PLENTY OF ELLIPTIC ISLANDS FOR THE STANDARD FAMILY OF AREA PRESERVING MAPS

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ABSTRACT. For the Standard Map, a well-known family of conservative diffeomorphisms on the torus, we construct *large* basic sets which fill in the torus as the parameter runs to ∞. Then we prove that, for a residual set of large parameters, these basic sets are accumulated by elliptic periodic islands. We also show that there exists a $k_0 > 0$ and a dense set of parameters in $[k_0, \infty)$ for which the standard map exhibits homoclinic tangencies.

1. INTRODUCTION

For surface diffeomorphisms the unfolding of a homoclinic tangency is a fundamental mechanism to understand nonhyperbolic dynamics. Infinitely many coexisting sinks is one of the surprising phenomena which occur, for dissipative systems, every time a homoclinic tangency is generically unfolded. This remarkable fact is due to S. Newhouse: he proved that arbitrarily close to a surface diffeomorphism with a homoclinic tangency, there are residual subsets of open sets of diffeomorphisms whose maps have infinitely many sinks. J. Palis conjectured that the same should hold for conservative systems with elliptic islands playing the role of sinks. In the present work we verify this is true in the context of the standard map family and prove there are "plenty" of elliptic islands for a residual set of large parameters. We were motivated by Palis' conjecture and also by the work in progress of Carleson and Spencer, as well as by an earlier question of Sinai to Palis about this family. This family of diffeomorphisms on $\mathbb{T}^2$ is given by,

$$f_k(x, y) = (-y + 2x + k \sin(2\pi x), x) \; mod \; \mathbb{Z}^2.$$

The orbits (x_n, x_{n-1}) of f_k correspond to solutions of the difference equation $\Delta^2 x_n = x_{n+1} - 2x_n + x_{n-1} = k \sin(2\pi x_n)$, which is a discrete version of the pendulum equation $\ddot{x}(t) = K \sin(2\pi x(t))$. But only for small values of k is the dynamics of the standard map an approximation of the pendulum's phase flow. In fact while the pendulum is always integrable, for any K, the standard map is integrable for $k = 0$, meaning $\mathbb{T}^2$ is completely foliated by invariant KAM curves. However as k

grows, all these curves gradually break up and the orbit behavior becomes increasingly "chaotic". Simple computer experiments may lead to the conjecture that for large k, in a measure theoretical sense, most points have nonzero Liapounov exponents. Nevertheless this question is completely open. There is no single parameter value k for which it is known that *Pesin*'s region, of nonzero Liapounov exponents, has positive Lebesgue measure. Carleson and Spencer have a work in progress in this direction: they plan to prove this conjecture for parameter values where no elliptic points exist. They also conjecture that for a set of parameters with full density at ∞ (in a measure sense), there are no elliptic points. Our work does not contradict this conjecture, but it certainly shows how subtle this subject is. It is interesting to point out that Sinai's question to Palis, made several years ago, concerned the possible abundance of elliptic islands in line with our present work.

Notice that, since f_k is conjugated to f_{-k} via the translation $(x, y) \mapsto \left(x + \frac{1}{2}, y + \frac{1}{2}\right)$, we can restrict our attention to the parameter half line $k \in [0, +\infty)$. The following theorems synthesize our main results. We begin constructing a family of large basic sets for f_k.

Theorem A *There is a family of basic sets Λ_k of f_k, such that:*

(1) *Λ_k is dynamically increasing, meaning for small $\epsilon > 0$, $\Lambda_{k+\epsilon}$ contains the continuation of Λ_k at parameter $k + \epsilon$.*

(2) *The thickness of Λ_k grows to ∞. For all sufficiently large k,*

$$\tau^s_{loc}(\Lambda_k)\ ,\ \tau^u_{loc}(\Lambda_k) \geq \frac{k^{1/3}}{9}.$$

(3) *The Hausdorf Dimension of Λ_k increases up to 2. For large k,*

$$HD(\Lambda_k) \geq 2\frac{\log 2}{\log\left(2 + \frac{9}{k^{1/3}}\right)}\ .$$

(4) *Λ_k is conjugated to a full Bernoulli shift in $2n_k$ symbols, where*

$$\lim_{k\to\infty} \frac{2n_k}{4k} = 1$$

(5) *Λ_k fills in the Torus, meaning that as k goes to ∞ the maximum distance of any point in $\mathbb{T}^2$ to Λ_k tends to 0. For large k, $\mathbb{T}^2 = B_{\delta_k}(\Lambda_k)$, where $\delta_k = \frac{4}{k^{1/3}}$.*

Then for this family of basic sets Λ_k we prove:

Theorem B *There exists $k_0 > 0$ and a residual subset $R \subseteq [k_0, \infty)$ such that for $k \in R$ the closure of the f_k's elliptic periodic points contains Λ_k.*

Theorem C *There exists $k_0 > 0$ such that given any $k \geq k_0$ and any periodic point $P \in \Lambda_k$, the set of parameters $k' \geq k$ at which the invariant manifolds $W^s(P(k'))$ and $W^u(P(k'))$ generically unfold a quadratic homoclinic tangency is dense in $[k, +\infty)$. $P(k')$ denotes the continuation of the periodic saddle P at parameter k'.*

We do not claim to be original in Theorem A which is rather a description of the basic set family Λ_k mentioned in theorems B and C. These results are proved through sections 4 to 6. To finish this introduction we present brief ideas of the proofs of theorems A to C. Given any periodic function $\varphi : \mathbb{R} \to \mathbb{R}$ with period 1, $\varphi(x+1) = \varphi(x) + l$, $l \in \mathbb{Z}$,

$$(1) \qquad \begin{cases} x' &= -y + \varphi(x) \\ y' &= x \end{cases}$$

defines an invertible area preserving dynamical system on $\mathbb{T}^2$, for which the following *hyperbolicity criterion* holds: An invariant set Λ is uniformly hyperbolic whenever there exists some constant $\lambda > 2$ such that for all $(x, y) \in \Lambda$, $|\varphi'(x)| \geq \lambda$. This type of system includes the Standard Map Family where $\varphi_k(x) = 2x + k\sin(2\pi x)$. For this family the *critical region* $\{|\varphi_k'(x)| < \lambda\}$, for some fixed $\lambda > 2$, shrinks to a pair of circles $\{x = \pm\frac{1}{4}\}$ as $k \to \infty$. Thus for all large k the maximal invariant set

$$\Lambda_k = \bigcap_{n \in Z} f^{-n}\{(x,y) \in \mathbb{T}^2 : |\varphi_k'(x)| \geq \lambda\}$$

will be a "big" hyperbolic set. Theorem B follows from theorem C using a renormalization scheme, showing that arbitrarily close to a tangency parameter an elliptic point is created through the unfolding of a saddle-node bifurcation. In order to prove theorem C we use the following version of Newhouse's "gap" lemma: any pair of Cantor sets K^s, K^u in the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, such that the product of their thicknesses is $\tau(K^s)\tau(K^u) > 1$, must intersect $K^s \cap K^u \neq \emptyset$. We apply this lemma extending the stable and unstable manifolds of Λ_k to global transversal foliations $\mathcal{F}^s, \mathcal{F}^u$ of $\mathbb{T}^2$. Remark that these foliations will be f-invariant only if restricted to a small neighborhood of Λ_k. Using that the leaves of $\mathcal{F}^u$ are almost horizontal, when we push $\mathcal{F}^u$ by the diffeomorphism f, we get a new foliation $\mathcal{G}^u = (f_k)_*\mathcal{F}^u$ which folds along the circles $\{y = \pm\frac{1}{4}\}$, thus making two circles of tangencies with the almost vertical foliation $\mathcal{F}^s$, see Fig (1). The Cantor sets K^s, K^u are then the projections of Λ_k to one of these tangency circles along the foliations $\mathcal{F}^s$ and $\mathcal{G}^u$. For large k, $\tau(K^s)\tau(K^u) \gg 1$ and so there will be a tangency between leaves of $W^s(\Lambda_k)$ and $W^u(\Lambda_k)$. A major difficulty is to give rigorous estimates of the thickness $\tau(K^s)$ and

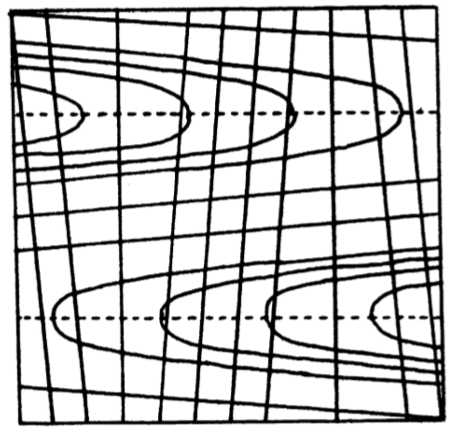

FIGURE 1. foliations $\mathcal{F}^s, \mathcal{G}^u$

$\tau(K^u)$, for which we must prove that the linear distortion of the one dimensional dynamical systems induced by the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ is bounded uniformly in k. To be able to do this we construct these globally defined foliations $\mathcal{F}^s$, $\mathcal{F}^u$ in the following way. We modify the function φ_k near its critical points into a new function ψ_k having a pole for each zero of φ_k' and such that $|\psi_k'(x)| \gg 2$. The new system (1) with ψ_k in place of φ_k is a singular area preserving diffeomorphism of $\mathbb{T}^2$. Although singular, it is hyperbolic in its maximal invariant domain, which has total measure, and most importantly it has smooth global invariant foliations.

