ABUNDANCE OF ELLIPTIC ISLES AT CONSERVATIVE BIFURCATIONS

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Abstract: *We prove a conservative analogue of a theorem of Newhouse on the abundance of wild hyperbolic sets: Arbitrarily close to some area preserving map with a homoclinic tangency there are "wild" hyperbolic sets with persistence of homoclinic tangencies. Furthermore, for a residual subset of nearby systems these basic sets are accumulated by periodic elliptic islands.*

1. Introduction

In 1969, see [AS-70], Abraham and Smale introduced, in a 4 dimensional model, the concept of *persistent homoclinic tangencies* in order to disprove the density of Ω-stable diffeomorphisms. Little after that S. Newhouse, see [N-70], in a much more elaborated construction, gave examples of open sets of surface diffeomorphisms with the same phenomenon. He uses the concept of *thickness of a Cantor set* to measure the stable and unstable Cantor like foliations of a horse-shoe. Then he shows that persistent homoclinic tangencies occur near every two dimensional diffeomorphism with a "thick" horse-shoe which has already one homoclinic tangency. Afterwards, in [N-74], Newhouse proved that, for dissipative models, these open sets with persistent homoclinic tangencies always contain residual subsets of diffeomorphisms with infinitely many coexisting sinks. Finally, in [N-79], he showed the abundance of the *persistence of homoclinic tangencies* phenomenon: Near every dissipative surface diffeomorphism with a homoclinic tangency there are open sets of maps with persistent homoclinic tangencies. In particular, in these open sets of diffeomorphisms, there are residual subsets with infinitely many sinks. The basic idea for this result is to show that "thick" horse-shoes appear at every one parameter unfolding of a quadratic homoclinic tangency. A proof of this theorem can be found in [PT-93], where a nice rescaling technique is used to show that for carefully chosen parameters there are high iterates of the initial map which, in some small regions near the homoclinic tangency and in suitable coordinates, are arbitrarily close to a map of the *quadratic family of endomorphisms*, $f_a(x,y) = (y, a - y^2)$. For the special parameter $a = 2$ the map f_2 has an invariant Cantor set, conjugated to the full Bernoulli shift in two symbols, with gaps of zero length, therefore an "infinitely thick" Cantor set. It follows that small invertible perturbations of f_2 must also exhibit hyperbolic basic sets with very large thickness. Notice that since the limit mapping f_a is infinitely dissipative, this argument uses disspativeness in a crucial way. This theorem has been generalised to higher dimensions, see [Rom-92] and [PV-94]. For the area preserving case and for a long time it has been conjectured by J. Palis that the same theorem should hold with elliptic islands playing the role of sinks. Such a generalisation is not obvious at all. Although the rescaling technique has been generalised in [MR1-97] to the conservative case, where the limit behaviour at the unfolding of a homoclinic tangency is given by the *conservative Henón family*, $H_a(x,y) = (y, -x + a - y^2)$, the major difficulty is that no special parameter, like $a = 2$, exists where the family H_a exhibits some distinctly "thick" horse-shoe.

In the present work we prove the conjecture of J. Palis working the Henón model at its first bifurcation, a saddle-centre bifurcation where a hyperbolic saddle and an elliptic centre are created. The crucial step for the proof was established in a previous work [D-98]. Recently we were told by D. Turaev of his work [T-98] where he proves the same result using a very different mechanism. We also mention the work of L. Mora and N. Romero in this direction [MR2-97] where the authors show that every area preserving diffeomorphism having one homoclinic tangency is in the boundary of an open set where maps with homoclinic tangencies are dense. During the years 94 and 95, the author was supported by JNICT grant PRAXIS/2/2.1/MAT/19/94. Since 1996 he was supported in part by JNICT grant PBIC/C/MAT/2140/95 and also by FCT and PRAXIS XXI through the Research Units Pluriannual Funding Program and Project 2/2.1/MAT/199/94.

2. Reference Definitions

Let P be a periodic hyperbolic point, with period n, of a diffeomorphism $f \in \mathrm{Diff}^r(M)$. A tangency between the stable and unstable manifolds, $W^s(P, f^n)$ and $W^u(P, f^n)$ of P, is called a *homoclinic tangency* of P. If Λ is a hyperbolic invariant set of $f \in \mathrm{Diff}^r(M)$, a tangency between stable and unstable leaves $W^s(x, f)$, $W^u(y, f)$ of two points $x, y \in \Lambda$ will be called a *homoclinic tangency* of Λ. Given a basic set Λ of a map f, there is always a compact neighbourhood U of Λ and a neighbourhood $\mathcal{U}$ of f in $\mathrm{Diff}^r(M)$ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^{-n}(U)$ and for every $g \in \mathcal{U}$, $\Lambda_g = \bigcap_{n\in\mathbb{Z}} g^{-n}(U)$ is also a basic set, topologically conjugated to Λ. From now on M^2 will always denote a compact two dimensional manifold and ω a non-degenerated 2-form on M^2, in other words a *symplectic* form. We will consider only smooth, that is of class C^∞, symplectic diffeomorphisms. A map $f : M^2 \to M^2$ is called symplectic if $f^\star\omega = \omega$, which means that f is area and orientation preserving. We denote by $\mathrm{Diff}^\infty\left(M^2, \omega\right)$ the space of all smooth symplectic diffeomorphisms and consider in it the usual weak C^∞ topology, of uniform convergence of derivatives on compact sets. The proof of next proposition is quite standard and will be omitted. See [PT-93], or [D-94] for a conservative argument.

Proposition 2.1. *Let Λ be a basic set of a symplectic map $f \in \mathrm{Diff}^\infty\left(M^2, \omega\right)$, $U \subseteq M^2$ be a neighbourhood of Λ and $\mathcal{U} \subseteq \mathrm{Diff}^\infty\left(M^2, \omega\right)$ a neighbourhood of f such that for every $g \in \mathcal{U}$ $\Lambda_g = \bigcap_{n\in\mathbb{Z}} g^{-n}(U)$ is a basic set conjugated to Λ. If for every $g \in \mathcal{U}$ the basic set Λ_g has some homoclinic tangency then*

(1) *Given a fixed point $P \in \Lambda$ of f there is a dense subset $\mathcal{D} \subseteq \mathcal{U}$, such that for every $g \in \mathcal{D}$ the unique fixed point $P_g \in \Lambda_g$ corresponding to P has some homoclinic tangency.*

(2) *there is a residual subset* $\mathcal{R} \subseteq \mathcal{U}$, *i.e. a countable intersection of open subsets dense in* $\mathcal{U}$, *such that for every* $g \in \mathcal{R}$ *the basic set* Λ_g *is contained in the closure of all generic elliptic periodic points of* g.

Definition 2.1. *Given* $f \in \mathit{Diff}^{\infty}(M)$ *and an invariant basic set* Λ *of* f *we say that* (f, Λ) *has* persistent homoclinic tangencies *if there are neighbourhoods* $\mathcal{U} \subseteq \mathit{Diff}^{\infty}(M)$ *of* f *and* $U \subseteq M$ *of* Λ *such that for every* $g \in \mathcal{U}$ *there is some homoclinic tangency of* $\Lambda_g = \bigcap_{n \in \mathbb{Z}} g^{-n}(U)$.

Definition 2.2. *A periodic point* P *of* $f \in \mathit{Diff}^{\infty}(M^2, \omega)$ *is said to be a* generic elliptic *point if both eigenvalues of* Df_P^n, *where* n *is the minimal period, sit in the unit circle without resonances of order* ≤ 3, *that is* $\lambda, \overline{\lambda} \in SS^1$ *with* $\lambda^2 \neq 1$ *and* $\lambda^3 \neq 1$, *and the first coefficient of* f*'s Birkhoff normal form at point* P *is nonzero.*

This implies that KAM theory applies and P is a full density point of "Cantor set" of invariant curves around P.

