\delta_{\sigma,\sigma'} - \frac{\chi^A_\sigma}{\chi^A_{\sigma_0}}\, \delta_{\sigma_0,\sigma'} \right)_{(\sigma,\sigma')\in F\times F} .$$

A sequence $\xi = (\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_n)$ is called a *chain* if $s(\gamma_i) = t(\gamma_{i-1})$, for every $i = 1, \ldots, n$. We call sub-chain of ξ to any initial subsequence $\xi_i = (\gamma_0, \ldots, \gamma_i)$ of ξ with $1 \le i \le n$. For each chain $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ we define the product matrix $M_\xi = M_{\gamma_n} \cdots M_{\gamma_1}$. Note M_{γ_0} is excluded from this product. The matrix M_ξ defines a linear operator on $\mathbb{R}^F$, which projects $\mathbb{R}^F$ onto the linear subspace spanned by the cone Π_{γ_n}. The chain $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ is called a *cycle* when $\gamma_n = \gamma_0$, in which case we have for every $X \in \Pi_\xi$, $R^\chi_{\gamma_0} X = M_\xi X$. The open convex cone Π_ξ can be characterized as the set of all $X \in \Pi^\chi_{\gamma_0}$ such that for each sub-chain $\xi_i = (\gamma_0, \ldots, \gamma_i)$ of ξ the vector $M_{\xi_i} X$ is interior to Π_{γ_i}.

4 Main Results

We are going to rescale the vector field X around the singularities at the vertices using some type of logarithmic coordinates. In [3] we single out a class of vector fields, that we call *regular vector fields*, for which these coordinates around the vertex singularities can be glued along the edges to obtain a global rescaling mapping. Regular vector fields include generic ones, with hyperbolic singularities at the vertices, but they also comprise many others with non-hyperbolic singularities. This generality is essential to embrace the Hamiltonian systems in which we are interested. Given $A \in V$ and $\sigma \in F$ such that $A \in \sigma$ we denote by $\gamma = \gamma_{A,\sigma}$ the edge opposed to σ at A, which is characterized by $\sigma \cap \gamma = \{A\}$. We refer to the pair (A,σ) as an *end corner of* γ. Notice each edge has exactly two end corners. Let $e_{A,\sigma} \in T_A \Gamma^d$ denote the unit vector tangent to the edge $\gamma_{A,\sigma}$ at A. To each vector field $X \in \mathscr{X}(\Gamma^d)$, $X \ne 0$, we associate an *order function* $\nu_X : F \to \mathbb{N}$

$$\nu_X(\sigma) = \max\{\, k \in \mathbb{N} : D(f_\sigma)_p D^i X_p \equiv 0,\ \forall\, i < k,\ \forall\, p \in \sigma \,\} ,$$

with the order of the first non-zero derivative at some of the face's vertices. Remark each face has finite order because the vector field X is analytic. Then we define the *character of X at the corner* (A,σ) by $\chi^A_\sigma = -\frac{1}{\nu!} D(f_\sigma)_A\, D^\nu X_A \cdot e_{A,\sigma}{}^{(\nu)}$, where $\nu = \nu_X(\sigma)$. We set $\chi^A_\sigma = 0$ if $A \notin \sigma$. The data $\chi = (\chi^A_\sigma)_{A\in V, \sigma\in F}$ determines a skeleton vector field we shall call the *skeleton of* X.

Definition 4. We say that a vector field $X \in \mathscr{X}(\Gamma^d)$ is *regular* iff for every edge γ of Γ^d, either $X = 0$ along γ or else $X \ne 0$ in the interior of γ and X has non-zero character at both end corners (A,σ) and (A',σ') of γ.

In particular, for the skeleton χ of a regular vector field, every edge γ of Γ^d is either χ-neutral or χ-flowing.

For each order function $\nu : F \to \mathbb{N}$ we define a one-parameter family of rescaling co-ordinates $\Psi_\varepsilon^\nu : \Gamma^d - \partial\Gamma^d \to \mathscr{C}^*(\Gamma^d)$ $(\varepsilon > 0)$ by $\Psi_\varepsilon^\nu(p) = (\Psi_\varepsilon^\sigma(p))_{\sigma\in F}$, where

$$\Psi_\varepsilon^\sigma(p) := \begin{cases} -\varepsilon \log f_\sigma(p) & \text{if } \nu_X(\sigma) = 1 \\ -\varepsilon \frac{1}{\nu-1}\left(1 - \frac{1}{f_\sigma(p)^{\nu-1}}\right) & \text{if } \nu_X(\sigma) \geq 2 \end{cases}$$

Actually, we take the domain of Ψ_ε^ν to be the union of a family of neighborhoods N_A, one for each vertex $A \in V$.