Section 2 is dedicated to the construction of the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. In section 3 we estimate the linear distortion of the one dimensional dynamics induced by these foliations. Section 4 is used to construct the family of basic sets and prove theorem A. Theorems C and B are then respectively proved in sections 5 and 6.

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2. Global Foliations

In this section we study the differentiability of the invariant foliations for a class of *singular hyperbolic* diffeomorphisms on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

2.1. **Singular Hyperbolic Diffeomorphisms.** Let $\psi:\mathbb{R}\to\mathbb{R}\cup\{\infty\}$ be a smooth function satisfying:

(1) ψ is periodic, $\psi(x+1) = \psi(x) + l \quad (l \in \mathbb{Z})$,
(2) ψ has a finite number of poles (all of them with finite order) in each fundamental domain,
(3) For some $\lambda > 2$, $|\psi'(x)| \geq \lambda$.

Define $f : D \subseteq \mathbb{T}^2 \to \mathbb{T}^2$, $f(x,y) = (-y + \psi(x), x) \quad mod\ \mathbb{Z}^2$. The domain of f is the complement of a finite union of vertical circles, one for each pole of ψ, $D = \{(x,y)\ mod\ \mathbb{Z}^2 :\ \psi(x) \neq \infty\}$, which is diffeomorphically mapped onto $D' = \{(x,y)\ mod\ \mathbb{Z}^2 :\ \psi(y) \neq \infty\}$. We call such f a *singular diffeomorphism*.

Now, given a pair $\nu_1 < \nu_2$ of consecutive poles of ψ, the vertical cylinder $C = \left\{(x,y)\ mod\,\mathbb{Z}^2 |\ \nu_1 < x < \nu_2\right\}$ is mapped onto the horizontal one $C' = \left\{(x,y)\ mod\,\mathbb{Z}^2 |\ \nu_1 < y < \nu_2\right\}$ with both ends infinitely twisted in opposite directions. To understand how f acts on C notice it is the composition $f = T \circ R$ of a 90 degree rotation $R(x,y) = (-y,x)\ mod\,\mathbb{Z}^2$, with $T(x,y) = (x + \psi(y), y)\ \ mod\,\mathbb{Z}^2$, a singular map which rotates each horizontal circle $\{y = y_0\}$ by $\psi(y_0)$. A similar description is true about $f^{-1}(x,y) = (y, -x + \psi(y)) \quad mod\ \mathbb{Z}^2$, which decomposes as $f^{-1} = T' \circ R'$ where $R'(x,y) = (y,-x)\ mod\,\mathbb{Z}^2$ is a 90 degree rotation and $T'(x,y) = (x, y + \psi(x))\ mod\,\mathbb{Z}^2$ preserves vertical circles.

The singular diffeomorphism f preserves area since

$$Df_{(x,y)} = \begin{pmatrix} \psi'(x) & -1 \\ 1 & 0 \end{pmatrix}$$

has determinant 1. Notice that the maximal invariant set

$$D_\infty = \bigcap_{n\in\mathbb{Z}} f^{-n}(D)$$

has full measure in $\mathbb{T}^2$. We are going to see now that $f : D_\infty \to D_\infty$ is uniformly hyperbolic.

Proposition 1. *There are continuous functions* $\alpha^s, \alpha^u : \mathbb{T}^2 \to \mathbb{R}$ *such that:*

(1)
$$|\alpha^s(x,y)|,\ |\alpha^u(x,y)| \leq \frac{1}{\lambda - 1} < 1$$

(2)
$$\alpha^s(x,y) = \alpha^u(y,x)$$

(3)
$$Df^{-1}_{(x,y)}\,(\alpha^s(x,y),1) = \frac{1}{\alpha^s f^{-1}(x,y)}\,\big(\alpha^s f^{-1}(x,y),1\big) \quad \forall (x,y)\in D'$$

(4)
$$Df_{(x,y)}\,(1,\alpha^u(x,y)) = \frac{1}{\alpha^u f(x,y)}\,(1,\alpha^u f(x,y)) \qquad \forall (x,y)\in D$$

Conditions 3 and 4 state that the line fields generated by $(\alpha^s(x,y),1)$ and $(1,\alpha^u(x,y)$ are fixed under the actions of f^{-1} and f. The existence of such continuous invariant line fields can be proved applying the Contraction fixed point Theorem to the action of f^{-1}, or f, on the space $\mathcal{C}^0(\mathbb{T}^2,[-1,1])$. We remark that 3 and 4 are respectively equivalent to

$$\alpha^s(x,y) = \frac{1}{\psi'(x) - \alpha^s(f(x,y))}\,, \tag{1}$$

$$\alpha^u(x,y) = \frac{1}{\psi'(y) - \alpha^u(f^{-1}(x,y))}\,. \tag{2}$$

Knowing that α^s and α^u are continuous and bounded a priori by 1, these expressions give us 1. Symmetry 2 follows from the reversible character of f. Denote by $I:\mathbb{T}^2\to\mathbb{T}^2$ the linear involution $I(x,y) = (y,x)$. Then *reversibility* of f simply means that $f(I(x,y)) = I(f^{-1}(x,y))$.

Defining the continuous line fields:

$$E^s(x,y) = \text{ line spanned by the vector } \ (\alpha^s(x,y),1)$$
$$E^u(x,y) = \text{ line spanned by the vector } \ (1,\alpha^u(x,y))$$

we have the following obvious consequence:

Corollary 2. *For any* $(x,y)\in\mathbb{T}^2$, $\mathbb{R}^2 = E^s(x,y)\bigoplus E^u(x,y)$ *and this is an invariant hyperbolic splitting for* $f:D_\infty\to D_\infty$.

Denote by $\mathcal{F}^s$ and $\mathcal{F}^u$ the foliations associated to the continuous line fields E^s and E^u. The two invariant foliations have a finite number of closed leaves, one for each pole of ψ. Since they are symmetric with respect to the linear involution $I(x,y) = (y,x)$ we only describe $\mathcal{F}^s$. For each pole ν of ψ, since $\alpha^s(\nu, y) \equiv 0$, the vertical singular circle $\{x = \nu\}$ is a leaf of $\mathcal{F}^s$. On the other hand given a pair $\nu_1 < \nu_2$ of consecutive poles of ψ, the vertical cylinder $C = \left\{(x, y) \; mod \, \mathbb{Z}^2 | \; \nu_1 < x < \nu_2 \right\}$ is foliated by open leaves winding around it with their ends accumulating on the two opposite boundary circles. This is because $\alpha^s(x, y)$ is nonzero, thus with constant sign, inside C. Notice that

$$0 < \frac{\lambda}{\lambda+1} \leq \alpha^s(x,y)\psi'(x) = \left(1 - \frac{\alpha^s f(x,y)}{\psi'(x)}\right)^{-1} \leq \frac{\lambda}{\lambda-1} < \infty.$$

2.2. **Differentiability of Foliations.** To study the differentiability of α^u and α^s we introduce the Lie derivatives along the vector fields $(\alpha^s(x, y), 1)$ and $(1, \alpha^u(x, y))$:

$$(\partial_s h)(x,y) = Dh_{(x,y)}(\alpha^s(x,y), 1)$$

$$(\partial_u h)(x,y) = Dh_{(x,y)}(1, \alpha^u(x,y))$$

We are going to prove that:

(1) α^u, α^s are C^1 functions.
(2) $\partial_u \alpha^u$, $\partial_s \alpha^s$ are also C^1 functions. It follows that $\partial_s \alpha^u$ is continuously differentiable along the vector field $(1, \alpha^u(x, y))$ with

$$\partial_u \partial_s \alpha^u = \partial_s \partial_u \alpha^u + [\partial_s, \partial_u]\alpha^u,$$

$\partial_u \alpha^s$ is continuously differentiable along the vector field $(\alpha^s(x, y), 1)$ with

$$\partial_s \partial_u \alpha^s = \partial_u \partial_s \alpha^s + [\partial_u, \partial_s]\alpha^s.$$

(3) $\partial_s \alpha^u$ is Hőlder continuous along the vector field $(\alpha^s(x, y), 1)$, $\partial_u \alpha^s$ is Hőlder continuous along the vector field $(1, \alpha^u(x, y))$.