Definition 2.3. *Given a symplectic diffeomorphism* $f \in \mathit{Diff}^{r}(M^2, \omega)$, *a periodic point* $f^n(P) = P$, *with minimal period* n, *is called* extremal *if:*

(1) $\mathit{Spec}(Df_P^n) = \{1\}$
(2) $\dim \mathit{Ker}(Df_P^n - Id) = 1$
(3) $D^2 f_P^n(v, v) \notin \mathit{Ker}(Df_P^n - Id)$ *for any* $v \neq 0$ *such that* $Df_P^n v = v$.

Item 3 should be read in coordinates. It is not difficult to check this is an intrinsic condition. Let now $f:]a_0 - \delta_0, a_0 + \delta_0[\times M^2 \to M^2$, $f_a(\,.\,) = f(a,\,.\,)$, be a smooth family of symplectic diffeomorphisms.

Definition 2.4. *We say that* (a_0, P_0) *is a* generic saddle-centre *bifurcation of the family* f_a *or, in other words that* f_a *unfolds generically at* (a_0, P_0) *a saddle-centre bifurcation, when:*

(1) P_0 *is an extremal fixed point of* $f_{a_0}^n$
(2) $\frac{\partial}{\partial a}\left(f_a^n(P)\right)_{a=a_0} \notin \mathit{Ker}\left(Df_{a_0}^n(P) - Id\right)$

The name "saddle centre" is justified by the following proposition. See [Mey-70].

Proposition 2.2. *There are smooth curves* $a: [-\delta, \delta] \to \mathbb{R}$ *and* $P: [-\delta, \delta] \to M^2$ *such that*

(1) $a(0) = a_0$, $a'(0) = 0$ *and* $a''(0) \neq 0$
(2) $P(0) = P_0$ *and* $P'(0) \in \mathit{Ker}\left(Df_{a_0}^n(P) - Id\right) - \{0\}$
(3) $f_{a(t)}(P(t)) = P(t)$
(4) *if* $\tau(t) = \mathit{trace}\left(Df_{a(t)}P(t)\right)$ *then* $\tau(0) = 2$ *and* $\tau'(0) \neq 0$.

By this proposition there is one side of a_0 in the parameter line where for each a we can find two fixed points $P(t_-)$ and $P(t_+)$ given by the solutions $t_- < 0 < t_+$ of the equation $a(t) = a$. By item 4. one is hyperbolic (saddle) and the other is elliptic (centre).

FIGURE 1. Hyperbolic and elliptic fixed point created at a saddle centre

3. RENORMALIZATION AT HOMOCLINIC TANGENCIES

Consider a one-parameter family of surface diffeomorphisms $\varphi_\mu : M^2 \to M^2$ of class C^∞, generically unfolding a quadratic homoclinic tangency at point Q and at parameter $\mu = 0$. Renormalization near the homoclinic tangency $(0, Q)$ means the following: For every large $n \geq 0$ one finds a small box near $(0, Q) \in \mathbb{R} \times M$, shrinking to this point as $n \to \infty$, which is mapped by $(\mu, x) \mapsto \big(\mu, \varphi_\mu^n(x)\big)$ near itself. Then in this tiny box one computes adequate rescaling changes in phase and parameter coordinates,

$$\mathbb{R}^3 \ni (a, x, y) \mapsto (\mu_n(a), \Psi_{n,a}(x, y)) \in \mathbb{R} \times M$$

such that in these new coordinates the map φ_μ^n ,

$$i.e. \quad \Psi_{n,a}^{-1} \circ \varphi_{\mu_n(a)}^n \circ \Psi_{n,a} \,,$$

converges to a normal form $f_a(x, y)$ in the C^∞ topology. Thus any feature or property of the dynamics of normal form f_a , which is stable under small perturbations, will also be present in the dynamics of φ_μ for parameter values very close to parameter $\mu = 0$. For dissipative systems, in fact it is enough to assume the saddle P associated to the tangency is dissipative $|\det D\varphi_\mu(P)| < 1$, the above scheme works having as limit the *quadratic family of endomorphisms*,

$$f_a(x, y) = (y, a - y^2).$$

Of course area expansive saddles $|\det D\varphi_\mu(P)| > 1$, reduce to dissipative ones considering φ_μ^{-1} . In the conservative case, that is if all φ_μ preserve the same area form, it turns out that the same scheme works having as limit the *conservative Henón family*

$$H_a(x, y) = (y, -x + a - y^2).$$

This was established by L. Mora and N. Romero, see [MR1-97].

Theorem A
Let $\{\varphi_\mu\} \subseteq \mathit{Diff}^\infty\big(M^2, \omega\big)$ *be a smooth family of area preserving maps unfolding generically a quadratic homoclinic tangency at the point* $Q_0 \in M$ *and parameter* $\mu = 0$ *. Then there are, for all large enough* $n \in \mathbb{N}$ *, reparametrizations* $\mu = \mu_n(a)$ *of the parameter variable* μ *and* a*-dependent coordinates*

$$(x, y) \mapsto Q = \Psi_{n\,,\,a}(x, y) \in M^2$$

such that

(1) *for each compact* K *, in the* (a,x,y)*-space, the images of* K *under the maps*
$$(a,x,y)\mapsto(\mu_n(a)\,,\,\Psi_{n\,,\,a}(x,y))$$
converge to $(0,Q_0)\in\mathbb{R}\times M^2$ *, as* $n\to\infty$ *,*

(2) *the domains of the maps*
$$(a,x,y)\mapsto\Big(a\,,\,\Psi_{n\,,\,a}^{-1}\circ\varphi_{\mu_n(a)}^{\;n}\circ\Psi_{n\,,\,a}\,(x,y)\Big)$$
converge to $\mathbb{R}^3$ *as* $n\to\infty$ *and the maps converge in the* C^∞ *topology to the conservative Henón map*
$$(a,x,y)\mapsto\left(y\,,\,-x+a-y^2\right)$$

For the Henón family we can easily compute that a pair of fixed points is created through the unfolding of a saddle centre bifurcation at the parameter $a=-1$. An elliptic fixed point $Q_a^e=\left(-1+\sqrt{1+a}\,,\,-1+\sqrt{1+a}\right)$ and a hyperbolic one $Q_a^h=\left(-1-\sqrt{1+a}\,,\,-1-\sqrt{1+a}\right)$. Then as a runs between -1 and 3, the eigenvalues of Q_a^e go through the unit circle from 1 to -1 and at parameter $a=3$ the point Q_a^e goes through a period doubling bifurcation becoming thereafter hyperbolic. In [MR1-97] the authors compute the first coefficient of the Birkhoff normal form at point Q_a^e and show that for almost all parameters $a\in]-1,3[$, Q_a^e is a generic elliptic point. Thus since generic elliptic points are persistent under conservative perturbations they derive the following conclusion.

Theorem B
Let $\varphi_\mu:M^2\to M^2$ *be a family of area preserving diffeomorphisms of class* C^∞*,* P_μ *be a continuous curve of periodic hyperbolic saddles of* φ_μ *with period* k*, and assume* $W^s(P_\mu)$ *and* $W^u(P_\mu)$ *generically unfold a quadratic homoclinic tangency at* $\mu=0$*. Then there is a sequence* $(\mu_n,Q_n)\in\mathbb{R}\times M$*, indexed in* $n\geq n_0$ *for some* $n_0\in\mathbb{N}$*, such that:*

- (μ_n,Q_n) *converges to* $(0,P)$*, as* $n\to\infty$*,*
- Q_n *is a generic elliptic periodic point of* φ_{μ_n} *with period* kn*.*

4. Statement of Results

In the present work the following theorems will be established:

Theorem 1. *Let* $f\in\mathit{Diff}^\infty\left(M^2,\omega\right)$ *have an extremal fixed point* P*. Then there is a sequence of basic sets* (f_n,Λ_n) *such that:*

(1) f_n *converges to* f *in* $\mathit{Diff}^\infty\left(M^2,\omega\right)$*.*
(2) (f_n,Λ_n) *has persistent homoclinic tangencies.*
(3) Λ_n *converges to* P *in the Hausdorff metric.*