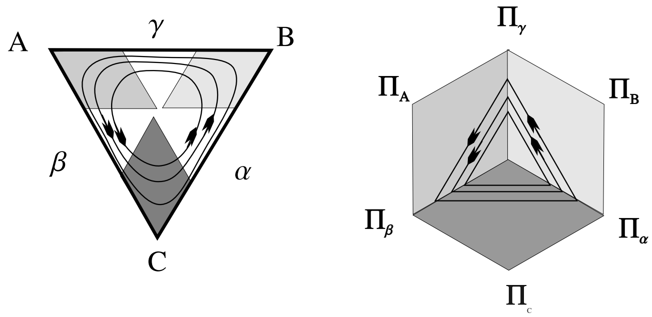

Fig. 7 The rescaling coordinates in the dual cone $\mathscr{C}^*(\Gamma^d)$.

The mapping Ψ_ε^ν *zooms in* a neighborhood of the union of all edges of Γ^d. The first theorem says, given $X \in \mathscr{X}(\Gamma^d)$, the rescaling limit of the flow ϕ_X^t is exactly the piecewise linear flow of the skeleton χ of X. Given a cycle ξ of χ, starting and ending with $\gamma \in G_\chi(\Gamma^d)$, we denote by P_ξ^X the Poincaré return map along ξ. This map is defined in a small cross section of ϕ_X^t which is mapped by every Ψ_ε^ν into the face $\Pi_\gamma \subset \mathscr{C}^*(\Gamma^d)$.

Theorem 1. *If $X \in \mathscr{X}(\Gamma^d)$ is a regular vector field with order ν, skeleton χ, and ξ is a cycle in $G_\chi(\Gamma^d)$ which starts and ends with γ, then for every compact subset $K \subset \Pi_\xi$, $(\Psi_\varepsilon^\nu) \circ P_\xi^X \circ (\Psi_\varepsilon^\nu)^{-1}$ converges to $R_\gamma^\chi : \Pi_\xi \to \Pi_\gamma$, in the C^∞-topology, uniformly over K, as $\varepsilon \to 0^+$.*

We consider in [3] the vector space, denoted by $\mathscr{H}(\Gamma^d)$, of analytic functions $h : \Gamma^d - \partial\Gamma^d \to \mathbb{R}$ such that for each face $\sigma \in F$, either h is essentially analytic on σ, or else dh has a pole of finite order along σ. We say that h is *essentially analytic on* σ if h has an analytic extension to a neighborhood of σ minus the union of all other faces $\sigma' \in F$, $\sigma' \neq \sigma$. A similar definition is given for analytic 1-forms. We say that dh has a *pole of order k along* σ iff there is a 1-form λ and function g, both analytic in $\Gamma^d - \partial\Gamma^d$ and essentially analytic on σ, such that $dh = \lambda + g\,\frac{df_\sigma}{(f_\sigma)^k}$. It follows from this definition that g is constant on σ. Each function $h \in \mathscr{H}(\Gamma^d)$ can be represented as

$$h = G + \sum_{\sigma\in F} c_{1,\sigma} \log f_\sigma + \frac{c_{2,\sigma}}{f_\sigma} + \cdots + \frac{c_{k_\sigma,\sigma}}{(f_\sigma)^{k_\sigma - 1}} \;, \tag{5}$$

where G is an analytic function on Γ^d, each $c_{i,\sigma}$ is a real constant, and $c_{k_\sigma,\sigma} \neq 0$. The function $\kappa : \sigma \mapsto k_\sigma$ is called *the order of h*.

We define now the *skeleton of $h \in \mathscr{H}(\Gamma^d)$* to be the piece-wise linear function $\lambda_h : \mathscr{C}^*(\Gamma^d) \to \mathbb{R}$, $\lambda_h(u_\sigma)_{\sigma\in F} = \sum_{\sigma\in F} c_{k_\sigma,\sigma}\, u_\sigma$, where $c_{k_\sigma,\sigma}$ is the main coefficient in (5). A function $h \in \mathscr{H}(\Gamma^d)$ with order κ is called *regular* if $\kappa(\sigma) \geq 1$, and all faces of order $\kappa(\sigma) \geq 2$ are pairwise disjoint. The second theorem states that the rescaling limit of a function $h \in \mathscr{H}(\Gamma^d)$ is precisely its skeleton λ_h.