Most of the *differentiability'* statements above follow in the same way as in the general theory of invariant foliations for smooth hyperbolic dynamical systems. See [HP], see also [HPS]. The main point in redoing this theory for this specific class of singular hyperbolic diffeomorphisms is that we need to have explicit bounds for the derivatives and Hőlder constants mentioned above. These bounds depend on the function ψ, but we will show that indeed they only depend on the following two parameters: $\lambda > 2$, and $\ell > 0$, such that $\lambda \geq \ell > 0$

$$\left|\frac{1}{\psi'(x)}\right| \leq \frac{1}{\lambda}, \tag{3}$$

$$\left|\frac{\psi'''(x)}{\psi'(x)^2}\right| + 2\left|\frac{\psi''(x)^2}{\psi'(x)^3}\right| \leq \frac{1}{\ell}. \tag{4}$$

This bound $1/\ell$ exists because $\frac{\psi'''(x)}{\psi'(x)^2}$ and $\frac{\psi''(x)^2}{\psi'(x)^3}$ are bounded functions, as follows easily from the fact that $\frac{1}{\psi'(x)}$ is a periodic C^∞ function (without poles). Also it is straightforward to check that

$$\left|\left(\frac{1}{\psi'(x)}\right)'\right| = \left|\frac{\psi''(x)}{\psi'(x)^2}\right| \leq \frac{1}{\sqrt{2\ell\lambda}} \tag{5}$$

$$\left|\left(\frac{1}{\psi'(x)}\right)''\right| = \left|\frac{\psi'''(x)}{\psi'(x)^2} + 2\frac{\psi''(x)^2}{\psi'(x)^3}\right| \leq \frac{1}{\ell}\,. \tag{6}$$

Notice

$$\left|\frac{\psi''(x)}{\psi'(x)^2}\right|^2 = \left|\frac{\psi''(x)^2}{\psi'(x)^3}\right| \left|\frac{1}{\psi'(x)}\right| \leq \frac{1}{2\ell}\frac{1}{\lambda}.$$

Finally we will make the following commodious assumption: $\lambda \geq 10$. Although statements 1, 2 and 3 should be true for any $\lambda > 2$ this assumption of a stronger hyperbolicity forces a stronger contraction of the derivatives by the action of f on the space $\mathcal{C}^0(\mathbb{T}^2, [-1,1])$ which simplifies the calculations.

Proposition 3. *α^s, α^u are of class C^1, and for $\alpha = \alpha^s, \alpha^u$*

$$|\partial_s \alpha(x,y)| \leq \sqrt{\frac{2}{\lambda\ell}} \qquad |\partial_u \alpha(x,y)| \leq \sqrt{\frac{2}{\lambda\ell}}.$$

Proposition 4. *$\partial_u\alpha^u$ and $\partial_s\alpha^s$ are of class C^1 and*

$$|\partial_s\partial_u\alpha^u(x,y)| \leq \frac{2}{\ell} \qquad |\partial_u\partial_u\alpha^u(x,y)| \leq \frac{2}{\ell}\,,$$

$$|\partial_s\partial_s\alpha^s(x,y)| \leq \frac{2}{\ell} \qquad |\partial_u\partial_s\alpha^s(x,y)| \leq \frac{2}{\ell}\,.$$

Propositions (3) and (4) are proved in the spirit of [HPS], using the Fiber Contraction Theorem to get the existence and continuity of these derivatives of $\alpha^s(x,y)$ and $\alpha^u(x,y)$.

Lemma 1. *(Fiber Contraction Theorem)*
Let $\mathcal{X}$ be a topological space and $T_0 : \mathcal{X} \to \mathcal{X}$ a map having one globally attracting fixed point $\alpha_0 \in \mathcal{X}$. Let $\mathcal{Y}$ be a complete metric space and $T : \mathcal{X}\times\mathcal{Y} \to \mathcal{X}\times\mathcal{Y}$ be a continuous map of the form $T(\alpha,\beta) = (T_0(\alpha), T_1(\alpha,\beta))$ where for all $\alpha \in \mathcal{X}$, $T_1(\alpha,\cdot) : \mathcal{Y} \to \mathcal{Y}$ is a Lipschitz contraction with Lipschitz constant $0 < \mu < 1$ uniform in $\alpha \in \mathcal{X}$, that is

$$\forall \alpha \in \mathcal{X} \ \ Lip\,(T_1(\alpha,\cdot)) \leq \mu < 1$$

Then if β_0 *is the unique fixed point of* $\gamma \mapsto T_1(\alpha_0, \gamma)$, (α_0, β_0) *is a globally attracting fixed point for* T.

See [HP, S] for a proof of this lemma. By symmetry 2 of proposition (1) we can restrict ourselves to study α^u. For instance to prove proposition (3) take $\mathcal{X} = \mathcal{C}^0\left(\mathbb{T}^2, [-1,1]\right)$ acting as the space of "horizontal" line fields $(1, \alpha)$ with $\alpha \in \mathcal{X}$, take $\mathcal{Y} = \mathcal{C}^0\left(\mathbb{T}^2, [-1,1]^2\right)$ as a space containing the derivatives $(\partial_s \alpha, \partial_u \alpha)$ of C^1 functions $\alpha \in \mathcal{X}$ and let T describe the action of f on the derivatives $\partial_s \alpha$, $\partial_u \alpha$ of the C^1 line fields $(1, \alpha)$ with $\alpha \in \mathcal{X}$. Now iterating some $(\alpha, \partial_s \alpha, \partial_u \alpha) \in \mathcal{X} \times \mathcal{Y}$ we obtain a sequence $(\alpha_n, \partial_s \alpha_n, \partial_u \alpha_n) \in \mathcal{X} \times \mathcal{Y}$ converging uniformly to the unique attracting fixed point $(\alpha^u, \beta_s, \beta_u) \in \mathcal{X} \times \mathcal{Y}$ given by lemma (1). This proves α^u is of class C^1. Since the proofs are quite standard we leave the calculations to the reader. We just remark that differentiating (2) with respect to ∂_s, ∂_u, $\partial_s \partial_u$ and $\partial_u \partial_u$, and using the following notation,

$$\widehat{\alpha}(x,y) = \alpha(y, -x + \psi(y)) = \alpha\left(f^{-1}(x,y)\right),$$

we obtain the relations

$$\partial_s \alpha^u(x,y) = \frac{\frac{1}{\widehat{\alpha^s}(x,y)} \widehat{\partial_s \alpha^u}(x,y) - \psi''(y)}{\left(\psi'(y) - \widehat{\alpha^u}(x,y)\right)^2} =$$

$$= \frac{\left(1 - \frac{\alpha^s(x,y)}{\psi'(y)}\right)}{\left(1 - \frac{\widehat{\alpha^u}(x,y)}{\psi'(y)}\right)^2} \frac{\widehat{\partial_s \alpha^u}(x,y)}{\psi'(y)} + \left(\frac{1}{\psi'(y)}\right)' \frac{1}{\left(1 - \frac{\widehat{\alpha^u}(x,y)}{\psi'(y)}\right)^2},$$

$$\partial_u \alpha^u(x,y) = \frac{\widehat{\partial_u \alpha^u}(x,y) - \psi''(y)}{\left(\psi'(y) - \widehat{\alpha^u}(x,y)\right)^2} \alpha^u(x,y) =$$

$$= \frac{1}{\left(1 - \frac{\widehat{\alpha^u}(x,y)}{\psi'(y)}\right)^2} \frac{\widehat{\partial_u \alpha^u}(x,y)}{\psi'(y)^2} \alpha^u(x,y) + \left(\frac{1}{\psi'(y)}\right)' \frac{\alpha^u(x,y)}{\left(1 - \frac{\widehat{\alpha^u}(x,y)}{\psi'(y)}\right)^2}$$

$$\partial_s \partial_u \alpha^u(x,y) = \frac{\left(\frac{1}{\widehat{\alpha^s}} \widehat{\partial_s \partial_u \alpha^u} - \psi'''(y)\right) \alpha^u}{(\psi'(y) - \widehat{\alpha^u})^2} + \frac{\left(\widehat{\partial_u \alpha^u} - \psi''(y)\right) \partial_s \alpha^u}{(\psi'(y) - \widehat{\alpha^u})^2}$$
$$-2 \frac{\left(\frac{1}{\widehat{\alpha^s}} \widehat{\partial_s \alpha^u} - \psi''(y)\right)\left(\widehat{\partial_u \alpha^u} - \psi''(y)\right) \alpha^u}{(\psi'(y) - \widehat{\alpha^u})^3}$$

$$\partial_u\partial_u\alpha^u(x,y) = \frac{\left(\widehat{\partial_u\partial_u\alpha^u} - \psi'''(y)\right)\alpha^u(x,y)^2}{(\psi'(y)-\widehat{\alpha^u})^2} + \frac{\left(\widehat{\partial_u\alpha^u} - \psi''(y)\right)\partial_u\alpha^u}{(\psi'(y)-\widehat{\alpha^u})^2}$$
$$-2\frac{\left(\widehat{\partial_u\alpha^u} - \psi''(y)\right)^2\alpha^u(x,y)^2}{(\psi'(y)-\widehat{\alpha^u})^3}$$