Theorem 2. *Let* $f\in\mathit{Diff}^\infty\left(M^2,\omega\right)$ *have an orbit* $\mathcal{O}$ *of homoclinic tangencies associated to some hyperbolic fixed point* P*. Then there is a sequence of basic sets* (f_n,Λ_n) *such that:*

(1) f_n *converges to* f *in* $\mathit{Diff}^\infty\left(M^2,\omega\right)$*.*
(2) (f_n,Λ_n) *has persistent homoclinic tangencies.*

(3) Λ_n *converges to* $\overline{\mathcal{O}}$ *in the Hausdorff metric.*
(4) *There is a sequence of fixed points* $P_n \in \Lambda_n$ *converging to* P *.*

Theorem 3. *Let* $f \in \mathit{Diff}^{\infty}\left(M^2, \omega\right)$ *have an orbit* $\mathcal{O}$ *of homoclinic tangencies associated to some hyperbolic fixed point* P *. Then there is an open set* $\mathcal{U} \subseteq \mathit{Diff}^{\infty}\left(M^2, \omega\right)$ *, with* $f \in \overline{\mathcal{U}}$ *, and a family of* g *invarinat basic sets* $\{\Lambda_g \,:\, g \in \mathcal{U}\}$ *such that:*

(1) Λ_g *contains the unique fixed point* P_g *near* P *, and*
(2) Λ_g *has a homoclinic tangency for every* $g \in \mathcal{U}$ *.*

Furthermore there are subsets $\mathcal{D}$ *,* $\mathcal{R} \subseteq \mathcal{U}$ *such that*

(1) $\mathcal{D}$ *is dense in* $\mathcal{U}$ *and for every* $g \in \mathcal{D}$ *there is some homoclinic tangency of the fixed point* P_g *.*
(2) $\mathcal{R}$ *is residual in* $\mathcal{U}$*, and for every* $g \in \mathcal{R}$ *,* Λ_g *is accumulated by* g *'s generic elliptic periodic points.*

In the next section we prove some abstract perturbation lemmas. Then in section 6 we state and prove a theorem which implies theorem 1. Finally in section 7 we prove theorem 2 from theorem 1 which in turn, and together with proposition 2.1, implies easily theorem 3.

As a final comment, a "one parameter" version of theorem 3 could be proved using, instead of theorem 1 in [D-98], the parametric version of this theorem stated in section 2, just after remark 2.4, of the cited work, if, and this is a big "if", one could estimate and control the geometry of the splitting of separatrices at the (first) bifurcation $(a, x, y) = (-1, -1, -1)$ of the conservative Henón family H_a . More precisely, if $\tilde{H}_\delta$ is the variation of the identity, c.f. next section, obtained rescaling as in section 6 the family H_a at $(a, x, y) = (-1, -1, -1)$, then it would be enough to know that the splitting of separatrices of the fixed point $(0, 0) = \tilde{H}_\delta(0, 0)$ is described by some Melnikov function of the form $\psi(\delta)\, \mu(t)$, where $\psi(\delta)$ has nonzero germ at $\delta = 0$ and $\mu(t)$ is some periodic Morse function.

5. Perturbations of the identity

Given a curve $\delta \mapsto f_\delta \in G$ in some finite dimensional Lie Group which goes through the unit element, $f_0 = 1 \in G$, we can always find an associated curve $\delta \mapsto F_\delta \in \mathcal{G}$, in the Lie algebra of G , such that $f_\delta = \exp\left(\delta\, F_\delta\right)$ for all sufficiently small δ . In this section we prove an abstract perturbation lemma for smooth one parameter families of diffeomorphisms unfolding the identity, which is an infinite dimensional analogue of the fact just stated. Let G be an infinite dimensional Lie group of smooth diffeomorphisms $f : \mathbb{R}^p \to \mathbb{R}^p$ and $\mathcal{G} = T_{Id}G$ be the associated Lie algebra. The flow of a vector field in $\mathcal{G}$ is a one parameter subgroup of G . A smooth map $f : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^p$, $f(\delta, x) = f_\delta(x)$, such that $f_\delta \in G$ for all δ , can be regarded as a smooth parameterized curve in G . If $f_0 = Id$ we call f a *variation of the identity* in G .

Proposition 5.1. *Given a smooth variation of the identity f_δ in G there is a smooth family $X_\delta(x) = X(\delta, x)$, of vector fields in $\mathcal{G}$, such that the family $\varphi_\delta = \phi^\delta_{X_\delta}$ of time one maps of δX_δ has the same infinite jet at $\delta = 0$ as the family f_δ, i.e. for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^p$,*

$$\frac{\partial^n f}{\partial \delta^n}(0, x) = \frac{\partial^n \varphi}{\partial \delta^n}(0, x) \, .$$

Proof. For each $n \in \mathbb{N}$ write $F_n(x) = \frac{d^n f}{d\delta^n}(0, x)$. Then we have the following formal Taylor development in δ

$$f_\delta(x) \sim \sum_{n=0}^{\infty} \frac{F_n(x)}{n!}\, \delta^n$$

Now, consider a formal vector field

$$X_\delta(x) \sim \sum_{n=0}^{\infty} \frac{X_n(x)}{n!}\, \delta^n$$

in the unknowns $X_0, X_1, \cdots$ to be found in $\mathcal{G}$. Let $\phi^t_{X_\delta}$ be the formal flow of X_δ. Set $t = \delta$ and compute the formal Taylor development in δ,

$$\phi^\delta_{X_\delta} \sim \sum_{n=0}^{\infty} \frac{Y_n}{n!}\, \delta^n \, .$$

Each coefficient Y_n in this formal series represents a smooth vector field in $\mathbb{R}^p$ which can be explicitly calculated in terms of the unknowns X_i

$$Y_n = Y_n(X_0, \cdots, X_{n-1})$$

Our task is then to show that the (infinite) system of equations

$$Y_n(X_0, X_1, \cdots, X_{n-1}) = F_n \, , \quad n = 0, 1, \cdots \tag{1}$$

has a unique solution $(X_0, X_1, \cdots) \in \mathcal{G}^{\mathbb{N}}$. We start looking for a solution of 1. in the Lie algebra $\mathcal{X}$ of all smooth vector fields in $\mathbb{R}^p$. The existence and uniqueness of a solution in $\mathcal{X}^{\mathbb{N}}$ will follow because Y_n depends on X_{n-1} in the following invertible way:

$$Y_n(X_0, X_1, \cdots, X_{n-1}) = n\, X_{n-1} + Z_n(X_0, X_1, \cdots, X_{n-2})$$

which shows that the sequence of unknowns can be recursively determined from system of equations 1. To compute the dependence of Y_n in the X_i variables we deduce the formal Taylor expansion in time of a smooth vector field's flow. Let ϕ^t be the flow of a vector field $X \in \mathcal{X}$. Given any other vector field $Y \in \mathcal{X}$ we have

$$\frac{d}{dt}(Y \circ \phi^t) = D_X Y \circ \phi^t$$

and since $\frac{d}{dt}\phi^t = X \circ \phi^t$ we obtain by induction

$$\frac{d^{n+1}}{dt^{n+1}}\phi^t = \overbrace{D_X\, D_X \cdots D_X}^{n}\, X \circ \phi^t$$

which shows that ϕ^t has the following power series expansion in t

$$\phi^t \sim Id + t\,X + \sum_{n=2}^{\infty} \frac{D_X^{\,n-1} X}{n!}\, t^n$$

Making the substitutions $t=\delta$, $X = X_\delta = \sum_{r=0}^{\infty} \frac{X_r}{r!}\,\delta^r$, and expanding, we obtain

$$\begin{aligned} \phi^\delta_{X_\delta} &\sim Id + \sum_{n=1}^{\infty} \frac{X_{n-1}}{(n-1)!}\,\delta^n + \\ &\quad + \sum_{k=2}^{\infty}\sum_{r_1=0}^{\infty}\cdots\sum_{r_k=0}^{\infty} \frac{D_{X_{r_1}}\cdots D_{X_{r_{k-1}}} X_{r_k}}{r_1!\,\cdots\, r_k!\,k!}\,\delta^{r_1+\cdots+r_k+k} \\ &\sim Id + \sum_{n=1}^{\infty} \frac{nX_{n-1}}{n!}\,\delta^n + \sum_{n=1}^{\infty}\frac{Z_n}{n!}\,\delta^n \sim \sum_{n=0}^{\infty}\frac{Y_n}{n!}\,\delta^n \end{aligned}$$

where $Y_0 = Id$, $Y_n = n\,X_{n-1} + Z_n$ and

$$Z_n = \sum_{k=2}^{n}\ \sum_{r_1+\cdots+r_k=n-k} \frac{n!}{r_1!\,\cdots\,r_k!\,k!}\, D_{X_{r_1}}\cdots D_{X_{r_{k-1}}} X_{r_k}$$

depends only on $X_0, X_1, \cdots, X_{n-2}$.