Theorem 2. *Given $h \in \mathscr{H}(\Gamma^d)$ regular with order κ, and $A \in V$, resp. $\gamma \in E$, as $\varepsilon \to 0^+$ the rescaled function $h \circ (\Psi_\varepsilon^\kappa)^{-1} : \mathscr{C}^*(\Gamma^d) \to \mathbb{R}$ tends in the C^∞-topology and uniformly on compact subsets in the interior of Π_A, resp. Π_γ, to the skeleton function $\lambda_h : \mathscr{C}^*(\Gamma^d) \to \mathbb{R}$.*

The class of Hamiltonian systems on polyhedra we are about to introduce uses Hamiltonian functions in the space $\mathscr{H}(\Gamma^{2d})$ and the class of algebraic symplectic structures we now discuss. Consider the finite dimensional space $\Omega^2(\Gamma^{2d})$ of algebraic 2-forms

$$\omega = \sum_{(\sigma_1,\sigma_2)\in F\times F} \omega_{\sigma_1,\sigma_2} \frac{d f_{\sigma_1} \wedge d f_{\sigma_2}}{f_{\sigma_1} f_{\sigma_2}} \;, \tag{6}$$

where $\Omega = (\,\omega_{\sigma_1,\sigma_2}\,)_{(\sigma_1,\sigma_2)\in F\times F}$ is a skew-symmetric matrix such that $\omega_{\sigma_1,\sigma_2} = 0$ whenever σ_1 and σ_2 are disjoint faces. Any algebraic form $\omega \in \Omega^2(\Gamma^{2d})$ determines the linear 2-form $\widehat{\omega} : \mathbb{R}^F \times \mathbb{R}^F \to \mathbb{R}$, $\widehat{\omega}(X,Y) = X^T\,\Omega\,Y$, which by restriction induces a piecewise linear 2-form on $\mathscr{C}^*(\Gamma^{2d})$ still denoted by $\widehat{\omega}$. Conversely, assume we are given a continuous piecewise linear 2-form $\widehat{\omega}$ on $\mathscr{C}^*(\Gamma^{2d})$. This is a family of linear 2-forms $\widehat{\omega}^A : \Pi_A \times \Pi_A \to \mathbb{R}$, one on each face Π_A with $A \in V$, such that $\widehat{\omega}^A = \widehat{\omega}^B$ on Π_γ, for every pair of vertices $A, B \in V$ connected by some edge γ. Under such conditions the piecewise linear 2-form $\widehat{\omega}$ is determined by a skew-symmetric matrix $\Omega = (\,\omega_{\sigma_1,\sigma_2}\,)_{(\sigma_1,\sigma_2)\in F\times F}$ as above, and is therefore associated to an algebraic 2-form $\omega \in \Omega^2(\Gamma^{2d})$. Given an algebraic 2-form $\omega \in \Omega^2(\Gamma^{2d})$, if ω is non-degenerate at every point interior to Γ^{2d} then ω is a symplectic structure on the interior of Γ^{2d}, that we refer as an algebraic symplectic structure. The third theorem says the symplectic gradient of a function in $\mathscr{H}(\Gamma^{2d})$ w.r.t. an algebraic symplectic structure in $\Omega^2(\Gamma^{2d})$ is, up to time reparametrization, a regular vector field in $\mathscr{X}(\Gamma^{2d})$.

Theorem 3. *Given an algebraic symplectic structure $\omega \in \Omega^2(\Gamma^{2d})$, and a regular function $h \in \mathscr{H}(\Gamma^{2d})$ of order κ, the symplectic gradient X_h of h w.r.t. ω is equivalent to the regular vector field $X = p X_h$ on Γ^{2d} with the same order $\nu_X = \kappa$, where $p = \prod_{\sigma\in F} (f_\sigma)^{\kappa(\sigma)-1} \geq 0$.*

Given an order function $\nu : F \to \mathbb{N}$ and $\omega \in \Omega^2(\Gamma^{2d})$ given by (6), we define the *reduced algebraic form*

$$\omega^\nu = \sum_{(\sigma_1,\sigma_2)\in F\times F} \omega^\nu_{\sigma_1,\sigma_2} \frac{d f_{\sigma_1}\wedge d f_{\sigma_2}}{f_{\sigma_1} f_{\sigma_2}} ,$$

where

$$\omega^\nu_{\sigma_1\,\sigma_2} = \begin{cases} \omega_{\sigma_1\,\sigma_2} & \text{if } \nu(\sigma_1)=\nu(\sigma_2)=1 \\ 0 & \text{otherwise} \end{cases} .$$

Next theorem says the rescaling limit of an algebraic form $\omega\in\Omega^2(\Gamma^{2d})$ is the piecewise linear reduced form $\widehat{\omega^\nu}$.