Remark that by items 3 and 4 of proposition (1) we have

$$\partial_s\widehat{\alpha}(x,y) = \frac{1}{\widehat{\alpha^s}(x,y)}\widehat{\partial_s\alpha}(x,y) \qquad \partial_s\left[\psi'(y)\right] = \psi''(y)$$

$$\partial_u\widehat{\alpha}(x,y) = \alpha^u(x,y)\widehat{\partial_u\alpha}(x,y) \qquad \partial_u\left[\psi'(y)\right] = \alpha^u(x,y)\psi''(y)$$

Also from (1) it follows that

$$\frac{1}{\widehat{\alpha^s}(x,y)\psi'(y)} = 1 - \frac{\alpha^s(x,y)}{\psi'(y)}.$$

This last equality is used in the first and third relations above. Now from these equalities, knowing that all the derivatives involved exist and are a priori bounded by 1, it is easy to deduce the estimations stated in propositions (3) and (4).

Corollary 5. *$\partial_s\alpha^u$ is continuously differentiable along the vector field $(1,\alpha^u(x,y))$ and*

$$|\partial_u\partial_s\alpha^u(x,y)| \le \frac{3}{\ell}\,,$$

$\partial_u\alpha^s$ is continuously differentiable along the vector field $(\alpha^s(x,y),1)$ and

$$|\partial_s\partial_u\alpha^s(x,y)| \le \frac{3}{\ell}\,.$$

The statements of differentiability follow at once from proposition (4) and next lemma, whose proof is an easy exercise in Differential Geometry. Once again we will leave the calculations to the reader.

Lemma 2. *Let M be a manifold, $f:M\to\mathbb{R}$ a C^1 function and X, Y C^1 vector fields on M. If $\partial_X f$ is of class C^1 then $\partial_Y f$ is differentiable along X and*

$$\partial_X\partial_Y f = \partial_Y\partial_X f + \partial_{[Y,X]}f\,.$$

2.3. **Hőlder Continuity.** Let us give precise definitions of what we mean by Hőlder continuity of a function $\theta : \mathbb{T}^2 \to \mathbb{R}$ along the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. Given constants $0 < \gamma < 1$ and $C > 0$ we say θ is (C, γ)-Hőlder continuous along $\mathcal{F}^s$, respectively $\mathcal{F}^u$, if for (x, y) and (x', y') in the same leaf of $\mathcal{F}^s$, respectively $\mathcal{F}^u$, we have

$$|\theta(x, y) - \theta(x', y')| \leq C|y - y'|^\gamma,$$

$$|\theta(x, y) - \theta(x', y')| \leq C|x - x'|^\gamma \text{ respectively.}$$

Remark that if (x, y) and (x', y') belong to the same leaf of $\mathcal{F}^s$, resp. $\mathcal{F}^u$, then

$$|x - x'| \leq \frac{1}{\lambda - 1} |y - y'| \ , \ \text{ resp. } \ |y - y'| \leq \frac{1}{\lambda - 1} |x - x'| \, .$$

Now given any fixed $0 < \gamma < 1$ assume that λ is large enough so that $(\lambda + 1)^2 < (\lambda - 1)^{3-\gamma}$ and define

$$C = C(\lambda, \gamma) = \left(1 - \frac{(\lambda + 1)^2}{(\lambda - 1)^{3-\gamma}}\right)^{-1} . \tag{7}$$

Proposition 6. *$\partial_s \alpha^u$ is $\left(\frac{4}{\ell} C \, , \, \gamma\right)$-Hőlder continuous along $\mathcal{F}^s$, and $\partial_u \alpha^s$ is $\left(\frac{4}{\ell} C \, , \, \gamma\right)$-Hőlder continuous along $\mathcal{F}^u$.*

Because of the usual symmetry it is enough to study $\partial_s \alpha^u$ along $\mathcal{F}^s$. We have

$$\partial_s \alpha^u = \frac{\frac{1}{\widehat{\alpha^s}} \widehat{\partial_s \alpha^u} - \psi''(y)}{(\psi'(y) - \widehat{\alpha^u})^2}$$

Define $F : \mathbb{T}^2 \times [-1, 1] \to \mathbb{R}$

$$\begin{aligned} F(x, y, z) \quad &= \quad \frac{\frac{1}{\widehat{\alpha^s}} z - \psi''(y)}{(\psi'(y) - \widehat{\alpha^u})^2} \\ &= \quad \left(1 - \frac{\widehat{\alpha^u}}{\psi'(y)}\right)^{-2} \left\{ \frac{z}{\psi'(y)} \left(1 - \frac{\alpha^s}{\psi'(y)}\right) + \left(\frac{1}{\psi'(y)}\right)' \right\} \end{aligned}$$

Clearly F is a C^1 function. Then we can rewrite the above relation

$$\partial_s \alpha^u(x, y) = F\left(x, y, \partial_s \alpha^u f^{-1}(x, y)\right)$$

Stating matters in this form we see $\partial_s \alpha^u$ is an invariant section of the trivial fiber bundle $\mathbb{T}^2 \times [-1, 1]$ by the fiber preserving map $(x, y, z) \mapsto \big(f(x, y), F_{f(x,y)}(z)\big)$. Although the base map $f : \mathbb{T}^2 \to \mathbb{T}^2$ is singular we can adapt the usual proof of Hőlder continuity for the unique invariant section of F. See [S]. For this we need the following technical lemma.

Lemma 3. *The function F satisfies:*

(1) *For every pole y_0 of $\psi'(y)$,* $|\partial_s \alpha^u(x,y)| \leq \frac{2}{\ell}|y-y_0|$,

(2) $|\partial_s F| \leq \frac{4}{\ell}$,

(3) $\left|\frac{\partial F}{\partial z}(x,y,z)\right| |\psi'(y) - \alpha^s(x,y)| \leq \left(\frac{\lambda+1}{\lambda-1}\right)^2$.

Proof:
For the proof of item 1 just remark that from (5) and (6), using the mean value theorem, we have, for any pole y_0 of ψ

$$\left|\frac{1}{\psi'(y)}\right| \leq \frac{1}{\sqrt{2\lambda\ell}}|y-y_0|$$

$$\left|\left(\frac{1}{\psi'(y)}\right)'\right| \leq \frac{1}{\ell}|y-y_0|.$$

Item 2 is an easy boring calculation. Item 3 follows because

$$\left|\frac{\partial F}{\partial z}\right| |\psi'(y) - \alpha^s(x,y)| =$$

$$= \frac{\left|1-\frac{\alpha^s}{\psi'(y)}\right| \left|\frac{1}{\psi'(y)}\right| |\psi'(y)-\alpha^s(x,y)|}{\left(1-\frac{\widehat{\alpha^u}}{\psi'(y)}\right)^2} = \frac{\left(1-\frac{\alpha^s}{\psi'(y)}\right)^2}{\left(1-\frac{\widehat{\alpha^u}}{\psi'(y)}\right)^2}.$$

■

Proof of proposition (6)**:**
Let (x,y) and (x',y') be two points in the same leaf of $\mathcal{F}^s$. We will use the following notation: for $n \geq 0$,

$$(x_n, y_n) = f^{-n}(x,y)$$
$$(x'_n, y'_n) = f^{-n}(x',y')$$
$$\theta_n = \partial_s \alpha^u(x_n, y_n)$$
$$\theta'_n = \partial_s \alpha^u(x'_n, y'_n)$$

Let N be the least integer $n \geq 0$ such that the interval $[y_n, y'_n]$ contains a pole of ψ. Notice that while $[y_n, y'_n]$ contains no pole the difference $y_n - y'_n$ grows exponentially with n because f^{-1} expands the stable leaves. By the mean value theorem for each $n < N$ there is a point