Let us now prove by induction that $X_n \in \mathcal{G}$ for all $n\in\mathbb{N}$. Of course $X_0 = F_1 = \frac{df}{d\delta}(0,\cdot) \in T_{f_0}G = \mathcal{G}$. Now assume that $X_0, X_1, \cdots, X_{n-1} \in \mathcal{G}$ and define ψ_δ to be the time-δ map of the flow of the vector field $X_{n-1,\,\delta} = \sum_{i=0}^{n-1} \frac{X_i}{i!}\,\delta^i \in \mathcal{G}$. The curve $\delta \mapsto \psi_\delta$ is a variation of the identity in G tangent to $\delta\mapsto f_\delta$ at the identity, and having order of contact $\geq n+1$

$$\frac{d^i f}{d\delta^i}(0,x) = F_i(x) = Y_i(X_0,\cdots,X_{i-1})(x) = \frac{d^i\psi}{d\delta^i}(0,x) \qquad i = 0,1,\cdots,n\,.$$

Thus the difference between the derivatives of order $n+1$ at $\delta=0$ is tangent to G at the identity:

$$\frac{d^{n+1}f}{d\delta^{n+1}}(0,x) - \frac{d^{n+1}\psi}{d\delta^{n+1}}(0,x) \in T_{Id}G = \mathcal{G}\,.$$

This is a well known fact from Differential Geometry. But since

$$\begin{aligned} \frac{d^{n+1}f}{d\delta^{n+1}}(0,\cdot) - \frac{d^{n+1}\psi}{d\delta^{n+1}}(0,\cdot) &= F_{n+1} - Y_{n+1}(X_0,\cdots,X_{n-1},0) = \\ &= F_{n+1} - Z_{n+1}(X_0,\cdots,X_{n-1}) \end{aligned}$$

we get

$$X_n = \frac{1}{n+1}\,\left(F_{n+1} - Z_{n+1}(X_0,\cdots,X_{n-1})\right) \in \mathcal{G}\,.$$

Now let, for each $n\in\mathbb{N}$, $\beta_n:\mathbb{R}\to\mathbb{R}$ be a smooth function such that $\beta_n(\delta)=1$ for all $\delta\in\left(-\frac{1}{n},\frac{1}{n}\right)$ and $\beta_n(\delta)=0$ when $\delta\notin\left(-\frac{2}{n},\frac{2}{n}\right)$. Consider the series

$$X_\delta(x) = \sum_{n=0}^{\infty} \beta_n(\delta)\,\delta^n\,\frac{X_n(x)}{n!}$$

For $\delta=0$ every summand above is zero and so the series adds up to $X_0(x)$. On the other hand in a neighbourhood of $\delta\neq 0$ not containing 0 there is only a finite number of nonzero terms in this series. $X_\delta(x)$ is thus a finite linear combination of vector fields in $\mathcal{G}$, and therefore belongs to $\mathcal{G}$. Of course this smooth family admits the following Taylor expansion at $\delta = 0$,

$$X_\delta(x) \sim \sum_{n=0}^{\infty} \delta^n \, \frac{X_n(x)}{n!} \; .$$

Thus, denoting by $\phi^t_{X_\delta}$ the flow of X_δ,

$$\phi^\delta_{X_\delta}(x) \sim \sum_{n=0}^{\infty} \delta^n \, \frac{F_n(x)}{n!} \sim f_\delta(x)$$

which proves the proposition. $\square$

For symplectic maps we have,

Proposition 5.2. *Let $X_\delta : \mathbb{R}^2 \to \mathbb{R}^2$, $\delta \in \mathbb{R}$, be a smooth family of Hamiltonian vector fields with a saddle connection γ_δ associated to some hyperbolic fixed point family P_δ. Denote by $\varphi_\delta : \mathbb{R}^2 \to \mathbb{R}^2$ the time δ flow of X_δ. Then there is a smooth family of area preserving diffeomorphisms $f_{\delta,\mu} : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

(1) *$f_{\delta,0} = \varphi_\delta$. Therefore $f_{\delta,\mu}$ and X_δ satisfy the assumptions H1 and H2 in* [D-98].
(2) *The Melnikov function of the family $f_{\delta,\mu}$, see definition 2.1 of* [D-98]*, has the form $M_\delta(t) = \psi(\delta)\, \sin(t)$ for some analytic function $\psi(\delta)$ which has at most a countable number of zeros accumulating at $\delta = 0$.*
(3) *There is some compact set $K \subseteq \mathbb{R}^2$ such that for all δ, μ and $(x, y) \notin K$, $f_{\delta,\mu}(x, y) = \varphi_\delta(x, y)$.*

Proof. Let $q_\delta(t)$ be a smooth family of solutions, $q'_\delta(t) = X_\delta\,(q_\delta(t))$, parametrizing the homoclinic connection γ_δ. Chose smooth symplectic coordinates $\Psi_\delta : U \subseteq \mathbb{R}^2 \to\,]-1, 1[^2$ such that $t \in]-1, 1[$ if and only if $q_\delta(t) \in U$ and for all $t \in]-1, 1[$, $\Psi_\delta\,(q_\delta(t)) = (t, 0)$. Take now any even nonzero smooth function $\beta : \mathbb{R} \to \mathbb{R}$, with compact support contained in $]-1, 1[$. The vector field $\tilde{Y}_\delta(x, y) = (\beta'(y)\, , \, -\beta'(x))$ is Hamiltonian and so is the pull-back

$$Y_\delta(x, y) = \begin{cases} D\Psi_\delta(x, y)^{-1} \tilde{Y}_\delta\,(\Psi_\delta(x, y)) & \text{if } (x, y) \in U \\ 0 & \text{if } (x, y) \notin U \end{cases}$$

which is a smooth vector field with compact support contained in U.