Theorem 4. *Given an order function $\nu : F\to\mathbb{N}$, $\omega\in\Omega^2(\Gamma^{2d})$, and $A\in V$, then as $\varepsilon\to 0^+$ the rescaled form $\varepsilon^2\left[(\Psi^\nu_\varepsilon)^{-1}\right]^*\omega$ tends in the C^∞-topology and uniformly on compact subsets in the interior of Π_A to the piecewise linear 2-form $\widehat{\omega^\nu}$.*

Corollary 1. *Consider $\omega\in\Omega^2(\Gamma^{2d})$, $h\in\mathscr{H}(\Gamma^{2d})$ and $X\in\mathscr{X}(\Gamma^{2d})$ as above. The skeleton χ of X is, up to some constant, the gradient of the skeleton λ_h w.r.t. $\widehat{\omega^\nu}$, i.e., for every $A\in V$ and $u\in\Pi_A$, $\lambda_h(\chi^A)=0$ and $\widehat{\omega^\nu}(\chi^A,u)=p(A)\,\lambda_h(u)$, where p is the function referred in theorem 3.*

Corollary 2. *Under the same assumptions, if all components of λ_h have the same sign (positive or negative), then every $A\in V$ is a χ-saddle and almost all orbits of χ are defined for all time.*

Two important subclasses of Lotka-Volterra systems, already studied by Volterra, are the so called dissipative and conservative systems. A Lotka-Volterra system, with interaction matrix A, is said to be *conservative* if there is a positive diagonal matrix D such that AD is skew-symmetric. On even dimensions, if conservative system (2) is Hamiltonian with respect to the symplectic structure on $\mathbb{R}^{2d}_+$

$$\omega = \sum_{i,j=1}^{2d} a^{-1}_{i,j} \frac{dx_i\wedge dx_j}{x_i\, x_j} ,$$

where $a^{-1}_{i,j}$ is the coefficient of the inverse matrix A^{-1}. In general, a conservative Lotka-Volterra system is Hamiltonian with respect to the Poisson structure on $\mathbb{R}^d_+$

$$\{f,g\} = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}\, x_i\, x_j \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j}\right) .$$

In any case, if q is a solution of the equation $r+A\,q=0$, where r and A are the Lotka-Volterra coefficient matrices, then the Hamiltonian function $h:\mathbb{R}^d_+\to\mathbb{R}$ is

$$h(x_1,\dots,x_d) = \sum_{i=1}^{d} (x_i - q_i\,\log x_i) , \tag{7}$$

which is, of course, a first integral for (2).

A Lotka-Volterra is called *dissipative* if there is a diagonal matrix $D > 0$ such that $AD \leq 0$. In this case, the system admits the global Lyapounov function (7). In [4] we have proved a result which further motivates the study of conservative Lotka-Volterra systems:

Theorem 5. *Every stably dissipative Lotka-Volterra system, with a singularity interior to $\mathbb{R}^d_+$, has a global attractor where the dynamics is that of a conservative Lotka-Volterra system.*

The non-zero entries of the matrix A determine a *food chain graph $G(A)$* with the eating relations within the ecosystem $\{1,2,\ldots,d\}$. A Lotka-Volterra system is said to be *stably dissipative* iff every nearby Lotka-Volterra system with the same food chain graph is still dissipative. Next theorem states that all linear replicator systems (3), in the simplex Δ^{2d}, which come from a conservative Lotka-Volterra system, fall in the scope of theorem 3, i.e., they are time reparametrizations of symplectic gradients of functions in $\mathscr{H}(\Delta^{2d})$ w.r.t. algebraic symplectic structures.

Theorem 6. *The replicator equation on Δ^{2d} corresponding to a conservative Lotka-Volterra system on $\mathbb{R}^{2d}$ given by some invertible coefficient matrix, is, up to equivalence, the symplectic gradient of a regular function $h \in \mathscr{H}(\Delta^{2d})$ of the form*

$$h(x_0,\ldots,x_{2d}) = \sum_{i=1}^{2d} \frac{x_i}{x_0} - q_i \log \frac{x_i}{x_0}$$

w.r.t. to some algebraic symplectic structure $\omega \in \Omega^2(\Delta^{2d})$.