(x_n^*, y_n^*), in the same leaf of $\mathcal{F}^s$ which contains (x_n, y_n) and (x_n', y_n'), such that

$$y_n^* \in [y_n, y_n']$$

$$\left|y_{n+1} - y_{n+1}'\right| = |\psi'(y_n^*) - \alpha^s(x_n^*, y_n^*)|\, |y_n - y_n'|$$

Thus writing, for $n < N$, $\lambda_n^* = |\psi'(y_n^*) - \alpha^s(x_n^*, y_n^*)|$ we have

$$1) \qquad |y_n - y_n'| = \left(\prod_{i=0}^{n-1} \lambda_i^*\right) |y - y'|$$

Now, abbreviating $a = \left(\frac{\lambda+1}{\lambda-1}\right)^2$, we will prove by induction that for $n \leq N$

$$2) \qquad |\theta_0 - \theta_0'| \leq \frac{4}{\ell} \sum_{k=0}^{n-1} \frac{a^k}{\prod_{i=0}^{k-1} \lambda_i^*} |y_k - y_k'|^\gamma \quad + \frac{a^n}{\prod_{i=0}^{n-1} \lambda_i^*} |\theta_n - \theta_n'|$$

Let $n = 0$. If $|y_0 - y_0'| \leq |y_0 - y_0'|^\gamma \leq 1$ then

$$\begin{aligned}
|\theta_0 - \theta_0'| &\leq \left|F_{(x_0,y_0)}\theta_1 - F_{(x_0',y_0')}\theta_1'\right| \leq \\
&\leq \left|F_{(x_0,y_0)}\theta_1 - F_{(x_0^*,y_0^*)}\theta_1\right| + \left|F_{(x_0^*,y_0^*)}\theta_1 - F_{(x_0^*,y_0^*)}\theta_1'\right| \\
&\quad + \left|F_{(x_0^*,y_0^*)}\theta_1' - F_{(x_0',y_0')}\theta_1'\right| \leq \\
&\leq \frac{4}{\ell}\{|y_0 - y_0^*| + |y_0^* - y_0'|\} + \left|\frac{\partial F}{\partial z}(x_0^*, y_0^*, z_0^*)\right| |\theta_1 - \theta_1'| \\
&\leq \frac{4}{\ell}|y_0 - y_0'|^\gamma + \frac{a}{\lambda_0^*}|\theta_1 - \theta_1'|,
\end{aligned}$$

otherwise it can be easily proved that

$$|\theta_0 - \theta_0'| \leq |\theta_0| + |\theta_0'| \leq \frac{4}{\ell} \leq \frac{4}{\ell}|y - y_0'|^\gamma.$$

Remark that $|y_0 - y_0'| = |y_0 - y_0^*| + |y_0^* - y_0'|$, because $y_0^* \in [y_0, y_0']$, and by item 3 of lemma (3),

$$\left|\frac{\partial F}{\partial z}(x_0^*, y_0^*, z_0^*)\right| \leq \frac{a}{\lambda_0^*}.$$

Other steps follow from item 2 of the same lemma. Now assume 2) holds for $n \leq N - 1$. The same argument we used above shows that

$$|\theta_n - \theta_n'| \leq \frac{4}{\ell}|y_n - y_n'|^\gamma + \frac{a}{\lambda_n^*}\left|\theta_{n+1} - \theta_{n+1}'\right|.$$

Then

$$\begin{aligned}|\theta_0-\theta_0'| &\leq \frac{4}{\ell}\sum_{k=0}^{n-1}\frac{a^k}{\prod_{i=0}^{k-1}\lambda_i^*}|y_k-y_k'|^\gamma + \\ &+\frac{a^n}{\prod_{i=0}^{n-1}\lambda_i^*}\left\{\frac{4}{\ell}|y_n-y_n'|^\gamma+\frac{a}{\lambda_n^*}\left|\theta_{n+1}-\theta_{n+1}'\right|\right\}\leq \\ &\leq \frac{4}{\ell}\sum_{k=0}^{n}\frac{a^k}{\prod_{i=0}^{k-1}\lambda_i^*}|y_k-y_k'|^\gamma + \frac{a^{n+1}}{\prod_{i=0}^{n}\lambda_i^*}\left|\theta_{n+1}-\theta_{n+1}'\right|\end{aligned}$$

proving that 2) also holds for $n+1$. From 2) we have

$$3) \qquad |\theta_0-\theta_0'|\leq\frac{4}{\ell}\sum_{k=0}^{N}\frac{a^k}{\prod_{i=0}^{k-1}\lambda_i^*}|y_k-y_k'|^\gamma$$

To see this choose a pole $y_N^*\in[y_N,y_N']$. By item 1 of lemma (3),

$$\begin{aligned}|\theta_N-\theta_N'| &\leq |\theta_N|+|\theta_N'| \\ &\leq \frac{2}{\ell}|y_N-y_N^*|+\frac{2}{\ell}|y_N^*-y_N'| \\ &= \frac{2}{\ell}|y_N-y_N'|\leq\frac{4}{\ell}|y_N-y_N'|\leq\frac{4}{\ell}|y_N-y_N'|^\gamma .\end{aligned}$$

The last inequality is clear if $|y_N-y_N'|\leq 1$. Otherwise, trivially,

$$|\theta_N-\theta_N'|\leq|\theta_N|+|\theta_N'|\leq\frac{4}{\ell}\leq\frac{4}{\ell}|y_N-y_N'|^\gamma$$

This proves 3). Thus using this inequality together with 1) we get

$$\begin{aligned}&|\partial_s\alpha^u(x,y)-\partial_s\alpha^u(x',y')| = |\theta_0-\theta_0'|\leq \\ &\leq \frac{4}{\ell}\sum_{k=0}^{N}\frac{a^k}{\prod_{i=0}^{k-1}\lambda_i^*}|y_k-y_k'|^\gamma \\ &\leq \frac{4}{\ell}\sum_{k=0}^{N}\frac{a^k}{\left(\prod_{i=0}^{k-1}\lambda_i^*\right)^{1-\gamma}}|y-y'|^\gamma \\ &\leq \frac{4}{\ell}\sum_{k=0}^{\infty}\left(\frac{a}{(\lambda-1)^{1-\gamma}}\right)^k|y-y'|^\gamma \\ &\leq \frac{4}{\ell}\frac{1}{1-\frac{a}{(\lambda-1)^{1-\gamma}}}|y-y'|^\gamma \qquad \blacksquare\end{aligned}$$

We finish this section by giving some estimations which will be needed in the next section. A straightforward calculation upon the estimatives of proposition (3) gives,

$$\left|\frac{\partial\alpha^s}{\partial x}\right|,\quad \left|\frac{\partial\alpha^s}{\partial y}\right|,\quad \left|\frac{\partial\alpha^u}{\partial x}\right|,\quad \left|\frac{\partial\alpha^u}{\partial y}\right| \le \frac{9}{8}\sqrt{\frac{2}{\lambda\ell}}. \tag{8}$$

Another important Hőlder continuity is, under the same assumptions and constants of proposition (6),

$$\left|\frac{\partial\alpha^s}{\partial x}(x,y) - \frac{\partial\alpha^s}{\partial x}(x',y)\right| \le \frac{6}{\ell}C(\lambda,\gamma)\,|x-x'|^\gamma . \tag{9}$$

To prove this write $\frac{\partial\alpha^s}{\partial x}$ in terms of the derivatives $\partial_u\alpha^s$ and $\partial_s\alpha^s$. Then it is enough to prove for these two that

(1)
$$|\partial_u\alpha^s(x,y) - \partial_u\alpha^s(x',y)| \le \frac{5}{\ell}C(\lambda,\ell)\,|x-x'|^\gamma ,$$

(2)
$$|\partial_s\alpha^s(x,y) - \partial_s\alpha^s(x',y)| \le \frac{3}{\ell}\,|x-x'| .$$