Consider now the rapidly oscillatory perturbation of X_δ,

$$\frac{dx}{dt} = X_\delta(x) + \mu \, \cos\left(\frac{t}{\delta}\right) \, Y_\delta(x) \; .$$

The slowing change in the time variable $\tau = \frac{t}{\delta}$ takes this equation into,

$$\frac{dx}{d\tau} = \delta \; (\, X_\delta(x) \, + \, \mu \, \cos(\tau) \; Y_\delta(x) \,) \; ,$$

which, in turn, is equivalent to the autonomous system

$$\begin{cases} x' & = \delta \; (\, X_\delta(x) \, + \, \mu \, \cos(\theta) \; Y_\delta(x) \,) \\ \theta' & = 1 \end{cases} .$$

Because this system is periodic in the variable θ it induces a flow in the cylinder $\phi^t_{\delta,\mu} : \mathbb{R}^2 \times (\mathbb{R}/2\pi\mathbb{Z}) \to \mathbb{R}^2 \times (\mathbb{R}/2\pi\mathbb{Z})$. We define $f_{\delta,\mu}$ to be the return map $\phi^{2\pi}_{\delta,\mu}$

to the cross-section $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\}$. Let us compute the Melnikov function of the family $f_{\delta,\mu}$. Notice $\tilde{q}_\delta(t) = q_\delta(\delta\, t)$ is a family of solutions of $\delta\, X_\delta$ parametrizing the homoclinic connection γ_δ . Using now a well known formula for the Melnikov function of a periodically perturbed system, see for instance [GH-83], we have

$$\begin{aligned}
M_\delta(\tau_0) &= \int_{-\infty}^{+\infty} \delta^2 \cos(\tau)\, X_\delta\left(\tilde{q}_\delta(\tau-\tau_0)\right) \wedge Y_\delta\left(\tilde{q}_\delta(\tau-\tau_0)\right)\, d\tau \\
&= \int_{-\infty}^{+\infty} \delta \cos(\tau_0 + \frac{t}{\delta})\, \underbrace{X_\delta\left(q_\delta(t)\right) \wedge Y_\delta\left(q_\delta(t)\right)}_{=-\beta'(t)}\, dt \\
&= -\int_{-\infty}^{+\infty} \delta\, \beta'(t) \cos(\tau_0 + \frac{t}{\delta})\, dt \\
&= -\delta \cos(\tau_0) \underbrace{\int_{-\infty}^{+\infty} \beta'(t) \cos(\frac{t}{\delta})\, dt}_{=0} + \delta \sin(\tau_0) \int_{-\infty}^{+\infty} \beta'(t) \sin(\frac{t}{\delta})\, dt \\
&= \psi(\delta) \sin(\tau_0)
\end{aligned}$$

where

$$\psi(\delta) = \delta \int_{-\infty}^{+\infty} \beta'(t) \sin(\frac{t}{\delta})\, dt\, .$$

Remark that since $\beta'(t)$ is an odd function $\int_{-\infty}^{+\infty} \beta'(t) \cos(\frac{t}{\delta})\, dt = 0$. Remark also that, since Ψ_δ is symplectic,

$$\begin{aligned}
X_\delta\left(q_\delta(t)\right) \wedge Y_\delta\left(q_\delta(t)\right) &= D\Psi_\delta(q_\delta(t)) \cdot \frac{dq_\delta}{dt}(t) \wedge \tilde{Y}_\delta(t,0) \\
&= \det \begin{pmatrix} 1 & \beta'(0) \\ 0 & \beta'(t) \end{pmatrix} = -\beta'(t)
\end{aligned}$$

for all $t \in]-1,1[$. For $t \notin]-1,1[$ this wedge product is zero since $Y_\delta(q_\delta(t)) = 0$.

The function $\psi(\delta)$ is analytic in $\mathbb{C} - \{0\}$ since $\beta'(t)$ has compact support. It is not identically zero because this would imply that the nonzero function $\beta'(t)$ would have zero Fourier Transform. Therefore it has at most a countable number of (real) zeros accumulating at $\delta = 0$. Finally item 3. is clear since the perturbation term $\mu \cos(\tau)\, Y_\delta(x)$ vanishes outside some compact subset of U . □

Proposition 5.3. *Let $X_0(x,y)$ be a Hamiltonian vector field with a saddle connection γ_0 associated to some hyperbolic fixed point P_0 and $g_\delta : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth variation of the identity in the group of symplectic diffeomorphisms of $\left(\mathbb{R}^2, \omega\right)$, $\omega = dx \wedge dy$, such that $g_\delta = Id + \delta\, X_0 + \cdots$.*

Given a compact set $K \subseteq \mathbb{R}^2$ and integers $r, p \in \mathbb{N}$, there is, for all small enough $\delta \neq 0$, a map $f_\delta \in Diff^\infty\left(\mathbb{R}^2, \omega\right)$ with a basic set Λ_δ , containing the unique fixed point P_δ near P_0 , such that:

1. *$(f_\delta, \Lambda_\delta)$ has persistent homoclinic tangencies.*
2. *$\|f_\delta - g_\delta\|_{K,\,C^r} = \max_{1\le i\le r} \max_{x\in K} \left\| D^i f_\delta(x) - D^i g_\delta(x) \right\| \le \delta^p$.*

Proof. By proposition 5.1 there is a smooth family of Hamiltonian vector fields $X_\delta : \mathbb{R}^2 \to \mathbb{R}^2$ such that the family φ_δ of time one maps of $\delta\, X_\delta$ has the same infinite jet at $\delta = 0$ as g_δ. Therefore, if δ is small enough, we can assume that $\|\varphi_\delta - g_\delta\|_{K,C^r} \le \delta^{p+1}$.

To the family φ_δ we can associate, by proposition 5.2, a two parameter family $f_{\delta,\mu}$ satisfying the assumptions H1 and H2 of [D-98]. Then, as we have proved in [D-98], if δ is small enough, and $\psi(\delta) \neq 0$, there are parameters μ arbitrarily close to 0 such that $f_{\delta,\mu}$ has a basic set $\Lambda_{\delta,\mu}$ with very large left-right thickness and such that some positive tangency occurs between stable and unstable leaves of a fixed point in $\Lambda_{\delta,\mu}$. Furthermore this tangency unfolds generically with μ. The same argument of proposition 3.2 in [D-98] proves that unfolding these tangencies we get pairs $(f_{\delta,\mu}, \Lambda_{\delta,\mu})$ with persistent homoclinic tangencies. Taking μ small enough we get $\|f_{\delta,\mu} - \varphi_\delta\|_{K,C^r} \le \delta^{p+1}$. Thus

$$\|g - g_\delta\|_{K,C^r} \le \|g - \varphi_\delta\|_{K,C^r} + \|\varphi_\delta - g_\delta\|_{K,C^r} \le 2\,\delta^{p+1} < \delta^p .$$

If $\psi(\delta) = 0$ we take some pair $(f_{\delta+h,\mu}, \Lambda_{\delta+h,\mu})$ having persistent homoclinic tangencies with $\psi(\delta+h) \neq 0$ and h small enough.

□

6. Perturbations near conservative bifurcations

Let $\{\varphi_\mu : \mathbb{R}^2 \to \mathbb{R}^2\}$ be a smooth family of area preserving maps unfolding a generic saddle centre bifurcation at $(\mu, x, y) = (0,0,0)$. Then after some linear change of (μ, x, y)-coordinates, non necessarily symplectic, the family φ_μ takes the following form

$$\varphi_\mu(x,y) = \big(x + y + g_1(\mu,x,y)\,,\; y + \mu - x^2 + g_2(\mu,x,y)\big) \tag{2}$$

where $g_1(\mu,x,y)$ and $g_2(\mu,x,y)$ are smooth functions such that

$$g_1(0,0,0) = \frac{\partial g_1}{\partial x}(0,0,0) = \frac{\partial g_1}{\partial y}(0,0,0) = 0 \tag{3}$$

$$g_2(0,0,0) = \frac{\partial g_2}{\partial x}(0,0,0) = \frac{\partial g_2}{\partial y}(0,0,0) = \frac{\partial g_2}{\partial \mu}(0,0,0) = \frac{\partial^2 g_2}{\partial x^2}(0,0,0) = 0$$

Proposition 6.1. *Given a smooth family of area preserving maps $\{\varphi_\mu : \mathbb{R}^2 \to \mathbb{R}^2\}$ unfolding a generic saddle centre bifurcation at $(\mu,x,y) = (0,0,0)$ there is a smooth rescaling in parameter-phase coordinates*

$$\Psi : \mathbb{R}^3 \to \mathbb{R}^3 \qquad (\mu,x,y) = \Psi(\delta,u,v) = (\mu(\delta), \psi_\delta(u,v))$$

such that

1. $\mu(\delta) = c\,\delta^4 + O(\delta^6)$*, with* $c \neq 0$*,*
2. *for each* $\delta \neq 0$ *the map* ψ_δ *is affine,*
3. *the rescaled family* $\tilde\varphi_\delta = \psi_\delta^{-1} \circ \varphi_{\mu(\delta)} \circ \psi_\delta$ *is a smooth variation of the identity, formed by area preserving maps,*
$$\tilde\varphi_\delta = Id + \delta\, X_0 + \cdots$$
with $X_0(u,v) = (v, 2\,u + u^2)$*,*
4. *for* $\delta \neq 0$*,* $\tilde\varphi_\delta(0,0) = (0,0)$ *is a hyperbolic fixed point of* $\tilde\varphi_\delta$*.*