5 An Application

In [4] we have analyzed the following Lotka-Volterra system, a four species food chain which couples two independent predator-prey systems

$$\begin{cases} y_1' = y_1\,(-1+y_2) \\ y_2' = y_2\,(1-y_1+\delta\,y_3) \\ y_3' = y_3\,(-1-\delta\,y_2+y_4) \\ y_4 = y_4\,(1-y_3) \end{cases} \tag{8}$$

where the coupling strength is controlled by the parameter δ. The coefficient matrices of this system are

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \delta & 0 \\ 0 & -\delta & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad r = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

We prove in [4] system (8) is non-integrable for any $\delta \neq 0$. There we pay special attention to a family of periodic orbits $\Gamma = \Gamma(\delta, E)$ defined, for all δ, as the intersection of the energy level $\{h = E\}$ with the following invariant 2-plane

$$\Pi = \{\, (y_1, y_2, y_3, y_4) \in \mathbb{R}^4_+ \, : \, y_1 = (1+\delta)\, y_3,\, y_4 = (1+\delta)\, y_2 \,\} \, .$$

There is a 3-plane containing Π, slicing transversally all energy levels in 2-spheres. The orbit Γ splits each of these 2-spheres in two disks transversal to the flow. The first return map, along the flow, to any of these disks is a continuous map which determines the dynamics in that energy level. Finally, the periodic orbit Γ has rotation number which tends to $+\infty$ with the energy level E, and its character alternates between stable (elliptic) and unstable (hyperbolic), as δ varies in $(0, +\infty)$. Furthermore, there is a sequence of small intervals of the parameter δ, where as $E \to +\infty$, the periodic orbit Γ becomes hyperbolic with arbitrary large trace.

In [3] we pursue the analysis of this system proving that

Theorem 7. *For $0 < \delta < 1$, the Lotka-Volterra system (8) has, in all sufficiently large energy level $\{\, h = E \,\}$, a non-trivial invariant hyperbolic basic set of saddle type.*

To prove this theorem we consider the replicator vector field $X \in \mathscr{X}(\Delta^4)$ in (3), associated with the Lotka-Volterra system (8). We denote by σ_i the face of Δ^4 opposed to vertex i, and by $\gamma_{i,j}$ the edge connecting the vertices i and j. We have $\nu_X(\sigma_0) = 2$ and $\nu_X(\sigma_i) = 1$, for $i = 1, 2, 3, 4$. Let χ be the skeleton of X.

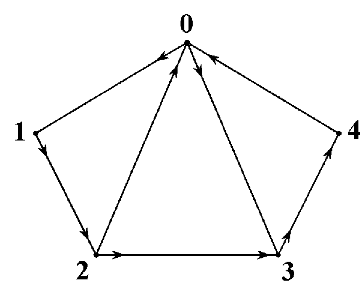

Fig. 8 The oriented graph $G_\chi(\Delta^4)$ consists of the 7 edges.

We can compute the following chains for χ, where $*$ stands for the chain concatenation operation.

$\xi^0 = (\gamma_{4,0}, \gamma_{0,1})$ $\qquad$ $\xi^1 = (\gamma_{0,1}, \gamma_{1,2}, \gamma_{2,0}, \gamma_{0,1})$

$\xi^2 = (\gamma_{0,1}, \gamma_{1,2}, \gamma_{2,3}, \gamma_{3,4}, \gamma_{4,0})$ $\qquad$ $\xi^3 = (\gamma_{0,1}, \gamma_{1,2}, \gamma_{2,0}, \gamma_{0,3}, \gamma_{3,4}, \gamma_{4,0})$

$\xi^4 = (\gamma_{4,0}, \gamma_{0,3}, \gamma_{3,4}, \gamma_{4,0})$

$\xi^5_n = \xi^0 * (\xi^1)^n * \xi^2$ $\qquad$ $\xi^6_n = \xi^0 * (\xi^1)^n * \xi^3 \qquad (n \geq 0)$

There are exactly four families of χ-cycles which start and end with γ_{40} but do not pass through this edge in between. They are $\{\xi^4\}$, $\{\,\xi^5_n : n \geq 0\}$ and $\{\,\xi^6_n : n \geq 0\}$. Whence the first return map $R^{\chi}_{\gamma_{4,0}}$ to Π_{40} is given by

$$R^{\chi}_{\gamma_{4,0}}(X) = \begin{cases} M_{\xi^4} X & \text{if } X \in \Pi_{\xi^4} \\ M_{\xi^5_n} X & \text{if } X \in \Pi_{\xi^5_n}, \quad n \geq 0 \\ M_{\xi^6_n} X & \text{if } X \in \Pi_{\xi^6_n}, \quad n \geq 0 \end{cases},$$

whose domain, the union of the open convex cones $\Pi_{\xi^4} \cup \bigcup_{n=0}^{\infty} \Pi_{\xi^5_n} \cup \bigcup_{n=0}^{\infty} \Pi_{\xi^6_n}$, can be characterized as follows.