To prove 1 let (x,y), (x',y) be points in $\mathbb{R}^2$ such that $|x-x'|<1$ and take (x^*,y^*) to be the unique intersection of the unstable leaf by (x,y) with the stable one by (x',y). Because (x,y) and (x^*,y^*) are on the same unstable leaf we have $|y-y^*| \le \dfrac{1}{\lambda-1}\,|x-x^*|$. Because (x^*,y^*) and (x',y) are on the same stable leaf we have $|x^*-x'| \le \dfrac{1}{\lambda-1}\,|y-y^*|$. Thus

$$|x-x^*| \le |x-x'| + |x'-x^*| \le |x-x'| + \frac{1}{(\lambda-1)^2}\,|x-x^*|\,,$$

$$|x-x^*| \le |x-x'|\left(1-\frac{1}{(\lambda-1)^2}\right)^{-1} \le \frac{(\lambda-1)^2}{\lambda(\lambda-2)}\,|x-x'|\,,$$

$$|y-y^*| \le \frac{\lambda-1}{\lambda(\lambda-2)}\,|x-x'|\,,$$

and so

$$\begin{aligned}
&|\partial_u\alpha^s(x,y) - \partial_u\alpha^s(x',y)| \le \\
&\le\ |\partial_u\alpha^s(x,y) - \partial_u\alpha^s(x^*,y^*)| + |\partial_u\alpha^s(x^*,y^*) - \partial_u\alpha^s(x',y)| \le \\
&\le\ \frac{4}{\ell}C\,|x-x^*|^\gamma + |\partial_s\partial_u\alpha^s|\,|y^*-y| \\
&\le\ \frac{4}{\ell}\frac{(\lambda-1)^2}{\lambda(\lambda-2)}C\,|x-x'|^\gamma + \frac{3}{\ell}\frac{\lambda-1}{\lambda(\lambda-2)}\,|x-x'| \le \frac{5}{\ell}C\,|x-x'|^\gamma .
\end{aligned}$$

To prove 2 we choose (x^*, y^*) in the same way. Then

$$
\begin{aligned}
&|\partial_s \alpha^s(x,y) - \partial_s \alpha^s(x',y)| \leq \\
&\quad \leq \ |\partial_s \alpha^s(x,y) - \partial_s \alpha^s(x^*,y^*)| + |\partial_s \alpha^s(x^*,y^*) - \partial_s \alpha^s(x',y)| \leq \\
&\quad \leq \ |\partial_u \partial_s \alpha^s|\,|x - x^*| + |\partial_s \partial_s \alpha^s|\,|y^* - y| \\
&\quad \leq \ \frac{2}{\ell} \left\{ \frac{(\lambda-1)^2}{\lambda(\lambda-2)} + \frac{\lambda-1}{\lambda(\lambda-2)} \right\} |x - x'| \leq \frac{3}{\ell}\,|x - x'|\,.
\end{aligned}
$$

3. Bounded Distortion

In this section we define the one dimensional dynamics on the circle SS^1 induced by the invariant foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. These dynamics are given by singular expansive maps $\Psi_s, \Psi_u : SS^1 \to SS^1$. The reversible character of f implies $\Psi_s = \Psi_u$ which we will simply call Ψ. This map lifts to a C^1 periodic function $\Psi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ having the same poles as ψ. In fact if λ is large Ψ is close to ψ. Our main goal here will be to prove a *modulus of Hőlder continuity* for the map $\log|\Psi'|$ and to deduce from it a bound for the linear distortion of Ψ which will depend only on the two parameters λ and ℓ. Finally we use the bound on the distortion to estimate the thickness of a given compact Ψ-invariant Cantor set containing no poles of Ψ and defined by some Markov Partition, in terms of the ratios between intervals and gaps of this Markov Partition.

3.1. **The map Ψ.** Consider the singular circles $C_s = \{(x,0)\, mod\, \mathbb{Z}^2 |\, x \in \mathbb{R}\}$ and $C_u = \{(0,y)\, mod\, \mathbb{Z}^2 \mid y \in \mathbb{R}\}$ respectively transversal to the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. We are assuming, where there is no loss of generality, that 0 is a pole of ψ. Now $\mathcal{F}^s$ induces on the cylinder $\mathbb{T}^2 - C_s$ a trivial fibration $\pi_s : \mathbb{T}^2 - C_s \to SS^1 \equiv C_s$ whose fibers are the connected components of the leaves of $\mathcal{F}^s$ in the cylinder $\mathbb{T}^2 - C_s$. This fibration is invariant by the action of f. To see this, use the factorization $f^{-1} = T' \circ R'$ described in section 2.1. We see at once that any given fiber $\pi_s^{-1}(x) \subseteq \mathbb{T}^2 - C_s$ when mapped by f^{-1} splits onto a finite number of complete leaves of $\mathcal{F}^s$ in $\mathbb{T}^2$, as many as the number of ψ's poles. See Fig (2). Thus the f image of every complete leaf of $\mathcal{F}^s$ in $\mathbb{T}^2$ is a piece of some fiber of π_s bounded between two horizontal consecutive singular circles. Also $\mathcal{F}^u$ induces on $\mathbb{T}^2 - C_u$ a trivial fibration $\pi_u : \mathbb{T}^2 - C_u \to SS^1 \equiv C_u$ which is invariant by the action of f^{-1}. In both cases we have natural dynamical systems describing the action of f and f^{-1} on the fibrations $\pi_s : \mathbb{T}^2 - C_s \to SS^1$ and $\pi_u : \mathbb{T}^2 - C_u \to SS^1$. These are the *singular*

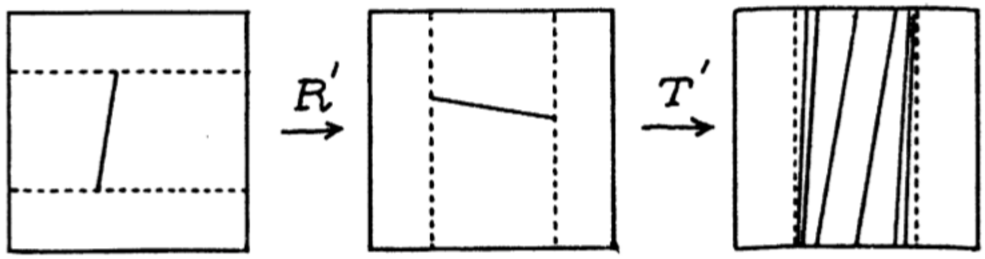

FIGURE 2. Action of f^{-1} on $\mathbb{T}^2 - C_s$

expansive maps $\Psi_s, \Psi_u : SS^1 \to SS^1$:

$$\Psi_s(x) = \pi_s(f(x,0)) = \pi_s(\psi(x), x)$$

$$\Psi_u(y) = \pi_u(f^{-1}(0,y)) = \pi_u(y, \psi(y))$$

The reversibility of f will imply that $\Psi_s = \Psi_u$, which we simply denote by Ψ. Using the above expressions for Ψ, we see that each interval I, bounded by consecutive poles of ψ, is expanded by Ψ onto SS^1 winding infinitely many times around it. In fact the restriction map $\Psi_I : I \to SS^1$ is an infinitely branched covering space of SS^1, the sign of $\psi'(x)$ in I giving the orientation character of Ψ_I. Over its maximal invariant domain,

$$\Delta_\infty = \bigcap_{n \geq 0} \Psi^{-n}(D) \tag{10}$$

where $D = \{x : \psi(x) \neq \infty\}$, the map $\Psi : \Delta_\infty \to \Delta_\infty$ is conjugated to a full shift in infinitely many symbols. Let m be the number of ψ's poles in each fundamental domain and denote by $I_1, \cdots, I_m \subseteq (0,1)$ all the connected components of $(0,1)-$poles of ψ. Then, since $\Psi_{I_i} : I_i \to SS^1$ is a covering space, $\Psi_{I_i}^{-1}(0,1)$ is a doubly infinite sequence of subintervals of I_i which we denote by $\cdots, I_{-1\,i}, I_{0\,i}, I_{+1\,i}, \cdots$. The set of all these subintervals $I_{l\,i}$, with $l \in \mathbb{Z}$ and $1 \leq i \leq m$, forms a Markov Partion for Ψ. Thus $\Psi : \Delta_\infty \to \Delta_\infty$ is conjugated to the full one sided shift in the infinite alphabet $\mathcal{A} = \mathbb{Z} \times \{1, \cdots m\}$.