Proof. We may assume the family φ_μ is already in the form 2. Let $\overline{\mu}(x)$ and $\overline{y}(x)$ be defined implicitly by the system of equations

$$\varphi_\mu(x,y) = (x,y) \quad \Leftrightarrow \quad \left\{ \begin{array}{lcl} y + g_1(\mu,x,y) & = & 0 \\ x + \mu - x^2 + g_2(\mu,x,y) & = & 0 \end{array} \right.$$

Differentiating these relations at $x = 0$, we get

$$(^*) \qquad \overline{\mu}(0) = \overline{\mu}'(0) = \overline{y}(0) = \overline{y}'(0) = 0 \quad \text{and} \quad \overline{\mu}''(0) = 2\ .$$

Then we define $\mu(\delta) = \overline{\mu}(\delta^2)$ and

$$\psi_\delta \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \delta^2 \\ \overline{y}(\delta^2) \end{pmatrix} + \begin{pmatrix} \delta^2 & 0 \\ 0 & \delta^3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

One can easily compute for the rescaled map $\psi_\delta^{-1} \circ \varphi_{\mu(\delta)} \circ \psi_\delta\ (u,v)$ the following expression

$$\left(\frac{\overline{y}(\delta^2) + \delta^2 u + \delta^3 v + \tilde{g_1}}{\delta^2}\ ,\ \frac{\delta^3 v + \overline{\mu}(\delta^2) - \delta^4 (1+u)^2 + \tilde{g_2}}{\delta^3} \right) =$$

$$= \left(u + \delta v + \frac{\overline{y}(\delta^2)}{\delta^2} + \frac{\tilde{g_1}}{\delta^2}\ ,\ v + \delta (2u + u^2) + \frac{\overline{\mu}(\delta^2)}{\delta^3} - \delta + \frac{\tilde{g_2}}{\delta^3} \right) =$$

$$= (u,v) + \delta X_0(u,v) + O(\delta^2)$$

where

$$\frac{\overline{y}(\delta^2)}{\delta^2} = O(\delta^2) \quad \text{and} \quad \frac{\overline{\mu}(\delta^2)}{\delta^3} - \delta = O(\delta^3)\ ,$$

because of $(^*)$, and

$$\tilde{g_1} = g_1 \left(\overline{\mu}(\delta^2)\,,\, \delta^2 (1+u)\,,\, \delta^3 (O(\delta) + v) \right) = O(\delta^4)\ ,$$

$$\tilde{g_2} = g_2 \left(\overline{\mu}(\delta^2)\,,\, \delta^2 (1+u)\,,\, \delta^3 (O(\delta) + v) \right) = O(\delta^5)\ .$$

Notice that, in general, given a smooth function $g(\mu,x,y)$, we have

$$g \left(\overline{\mu}(\delta^2)\,,\, \delta^2 (1+u)\,,\, \delta^3 (O(\delta) + v) \right) = \delta^p\, G(\delta,u,v)\ ,$$

for some $G(\delta,u,v)$ smooth and

$$p = \min \left\{ 4\alpha_1 + 2\alpha_2 + 3\alpha_3\ :\ \frac{\partial^{\alpha_1+\alpha_2+\alpha_3} g}{\partial \mu^{\alpha_1} x^{\alpha_2} y^{\alpha_3}}(0,0,0) \neq 0 \right\}\ .$$

Finally to prove item 4. just remark that, for each δ, $\psi_\delta(0,0) = (\overline{\mu}(\delta^2), \delta^2, \overline{y}(\delta^2))$ is a fixed point of $\varphi_{\mu(\delta)}$. It is hyperbolic since $(0,0)$ is a hyperbolic fixed point of X_0. □

Theorem 1 follows easily from the next one.

Theorem 4. *Let $\{\varphi_\mu : \mathbb{R}^2 \to \mathbb{R}^2\}_\mu$ be a smooth family of area preserving maps unfolding a generic saddle-centre bifurcation at $(\mu,x,y) = (0,0,0)$. Given a compact set $K \subseteq \mathbb{R}^2$ and positive integers $r, p \in \mathbb{N}$ there is, for all small enough $\mu \neq 0$, a map $f_\mu \in \mathit{Diff}^\infty(\mathbb{R}^2,\omega)$ and a basic set Λ_μ, containing the unique fixed point P_μ near $(0,0)$, such that:*

1. *(f_μ, Λ_μ) has persistent homoclinic tangencies.*
2. $\|f_\mu - \varphi_\mu\|_{K,C^r} = \max_{1\le i\le r} \max_{x\in K} \left\| D^i f_\mu(x) - D^i \varphi_\mu(x) \right\| \le \mu^p$.

Proof. Let $\tilde{\varphi}_\delta = \mathrm{Id} + \delta X_0 + \cdots = \psi_\delta^{-1} \circ \varphi_{\mu(\delta)} \circ \psi_\delta$ be the variation of the identity obtained rescaling the family φ_μ as in proposition 6.1. Then given integers $r, p_0 \in \mathbb{N}$, by proposition 5.3 there is a map $\tilde{g}_\delta \in \mathrm{Diff}^\infty(\mathbb{R}^2,\omega)$ and a basic set $\tilde{\Lambda}_\delta$ of $\tilde{g}_\delta$ such that $\left(\tilde{g}_\delta, \tilde{\Lambda}_\delta\right)$ has persistent homoclinic tangencies and $\|\tilde{g}_\delta - \tilde{\varphi}_\delta\|_{B(10),\,C^r} \leq \delta^{p_0}$. Consider the mapping Φ obtained from lemma 6.1 with $R=5$ and define $\tilde{f}_\delta = \Phi\left(\tilde{g}_\delta \circ \tilde{\varphi}_\delta^{-1}\right) \circ \tilde{\varphi}_\delta$. Now, given $p_1 \in \mathbb{N}$, since Φ is continuous, taking p_0 large enough we will have $\left\|\tilde{f}_\delta - \tilde{\varphi}_\delta\right\|_{B(10),\,C^r} < \delta^{p_1}$. We also have $\tilde{f}_\delta = \tilde{\varphi}_\delta$ outside $B(10)$ and $\tilde{f}_\delta = \tilde{g}_\delta$ inside $B(5)$. Notice that $B(5)$ is sufficiently large to contain $\tilde{\Lambda}_\delta$ and all the local stable and unstable manifolds of $\tilde{\Lambda}_\delta$ with persistent tangencies. Just observe that $B(5)$ contains the homoclinic connection of $X_0(u,v) = \left(v, 2u + u^2\right)$. Therefore the pair $\left(\tilde{f}_\delta, \tilde{\Lambda}_\delta\right)$ also has persistent homoclinic tangencies. We use the affine rescaling operator $R_\delta : \tilde{h} \mapsto \psi_\delta \circ \tilde{h} \circ \psi_\delta^{-1}$, to define $f_\mu = R_\delta\left(\tilde{f}_\delta\right)$, where $\mu = \mu(\delta)$. Then $f_\mu = \varphi_\mu$ outside $\psi_\delta\left(B(10)\right)$, because $\varphi_\mu = R_\delta\left(\tilde{\varphi}_\delta\right)$ and $\tilde{f}_\delta = \tilde{\varphi}_\delta$ outside $B(10)$. Given $p \in \mathbb{N}$ we set $p_1 = 5p + 3r$. Since R_δ clearly satisfies

$$\|R_\delta(\tilde{g}) - R_\delta(\tilde{h})\|_{\psi_\delta(K),\,C^r} \leq \frac{1}{\delta^{3r}} \|\tilde{g} - \tilde{h}\|_{K,\,C^r},$$

we have

$$\begin{aligned} \|f_\mu - \varphi_\mu\|_{C^r} &= \|f_\mu - \varphi_\mu\|_{\psi_\delta(B(10)),\,C^r} \\ &\leq \frac{1}{\delta^{3r}} \|\tilde{f}_\delta - \tilde{\varphi}_\delta\|_{B(10),\,C^r} \leq \delta^{5p} \leq \mu^p . \end{aligned}$$