Proposition 1. *The open cones Π_{ξ^4}, $\Pi_{\xi^5_n}$ and $\Pi_{\xi^6_n}$ ($n \geq 0$) are defined by the following inequalities:*

1. Π_{ξ^4} *by* $u_0 = u_4 = 0$, $-u_1 + u_3 < 0$, $u_1 > 0$ *and* $u_2 > 0$.
2. $\Pi_{\xi^5_n}$ *by* $u_0 = u_4 = 0$, $u_1 > 0$, $u_2 > 0$ *and*

$$\frac{-u_1 + u_3}{(1+\delta)(u_1+u_2)} - \frac{\delta}{1+\delta} < n < \frac{-u_1 + u_3}{(1+\delta)(u_1+u_2)}.$$

3. $\Pi_{\xi^6_n}$ *by* $u_0 = u_4 = 0$, $u_1 > 0$, $u_2 > 0$ *and*

$$\frac{-u_1 + u_3}{(1+\delta)(u_1+u_2)} - 1 < n < \frac{-u_1 + u_3}{(1+\delta)(u_1+u_2)} - \frac{\delta}{1+\delta}.$$

A simple computation shows that

$$M_{\xi^4} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1+\delta & \delta \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{\xi^5_n} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ * & -n - \frac{n+1}{\delta} & -n - \frac{n}{\delta} & \frac{1}{\delta} & 0 \\ * & (n+1)\delta & n\delta & 0 & \delta \\ * & 2n+2+\frac{n+1}{\delta} & 2n+1+\frac{n}{\delta} & -\frac{1}{\delta} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$M_{\xi^6_n} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ * & n+2+(n+1)\delta & n+1+(n+1)\delta & -1 & 0 \\ * & -(n+1)-(n+1)(\delta+\delta^2) & -n-(n+1)(\delta+\delta^2) & 1+\delta & \delta \\ * & -(n+1)\delta & -(n+1)\delta & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The '$*$' entries are not important, since we are only interested in the action of these matrices on the 3-plane $u_0 = u_4 = 0$ spanned by the cone $\Pi_{\gamma_{0,4}}$. Actually, the action of these matrices on $\Pi_{\gamma_{0,4}}$ is determined by the inner 3×3 submatrices of the above ones. By theorem 6 this system is, up to a time reparametrization, the symplectic gradient of the following Hamiltonian

$$h(x_0,\dots,x_4) = \frac{x1+x2+x3+x4}{x_0} + (1+\delta)\,\log\frac{x_1}{x_0} + \log x_2 + \log x_3 + (1+\delta)\,\log\frac{x_4}{x_0}$$

w.r.t. some algebraic symplectic structure. Whence by corollary 1, $\lambda_h : \mathscr{C}^*(\Gamma^d) \to \mathbb{R}$ is invariant under the flow of χ. We have $\lambda_h(u) = (1+\delta)\,u_1 + u_2 + u_3$, for every $u \in \Pi_{\gamma_{0,4}}$. Consider now the 2-simplex $\Delta^2 = \{\, u \in \Pi_{\gamma_{0,4}} : \lambda_h(u) = 1\}$, which is invariant under $R^{\chi}_{\gamma_{0,4}}$, and denote by $T : \Delta^2 \to \Delta^2$ the restriction of $R^{\chi}_{\gamma_{0,4}}$ to this simplex. For each cycle ξ through $\gamma_{0,4}$ we define $\Delta_\xi = \{\, u \in \Pi_\xi : \lambda_h(u) = 1\}$. Each restriction $T_\xi = T|_{\Delta^\xi}$ is an affine map, which we can compute explicitly, as well as its domain Δ_ξ and range $T_\xi(\Delta_\xi)$, for every cycle ξ through $\gamma_{4,0}$. With this notation, Δ^2 is the disjoint union (mod 0) of the polygons

$$\Delta_{\xi^4},\, \Delta_{\xi_0^5},\, \Delta_{\xi_0^6},\, \Delta_{\xi_1^5},\, \Delta_{\xi_1^6},\, \Delta_{\xi_2^5},\, \Delta_{\xi_2^6},\, \cdots \ .$$

Figure 9 shows these polygons, as well as their T-images, labeled in this order.

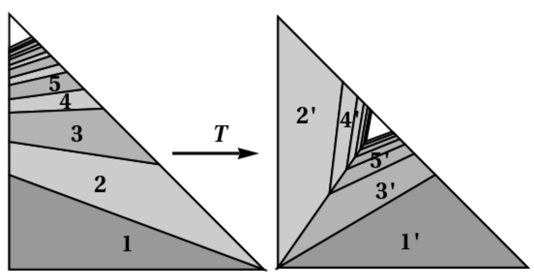

Fig. 9 Domain and range of the return map $T : \Delta^2 \to \Delta^2$.

We can check that the affine map $T_\xi : \Delta_\xi \to \Delta^2$ is

1. parabolic for $\xi = \xi^4$, for all $0 < \delta < 1$,
2. elliptic for $\xi = \xi_n^6$, $n \geq 0$, for some $0 < \delta < 1$,
3. hyperbolic with negative trace for $\xi = \xi_n^5$, $n \geq 0$, $0 < \delta < 1$.