We give now a precise definition of natural liftings for these projections. Let $g_s, g_u : \mathbb{R}^2 \to \mathbb{R}$ be the C^1 functions whose graphs $\{(g_s(x,y), y)\}$ and $\{(x, g_u(x,y))\}$ are liftings of leaves of the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. They can be defined by

$$1) \qquad \begin{cases} g_s(x,0) = x \\ \dfrac{\partial g_s}{\partial y}(x,y) = \alpha^s\,(g_s(x,y), y) \end{cases}$$

and

$$2) \qquad \begin{cases} g_u(0,y) = y \\ \dfrac{\partial g_u}{\partial x}(x,y) = \alpha^u\left(x, g_u(x,y)\right) \end{cases}$$

Then π_s and π_u are defined implicitly by

$$3) \qquad \begin{cases} g_s\left(\pi_s(x,y), y + k(y)\right) = x \\ g_u\left(x + k(x), \pi_u(x,y)\right) = y, \end{cases}$$

where $k(x) \in \mathbb{Z}$ is the only integer such that $0 \leq x + k(x) < 1$. Notice that 3) is equivalent to $(\pi_s(x,y), 0)$ and $(x, y + k(y))$ belonging to the same leaf of $\mathcal{F}^s$, and $(0, \pi_u(x,y))$, $(x + k(x), y)$ belonging to the same leaf of $\mathcal{F}^u$. From the definitions 1) and 2) and the symmetry $\alpha^s(x,y) = \alpha^u(x,y)$ it follows easily that

$$g_s(x,y) = g_u(y,x).$$

Then from the definition 3) we get

$$\pi_s(x,y) = \pi_u(y,x). \tag{11}$$

The projections $\pi_s, \pi_u : \mathbb{R}^2 \to \mathbb{R}$ are respectively discontinuous along the horizontal lines $\{\, y = k \,\}$ $(k \in \mathbb{Z})$ and the vertical ones $\{\, x = k \,\}$ $(k \in \mathbb{Z})$, and everywhere else of class C^1. Also they are periodic with period 1 in both variables:

$$\pi_s(x+1,y) = \pi_s(x,y) + 1$$
$$\pi_u(x,y+1) = \pi_u(x,y) + 1$$
$$\pi_s(x,y+1) = \pi_s(x,y)$$
$$\pi_u(x+1,y) = \pi_u(x,y).$$

as follows from the periodicity of the functions g_s and g_u:

$$g_s(x+1,y) = g_s(x,y) + 1$$
$$g_u(x,y+1) = g_u(x,y) + 1.$$

Both sides of these relations solve the same Cauchy problem. We still have to prove that π_s and π_u are well defined. By symmetry we may stick to π_s. We can prove the following relation, again by checking that both sides are solutions of the same Cauchy problem,

$$\frac{\partial g_s}{\partial x}(x,y) = \exp\left\{ \int_0^y \frac{\partial \alpha^s}{\partial x}\left(g_s(x,t), t\right)\, dt \right\} \neq 0. \tag{12}$$

Thus, by the Implicit Function Theorem, π_s is well defined. Then we define $\Psi_s, \Psi_u : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ putting

$$4) \qquad \begin{cases} \Psi_s(x) = \pi_s(f(x,0)) = \pi_s(\psi(x), x) \\ \Psi_u(y) = \pi_u(f^{-1}(0,y)) = \pi_u(y, \psi(y)) \end{cases}$$

Symmetry (11) implies that $\Psi_s = \Psi_u$, which we simply denote by Ψ. It is a C^1 function outside the poles of ψ. By the periodicity of the projections and also that of ψ it is clear that the function Ψ is periodic with period 1.

$$\Psi(x+1) = \Psi(x) + l$$

From definitions 3) and 4) it follows at once that for $0 \leq x < 1$,

$$\begin{aligned} g_s\left(\Psi(x), x\right) &= \psi(x) \\ g_u\left(y, \Psi(y)\right) &= \psi(y). \end{aligned} \tag{13}$$

These relations show us how close Ψ is to ψ. For large λ, the leaves of $\mathcal{F}^s$ are almost vertical because $|\alpha^s| \leq \frac{1}{\lambda-1}$. Thus $\psi(x) = g_s\left(\Psi(x), x\right)$ is close to $\Psi(x)$.

Proposition 7.

(1) $|\Psi(x) - \psi(x)| \leq \dfrac{1}{\lambda - 1}$

(2) $\Psi'(x) = \dfrac{\psi'(x) - \alpha^s\left(\psi(x), x\right)}{\exp\left\{\int_0^x \frac{\partial \alpha^s}{\partial x}\left(g_s(x,t), t\right)\, dt\right\}}$

Proof:

$$\begin{aligned} |\Psi(x) - \psi(x)| &= |\Psi(x) - g_s\left(\Psi(x), x\right)| = |g_s\left(\Psi(x), 0\right) - g_s\left(\Psi(x), x\right)| \\ &\leq \int_0^x |\alpha^s\left(g_s(\Psi(x), t), t\right)|\, dt \leq |x|\, |\alpha^s| \leq \frac{1}{\lambda - 1} \end{aligned}$$

Differentiating the relation $\psi(x) = g_s\left(\Psi(x), x\right)$, by (12) we get

$$\begin{aligned} \psi'(x) &= \frac{\partial g_s}{\partial x}\left(\Psi(x), x\right) \Psi'(x) + \frac{\partial g_s}{\partial y}\left(\Psi(x), x\right) \\ &= \frac{\partial g_s}{\partial x}\left(\Psi(x), x\right) \Psi'(x) + \alpha^s\left(g_s(\Psi(x), x), x\right) \\ &= \Psi'(x) \exp\left\{\int_0^x \frac{\partial \alpha^s}{\partial x}\left(g_s(\Psi(x), t), t\right)\, dt\right\} + \alpha^s\left(\psi(x), x\right) \end{aligned}$$

■

By item 2 of proposition (7) setting

$$\mu = \exp\left\{\frac{9}{8}\sqrt{\frac{2}{\lambda \ell}}\right\} \tag{14}$$

we have, recall (8), $|\Psi'(x)| \geq \dfrac{\lambda - 1}{\mu}$.

Finally, it is geometrically clear that the projections π_s and π_u semi-conjugate f resp. f^{-1} with the expansive map Ψ, that is $\pi_s \circ f = \Psi \circ \pi_s$ and $\pi_u \circ f^{-1} = \Psi \circ \pi_u$.

3.2. **Distortion Estimates.** We prove a *modulus of Hőlder continuity* for the function $\log |\Psi'(x)|$, which is the main tool to get the boundness of Ψ's linear distortion. Assume that $0 < \gamma < 1$ is fixed and $\lambda > 0$ is large enough so that

$$(\lambda + 1)^2 < (\lambda - 1)^{3-\gamma}, \quad \text{and} \quad \mu < \lambda - 1.$$

See (14) for the definition of μ. Then set

$$C_0 = C_0(\lambda, \ell, \gamma) = \frac{8\mu}{\ell} C(\lambda, \gamma). \tag{15}$$

$$C_1 = C_1(\lambda, \ell, \gamma) = \frac{\mu^\gamma}{(\lambda - 1)^\gamma - \mu^\gamma} C_0(\lambda, \ell, \gamma) \tag{16}$$

where $C(\lambda, \gamma)$ was defined (7).

Lemma 4. *If* $[x, y] \subseteq \mathbb{R}$ *contains no pole of* ψ *and* $|\Psi(x) - \Psi(y)| \leq 1$ *then*

$$|\log |\Psi'(x)| - \log |\Psi'(y)|| \leq C_0 \, |\Psi(x) - \Psi(y)|^\gamma$$

Proposition 8. *Bounded Distortion Property*
Given $x, y \in \mathbb{R}$, *if for* $i = 0, 1, \cdots, n-1$

(1) $[\Psi^i(x), \Psi^i(y)]$ *contains no pole of* ψ, *and*
(2) $|\Psi^n(x) - \Psi^n(y)| \leq 1$, *then*

$$\exp\left\{-C_1 \, |\Psi^n(x) - \Psi^n(y)|^\gamma\right\} \leq \frac{\left|(\Psi^n)'(x)\right|}{\left|(\Psi^n)'(y)\right|} \leq \exp\left\{C_1 \, |\Psi^n(x) - \Psi^n(y)|^\gamma\right\}$$

Proof:

$$
\begin{aligned}
\left|\log\left|(\Psi^n)'(x)\right| - \log\left|(\Psi^n)'(y)\right|\right| &= \left|\sum_{i=0}^{n-1} \log\left|\Psi'(\Psi^i(x))\right| - \log\left|\Psi'(\Psi^i(y))\right|\right| \\
&\leq \sum_{i=0}^{n-1} \left|\log\left|\Psi'(\Psi^i(x))\right| - \log\left|\Psi'(\Psi^i(y))\right|\right| \\
&\leq \sum_{i=0}^{n-1} C_0 \left|\Psi(\Psi^i(x)) - \Psi(\Psi^i(y))\right|^\gamma \\
&= C_0 \sum_{i=1}^{n} \left|\Psi^i(x) - \Psi^i(y)\right|^\gamma \\
&\leq C_0 \sum_{i=1}^{n} \left(\frac{\mu}{\lambda-1}\right)^{\gamma(n-i)} |\Psi^n(x) - \Psi^n(y)|^\gamma \\
&\leq C_0 \frac{\mu^\gamma}{(\lambda-1)^\gamma - \mu^\gamma} |\Psi^n(x) - \Psi^n(y)|^\gamma .
\end{aligned}
$$