Finally the rescaled pair $(f_\mu, \Lambda_\mu) = \left(\psi_\delta \circ \tilde{f}_\delta \circ \psi_\delta^{-1},\ \psi_\delta\left(\tilde{\Lambda}_\delta\right)\right)$ has also persistent homoclinic tangencies. □

Lemma 6.1. *Given $R > 1$ there is a neighbourhood $\mathcal{U}$ of the identity in $\mathit{Diff}^\infty(\mathbb{R}^2,\omega)$ and a continuous mapping $\Phi : \mathcal{U} \to \mathit{Diff}^\infty(\mathbb{R}^2,\omega)$ such that:*

1. *$\Phi(Id) = Id$,*
2. *$\Phi(f)(x,y) = (x,y)$ if $\|(x,y)\| \geq 2R$, and*
3. *$\Phi(f)(x,y) = f(x,y)$ if $\|(x,y)\| \leq R$.*

Proof. A smooth function $S(x,Y)$, $S : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, is called a *generating function* of $f \in \mathrm{Diff}^\infty\left(M^2,\omega\right)$ if

$$f\left(x\,,\, Y + \frac{\partial S}{\partial x}(x,Y)\right) = \left(x + \frac{\partial S}{\partial Y}(x,Y)\,,\, Y\right),$$

for every $(x,Y) \in U$. Notice that if $S : \mathbb{R}^2 \to \mathbb{R}$ is a generating function defined everywhere with uniformly small first and second order derivatives then the maps

$$\varphi_S : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \varphi_S(x,Y) = \left(x\,,\, Y + \frac{\partial S}{\partial x}(x,Y)\right),$$

$$\psi_S : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \psi_S(x,Y) = \left(x + \frac{\partial S}{\partial Y}(x,Y)\,,\, Y\right),$$

are diffeomorphisms and $f = \psi_S \circ \varphi_S^{-1}$.

Define $\mathcal{V}_\epsilon$ as the neighbourhood of the identity formed by all maps $f \in \mathrm{Diff}^\infty\left(\mathbb{R}^2, \omega\right)$ such that for all $\|(x,y)\| \leq 3\,R$,

$$|f_2(x,y) - y| < \epsilon \quad \text{and} \quad \frac{\partial f_2}{\partial y}(x,y) > 1 - \epsilon\,.$$

Then if $\epsilon > 0$ is small enough we can associate to each $f = (f_1, f_2) \in \mathcal{V}_\epsilon$ a generating function S_f over the ball $B(2R) = \{\,(x,Y) : \|(x,Y)\| \leq 2\,R\}$. Let $y_f(x,Y)$ and $X_f(x,Y)$ be defined implicitly by

$$f\,(\,x\,,\,y_f(x,Y)\,) = (\,X_f(x,Y)\,,\,Y\,)$$

or, equivalently, by

$$f_2\,(\,x\,,\,y_f(x,Y)\,) = Y \quad \text{and} \quad X_f(x,Y) = f_1\,(x, y_f(x,Y))\ .$$

The condition on $\mathcal{V}_\epsilon$ implies that for small $\epsilon > 0$ $y_f(x,Y)$ is well defined all over $B(2R)$. On the other hand, the symplectic character of f implies that

$$\frac{\partial y_f}{\partial Y}(x,Y) = \frac{\partial X_f}{\partial x}(x,Y)\ ,$$

and so there is a unique function $S = S_f$ on $B(2R)$ such that $S(0,0) = 0$ and

$$\frac{\partial S}{\partial x}(x,Y) = y_f(x,Y) - Y \quad \text{and} \quad \frac{\partial S}{\partial Y}(x,Y) = X_f(x,Y) - x\ ,$$

which implies that S is the generating function of f. The mapping $f \mapsto S_f$ from $\mathcal{V}_\epsilon$ to $C^\infty\left(B(2R), \mathbb{R}\right)$ is clearly continuous.

Take a smooth function $\rho : \mathbb{R}^2 \to [0,1]$ vanishing outside $B\left(\frac{7R}{4}\right)$ and constant equal to 1 inside $B\left(\frac{5R}{4}\right)$. Taking a smaller neighbourhood of the identity $\mathcal{V} \subseteq V_\epsilon$ we can make $\rho\,S_f$ to have small first and second order derivatives for all $f \in \mathcal{V}$. Just notice $S_{\mathrm{Id}} = 0$. This implies that $\varphi_{\rho S_f}$ and $\psi_{\rho S_f}$, defined above, are diffeomorphisms. Setting $\Phi(f) = \psi_{\rho S_f} \circ \varphi_{\rho S_f}^{-1}$, $\Phi(f)$ is a symplectic diffeomorphism with generating function $\rho\,S_f$ and, of course, $\Phi(\mathrm{Id}) = \mathrm{Id}$. Since the mappings $f \mapsto \varphi_{\rho S_f}$ and $f \mapsto \psi_{\rho S_f}$ are continuous so is $\Phi : \mathcal{V} \to \mathrm{Diff}^\infty\left(\mathbb{R}^2, \omega\right)$. Since $\rho = 1$ on $B\left(\frac{5R}{4}\right)$ we have $\Phi(f)(x,y) = f(x,y)$ over $\varphi_{\rho S_f}\left(B\left(\frac{5R}{4}\right)\right)$. Similarly $\Phi(f)(x,y) = (x,y)$ outside $\varphi_{\rho S_f}\left(B\left(\frac{7R}{4}\right)\right)$, because $\rho = 0$ out of $B\left(\frac{7R}{4}\right)$. But since $\varphi_{\rho S_{\mathrm{Id}}} = \mathrm{Id}$ taking, if necessary, a smaller neighbourhood $\mathcal{V}$ we may assume that $B(R) \subseteq \varphi_{\rho S_f}\left(B\left(\frac{5R}{4}\right)\right)$ and $\varphi_{\rho S_f}\left(B\left(\frac{7R}{4}\right)\right) \subseteq B(2R)$ for all $f \in \mathcal{V}$. Items 2. and 3. then follow. $\square$

7. Proof of Theorems 2. and 3

Proof of theorem 3. Follows from Theorem 2 by standard arguments, since the unfolding of a homoclinic tangency creates elliptic periodic orbits which shadow the orbit of homoclinic tangencies. The creation of these generic elliptic points can be seen from the renormalization at conservative homoclinic tangencies, outlined in section 1.4. See [MR1-97]. $\square$