For $0 < \delta < 1$ we compute the following two hyperbolic fixed points:

1. $P_0 = \left(\frac{1}{2+3\delta}, \frac{\delta}{2+3\delta}, \frac{1+\delta}{2+3\delta}\right) \in \Delta_{\xi_0^5}$, $\quad P_0 = T_{\xi_0^5}(P_0)$, and
2. $P_1 = \left(\frac{1-\delta}{3+4\delta-\delta^2}, \frac{2\delta}{3+4\delta-\delta^2}, \frac{2+2\delta}{3+4\delta-\delta^2}\right) \in \Delta_{\xi_1^5}$, $\quad P_1 = T_{\xi_1^5}(P_1)$.

We define the local invariant manifolds of these hyperbolic fixed points as follows: $W^s_{\mathrm{loc}}(P_i)$ is the intersection of the line through P_i parallel to the contracting eigenspace of P_i, w.r.t. the linear part of $T_{\xi_i^5}$, with the polygon $\Delta_{\xi_i^5}$, while $W^u_{\mathrm{loc}}(P_i)$ is the intersection of the line through P_i parallel to the expanding eigenspace of P_i with the image polygon $T_{\xi_i^5}(\Delta_{\xi_i^5})$. Using them we define the global manifolds

$$W^s(P_i) = \bigcup_{n\geq 0} T^{-n} W^s_{\mathrm{loc}}(P_i) \quad \text{and} \quad W^u(P_i) = \bigcup_{n\geq 0} T^{n} W^u_{\mathrm{loc}}(P_i)\ .$$

Then we can prove that

Proposition 2. *For all $\delta \in (0,1)$,*

$$W^s_{loc}(P_0) \cap W^u(P_1) \neq \emptyset \quad \textit{and} \quad W^s_{loc}(P_1) \cap W^u_{loc}(P_0) \neq \emptyset\ ,$$

with transversal intersections.

In figure 10, the filled lines represent unstable manifolds of P_0 and P_1, while the dashed lines represent stable manifolds.

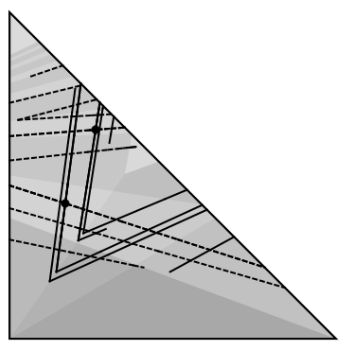

Fig. 10 Heteroclinic intersections of the return map $T : \Delta^2 \to \Delta^2$.

By proposition 2, the map T has a transversal heteroclinic cycle formed of two heteroclinic orbits. Because these orbits accumulate on the fixed points they stay at positive distance of the boundaries $\partial\Delta_{\xi_i^5}$ $(i = 0,1)$. Using them we can construct an invariant hyperbolic basic set of saddle type $\Lambda \subset \Delta_{\xi_0^5} \cup \Delta_{\xi_1^5}$, for the map T, still at a positive distance of $\partial\Delta_{\xi_0^5} \cup \partial\Delta_{\xi_1^5}$. By theorem 1, in all sufficiently large energy level surface the system must have a conjugate invariant hyperbolic basic set of saddle type $\Lambda_E \subset \{h = E\}$, which concludes the argument for theorem 7.

6 Conclusions

We finish with some related questions and possible generalizations.

The analyticity assumption was mainly aesthetic, everything works fine for smooth systems. One can also adapt the argument to work with compact manifolds with boundary, instead of simple polyhedra. Recall that a compact manifold with boundary, say M^d of dimension d, is one which at every point is locally diffeomorphic to a model $\left(\mathbb{R}^k \times \mathbb{R}_+^{d-k}, 0\right)$, for some $0 \leq k \leq d$. The integer k is called the index of M^d at that point. The set of all points with index k, denoted by $\partial_k(M^d)$, is exactly the union of all interiors of k-dimensional faces of the manifold M^d.

General algorithms can be developed to facilitate the analysis of a skeleton vector field's dynamics. In [4] we describe how to derive the skeleton vector field components from the payoff matrix of a replicator system. Similar relations can be driven for other Game Theory systems.

Vicinity relations of a cone domain Π_ξ should translate to symbolic kneading relations of the corresponding chain, or cycle, ξ. Such a kneading theory would be a very useful instrument of analysis.

Given a skeleton vector field, can we realize it as the edge asymptotics of some regular vector field? This realization is important to construct examples with prescribed dynamical behavior along the edges. For general regular vector fields the answer to this problem is positive. Every regular skeleton vector field χ in $\mathscr{C}^*(\Gamma^d)$ is the skeleton of some regular vector field $X \in \mathscr{X}(\Gamma^d)$. For conservative skeleton vector fields, the answer is yes locally, in a neighborhood of the 1-dimensional skeleton of Γ^{2d}. If a skeleton vector field χ of $\mathscr{C}^*(\Gamma^d)$ is the symplectic gradient of a skeleton function $\lambda : \mathscr{C}^*(\Gamma^d) \to \mathbb{R}$ w.r.t. a piecewise linear symplectic structure $\widehat{\omega}$ on $\mathscr{C}^*(\Gamma^d)$ then λ is the skeleton of a function $h \in \mathscr{H}(\Gamma^{2d})$ and $\widehat{\omega}$ is associated to some algebraic form $\omega \in \Omega^2(\Gamma^{2d})$. Whence, the symplectic gradient X_h of h w.r.t. ω is, as in theorem 3, equivalent to a regular vector field X whose skeleton will be χ. The problem with this approach is that it's not clear if ω is non degenerate everywhere, i.e., if ω is a symplectic structure on the interior of Γ^{2d}. In this case the gradient X_h may not be defined everywhere in Γ^{2d}. This raises the question of characterizing the subset of symplectic structures in $\Omega^2(\Gamma^{2d})$. We can avoid this problem dealing with Poisson structures instead of symplectic ones. We believe that a concept of "algebraic Poisson structure" can be defined on the polyhedron Γ^d, as well as a class of Hamiltonian systems with Hamiltonians in $\mathscr{H}(\Gamma^d)$ w.r.t. such algebraic Poisson structures, which up to equivalence give rise to regular vector fields in $\mathscr{X}(\Gamma^d)$. Then theorems 3, 4 and 6 should generalize to arbitrary dimensions.

Skeleton vector field's bifurcations is another interesting subject of study. These bifurcations are caused by changes in the geometry and combinatorics of the domain and image partitions of the return map $R^\chi_\gamma : \Pi_\gamma \to \Pi_\gamma$, respectively $\{\Pi_\xi\}_\xi$ and $\{R^\chi_\gamma(\Pi_\xi)\}_\xi$ where ξ varies on the set of all χ-cycles which start and end with γ but do not pass through γ in between. Considering skeletons of regular vector fields in $\mathscr{X}(\Gamma^d)$, it should be possible to relate these skeleton bifurcations with the bifurcations of the underlying vector field.

In theorem 7, for simplicity, we have assumed $\delta \in (0,1)$, but we believe that the same holds for all $\delta > 0$. The reason we made such restriction is that for $\delta > 1$ the dynamics is harder to analyze due to the presence of the elliptic fixed point P_0 in the main branch $T_{\xi_0^5} : \Delta_{\xi_0^5} \to \Delta^2$.

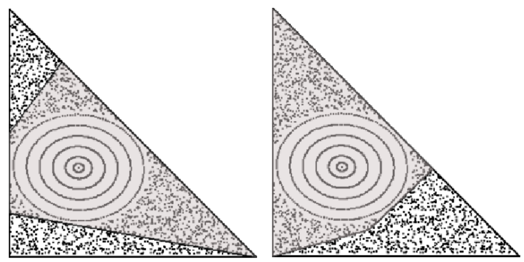

Fig. 11 An elliptic fixed point P_0 at $\delta = 3.7$

Figure 11 shows ten different orbits, with a couple of hundred iterates each, for a particular parameter. The shaded regions represent the polygon $\Delta_{\xi_0^5}$, on the left, and its image $T_{\xi_0^5}(\Delta_{\xi_0^5})$, on the right. The invariant curves break up as they touch the boundary of their domains. Outside these curves, the dynamics seems to be chaotic, which indicates the presence of hyperbolicity. Concerning the parameter interval $(0,1)$, we pose some more questions. Are there elliptic periodic points for parameters $0 < \delta < 1$? Is this true for many parameters? The Newhouse phenomenon, of persistent homoclinic tangencies associated with large thickness hyperbolic sets, is a mechanism for the appearance of many elliptic structures in the dynamics of the underlying Hamiltonian vector field. See for instance [5]. As $\delta \to 0^+$ can one find large uniformly hyperbolic basic sets with very large thickness? Is this also a mechanism for the creation of many elliptic periodic points of the skeleton vector field? It would be interesting to understand, for conservative skeleton vector fields, the mechanism for the creation of elliptic structures, and then relate it with the corresponding homoclinic bifurcation mechanism of the underlying dynamics of vector fields in $\mathscr{X}(\Gamma^{2d})$.

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