Proof of theorem 2. Assume f has a homoclinic tangency q between the branches of invariant manifolds, $\gamma^s(P) \subseteq W^s(P)$ and $\gamma^u(P) \subseteq W^u(P)$, of some hyperbolic fixed point P, as well as a transversal intersection between the same branches $\gamma^s(P)$ and $\gamma^u(P)$. If these branches do not have transversal intersections we perturb f creating transversal homoclinic orbits together with a new tangency. Second we perturb f to make the homoclinic tangency quadratic and then embed the perturbed diffeomorphism, still denoted by f, in a one parameter family of diffeomorphisms $\{\varphi_\mu\}$ which unfolds generically at $\varphi_0 = f$ the quadratic homoclinic tangency q. The conservative Henón family, $\{H_a : \mathbb{R}^2 \to \mathbb{R}^2 |\, a \in \mathbb{R}\}$, $H_a(x, y) = (y, -x + a - y^2)$, shows up through renormalization, infinitely often in very small scales near the orbit of homoclinic tangencies and for carefully chosen parameters very near the tangency at $\mu = 0$. The Henón family unfolds at $a = -1$ its first bifurcation, a generic saddle-centre, where a pair of fixed points is created. Thus, using the notation of Mora-Romero's Renormalization Theorem stated in section 3 , the family of diffeomorphisms φ_μ goes through a sequence of generic saddle-centre bifurcations $(\nu_n, Q_n) \approx (\mu_n(-1), \Psi_{n\,,-1}(-1, -1))$, where $(\nu_n, Q_n) \to (0, q)$. Denote by $I_n \approx \mu_n([-1, 3])$ a sequence of one sided neighbourhoods of ν_n. For each $\mu \in I_n$ let $Q_n(\mu)$ be the unique $n-$periodic point of φ_μ close to Q_n, which is hyperbolic for $\mu \neq \nu_n$. From Theorem 1 there is a map f_n near φ_{ν_n} and some basic set Λ_n containing the hyperbolic periodic point Q_n and such that the pair (f_n, Λ_n) has persistent homoclinic tangencies. The sequence (f_n, Λ_n) satisfies items 1.,2. and 3. but does not satisfy item 4. . To fulfil this last item we just need to enlarge the basic set Λ_n in order to contain the unique hyperbolic fixed point $P_n = f_n(P_n)$ near P. This is be possible since P_n and Q_n are homoclinically related. In the following, and last, lemma we prove this fact. $\square$

Lemma 7.1. *For each $\mu \in I_n$ and all sufficiently large $n \in \mathbb{N}$, the hyperbolic $n-$periodic point $Q_n(\mu)$ of φ_μ is homoclinically related with the unique hyperbolic fixed point P_μ of φ_μ near P, meaning that*

(1) *$W^s(P_\mu)$ intersects transversally $W^u(Q_n(\mu))$, and*
(2) *$W^u(P_\mu)$ intersects transversally $W^s(Q_n(\mu))$.*

FIGURE 2. Stable and unstable manifolds of the periodic point Q_n

Proof. This fact has a dissipative analogue which is used in the proof of Newhouse's Theorem to be found in [PT-93], c.f. proposition 1 of section 6.4. Roughly, we look at the open branches of $Q_n(\mu)$, i.e. the opposite branches of those involved in the basic set Λ_n, and see large pieces of them accumulating, as $n \to \infty$, on corresponding large arcs of the branches $\gamma^s(P_\mu)$ and $\gamma^u(P_\mu)$. Large here means covering many fundamental domains of the invariant manifolds. But $\gamma^s(P_\mu)$ and $\gamma^u(P_\mu)$ do intersect transversally and so the lemma follows. Because the proof is entirely analogous to that of proposition 6.4-1 in [PT-93] we just outline some of the differences here. The neighbourhood U of q and the C^1 invariant stable and unstable foliations $\mathcal{F}^s_\mu$, and $\mathcal{F}^u_\mu$ in a neighbourhood of the fixed point P are taken exactly as in [PT-93]. As there, the proof is worked in three different scales. The first and smaller scale is the one covered by the coordinate system $(\tilde{x}, \tilde{y}) = \Psi_{n,a}^{-1}$. The same picture depicted in figure 6.6 holds but now $\sigma = \lambda^{-1}$. The foliations $\mathcal{F}^s_\mu$ and $\mathcal{F}^u_\mu$, expressed in the $(\tilde{x}, \tilde{y})$ coordinates, converge as $n \to +\infty$ to the horizontal foliation and the foliation by parabolas $\{\tilde{y} = a - \tilde{x}^2 : a \in \mathbb{R}\}$. The convergence is uniform in the C^1 topology on compact parts of the $(\tilde{x}, \tilde{y})-$ plane. The explanation of these facts is simple. In the definition of the rescaling map $\Psi_{n,a}$ one first chooses coordinates (x, y) around P_μ in which φ_μ is almost linear, $\varphi_\mu(x, y) = (\lambda\, x + O(x^2\, y), \lambda^1 y + O(x\, y^2))$ c.f. lemma 2.3 in [MR1-97]. Then with respect to these coordinates $(\tilde{x}, \tilde{y}) = \Psi_{n,a}(x, y)$ is almost affine with almost diagonal linear part, and it maps a large region of the $(\tilde{x}, \tilde{y})-$ plane onto a microscopic rectangle located just above the point of homoclinic tangency in the xx axes. This should be understood as an asymptotic statement as $n \to +\infty$. Thus the stable foliation in the $(\tilde{x}, \tilde{y})-$coordinates converges to the horizontal foliation because its leaves in the $(x, y)-$coordinates accumulate on the horizontal axes. Since the iterates $\varphi^i_\mu(x, y)$, $i = 0, 1, \cdots, n$, of points (x, y) in the "microscopic" rectangular domain of $\Psi_{n,a}^{-1}$ stay very close to the union of the axes, where the nonlinearity of φ_μ is negligible, the iterates of both vertical lines, as well as of unstable leaves, accumulate on the yy axes. Thus the unstable leaves approach, in the $(\tilde{x}, \tilde{y})-$ coordinates, to the $\widehat{\Psi}_{n,a} = \Psi_{n,a}^{-1} \circ \varphi_{\mu_n(a)} \circ \Psi_{n,a}$ images of vertical lines, which in turn converge to the parabola foliation above, since $\widehat{\Psi}_{n,a} \to H_a$ as $n \to +\infty$.

Given $K > 0$ large, for all sufficiently large n, say $n \geq n(K)$, and all $\mu \in I_n$, there are compact arcs $\sigma^s_n(\mu) \subseteq W^s(Q_n(\mu))$ and $\sigma^u_n(\mu) \subseteq W^u(Q_n(\mu))$, see figure 2, and such that:

- these arcs contain at least K fundamental domains of φ^n_μ,
- the angles between leaves of $\mathcal{F}^s_\mu$ and $\sigma^u_n(\mu)$, respectively between leaves of $\mathcal{F}^u_\mu$ and $\sigma^s_n(\mu)$, are at least $\arctan K$, when measured in the $(\tilde{x}, \tilde{y})-$ coordinates,
- the angles between leaves of $\mathcal{F}^s_\mu$ and $\sigma^s_n(\mu)$, respectively between leaves of $\mathcal{F}^u_\mu$ and $\sigma^u_n(\mu)$, are at most $1/K$, when measured in the $(\tilde{x}, \tilde{y})-$ coordinates.

These facts are consequences of the following one which can be proved with a simple argument. The open separatrices of the fixed point $Q^h_a = H_a(Q^h_a)$, for any $a \geq -1$, are graphs $\gamma^u(Q^h_a) = \{(x, g_a(x)) : x < A_a\}$ and $\gamma^s(Q^h_a) = \{(g_a(x), x) : x < A_a\}$

where $A_a = -1 - \sqrt{1+a}$ and $g_a :]-\infty, A_a[\to \mathbb{R}$ is a smooth function satisfying,

$$\lim_{x \to -\infty} \frac{g_a(x)}{x^2} = -1 \quad \text{and} \quad \lim_{x \to -\infty} \frac{(g_a)'(x)}{2\,x} = -1\,,$$

and where the convergence is uniform for $a \in [-1, R]$ with $R > -1$.

The argument in the larger scales is now entirely similar to that of proposition 6.4-1 in [PT-93]. □

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Abondance des îles elliptiques aux bifurcations conservatives

Résumé: On démontre une version conservative d' un théorème de Newhouse sur l' abondance des ensembles hyperboliques sauvages: Arbitrairement proche d' une application qui conservent l' aire avec une tangence homoclinique il y a des ensembles hyperboliques sauvages avec persistence des tangences homocliniques. En outre, pour un ensemble résiduel des systèmes proches ces ensembles basiques sont accumulés par des îles elliptiques périodiques.