Remark that

$$
|\Psi^n(x) - \Psi^n(y)| \geq \left(\frac{\lambda-1}{\mu}\right)^{n-i} \left|\Psi^i(x) - \Psi^i(y)\right| .
$$

■

Proof of lemma 4:
Consider the expression for $\Psi'(x)$ given on item 2 of proposition (7). Taking logarithms we have

$$
\log|\Psi'(x)| = \log|\psi'(x)| + \log\left(1 - \frac{\alpha^s(\psi(x),x))}{\psi'(x)}\right) - \int_0^x \frac{\partial\alpha^s}{\partial x}\left(g_s(\Psi(x),t),t\right)\,dt.
$$

Thus

$$
|\log|\Psi'(x)| - \log|\Psi'(y)|| \leq \Delta_0 + \Delta_1 + \Delta_2
$$

where

$$
\Delta_0 = |\log|\psi'(x)| - \log|\psi'(y)||
$$

$$
\Delta_1 = \left|\log\left(1 - \frac{\alpha^s(\psi(x),x))}{\psi'(x)}\right) - \log\left(1 - \frac{\alpha^s(\psi(y),y))}{\psi'(y)}\right)\right|
$$

$$
\Delta_2 = \left|\int_0^x \frac{\partial\alpha^s}{\partial x}\left(g_s(\Psi(x),t),t\right)\,dt - \int_0^y \frac{\partial\alpha^s}{\partial x}\left(g_s(\Psi(y),t),t\right)\,dt\right|
$$

From the estimative (8) and the Hőlder continuity relation (9) one can easily conclude that

$$\Delta_2 \leq \frac{6.5\,\mu}{\ell} C \left|\Psi(x)-\Psi(y)\right|^\gamma .$$

To estimate Δ_1 remark $\log\left(1-\frac{\alpha^s(\psi(x),x)}{\psi'(x)}\right)$ is of class C^1. A simple computation shows that this function has derivatives smaller than $\frac{2}{\ell}$. Thus

$$\Delta_1 \leq \frac{2}{\ell}\left|x-y\right| \leq \frac{2\,\mu}{(\lambda-1)\ell}\left|\Psi(x)-\Psi(y)\right| .$$

Now suppose that the interval $[x,y]$ does not contain any pole of ψ. Let $z_t = x + t(y-x)$ for $t\in[0,1]$. By the Mean Value Theorem,

$$\begin{aligned} \left|\psi(x)-\psi(y)\right| &= \left|\int_0^1 \psi'(z_t)\,dt\right| \left|x-y\right| \\ &= \int_0^1 \left|\psi'(z_t)\right|\,dt\,\left|x-y\right| . \end{aligned}$$

Notice that, as ψ has no poles inside $[x,y]$, the sign of $\psi'(z_t)$ keeps unchanged for $t\in[0,1]$. Again by the Mean Value Theorem, using (5),

$$\begin{aligned} &\left|\log\left|\psi'(x)\right| - \log\left|\psi'(y)\right|\right| \leq \int_0^1 \left|\frac{\psi''(z_t)}{\psi'(z_t)}\right|\,dt\,\left|x-y\right| \\ &\leq \int_0^1 \frac{1}{\sqrt{2\lambda\ell}}\left|\psi'(z_t)\right|\,dt\,\left|x-y\right| = \frac{1}{\sqrt{2\lambda\ell}}\left|\psi(x)-\psi(y)\right| . \end{aligned}$$

On the other hand,

$$\begin{aligned} &\left|\psi(x)-\psi(y)\right| \leq \left|g_s(\Psi(x),x) - g_s(\Psi(y),y)\right| \\ &\leq \left|\frac{\partial g_s}{\partial x}\right| \left|\Psi(x)-\Psi(y)\right| + \left|\frac{\partial g_s}{\partial y}\right| \left|x-y\right| \\ &\leq \left\{\mu + \frac{\mu}{(\lambda-1)^2}\right\} \left|\Psi(x)-\Psi(y)\right| \leq \frac{\lambda^2}{(\lambda-1)^2}\,\mu\left|\Psi(x)-\Psi(y)\right| , \end{aligned}$$

because $\left|\frac{\partial g_s}{\partial x}\right| \leq \mu$, see (12), and by definition of g_s, $\left|\frac{\partial g_s}{\partial y}\right| \leq \left|\alpha^s\right| \leq \frac{1}{\lambda-1}$. Thus

$$\Delta_0 \leq \frac{\mu}{\ell} C \left|\Psi(x)-\Psi(y)\right| .$$

Adding all these inequalities we prove the lemma. ■

3.3. **Invariant Cantor Sets and Thickness Estimates.** Let K be a closed subset of SS^1 or $\mathbb{R}$. The *thickness* of K can be defined as follows. See [N 3] and also [PT]. Any bounded component of the complement of K, $SS^1 - K$ or $\mathbb{R} - K$, will be called a *gap* of K. For every triple (U_1, C, U_2) formed by a pair of gaps U_1, U_2 and a bounded component C of $SS^1 - (U_1 \cup U_2)$ resp. $\mathbb{R} - (U_1 \cup U_2)$ we define

$$\tau\left(U_1, C, U_2\right) = \max\left\{\frac{|C|}{|U_1|}, \frac{|C|}{|U_2|}\right\},$$

where $|U|$ denotes the length of U. Then the thickness of K is the infimum

$$\tau(K) = \inf \tau\left(U_1, C, U_2\right)$$

taken over all possible triples (U_1, C, U_2).

Suppose now we are given a Ψ-invariant Cantor set $K \subseteq SS^1$ defined as the maximal invariant set,

$$K = \bigcap_{n \geq 0} \Psi^{-n}\left(\cup_{i=1}^{m} I_i\right)$$

over a finite disjoint union of closed intervals $I_1 \dot{\cup} I_2 \dot{\cup} \cdots \dot{\cup} I_m$ containing no poles of ψ. Further more we will assume that, $\mathcal{P} = \{I_1, I_2, \cdots, I_m\}$ is a *Markov Partition* for $\Psi : K \to K$. Our goal here is to give an estimation for the thickness $\tau(K)$ in terms of the easily computable thickness $\tau(\mathcal{P})$ of the Markov Partition $\mathcal{P}$, which we define to be the minimum,

$$\tau(\mathcal{P}) = \min \frac{|I_i|}{|U|}$$

taken over all $I_i \in \mathcal{P}$ and over the gaps U of $\mathcal{P}$ adjacent to I_i, where a gap of $\mathcal{P}$ simply means a gap of $\bigcup_{i=1}^{n} I_i$ in SS^1. Now under the same assumptions of proposition (8) which states the *Bounded Distortion Property* the following estimation holds.

Proposition 9.

$$\tau(K) \geq e^{-C_1} \tau(\mathcal{P})$$

We now make precise our assumptions on $\mathcal{P}$. Lift the Markov Partition $\mathcal{P}$ to $\mathbb{R}$, the universal covering of SS^1. We obtain a countable disjoint union of intervals. These intervals will still be said intervals of $\mathcal{P}$. Also we keep calling gaps of $\mathcal{P}$ to gaps of this countable union. With this terminology we assume:

(1) The gaps of $\mathcal{P}$ contain all the poles of ψ.

(2) For each interval I of $\mathcal{P}$, $\Psi(I)$ is the convex hull of a finite union of intervals of $\mathcal{P}$, covering at least one fundamental domain of SS^1.

It follows from 2 that the set $\partial\mathcal{P}$, of all boundary points of the intervals $I_1, I_2, \cdots I_m$ in $\mathcal{P}$, is invariant by $\Psi : SS^1 \to SS^1$. Thus it consists of periodic and preperiodic orbits of Ψ.

The proof runs as follows.

Proof:
We begin with some notations, comments and definitions which will be very useful. Denote by $\mathcal{G}$ the set of all gaps of K. Then define *order of a gap*. The gaps of $\mathcal{P}$ will be said to have order 0. We denote by $\mathcal{G}_0$ the set of all these gaps. Now remark that as these gaps contain all poles of ψ the restriction of Ψ to any interval which intersects no gap of order 0 is an expansive diffeomorphism. Thus, by invariance of K, if $U \in \mathcal{G}$ is not of order 0 then $\Psi(U)$ is another and longer gap of K. If $U \in \mathcal{G} - \mathcal{G}_0$ and $\Psi(U) \in \mathcal{G}_0$ we say U is a gap of order 1. $\mathcal{G}_1$ will denote the set of all gaps with order 1. Notice that Ψ^2 is an expansive diffeomorphism over any interval which intersects no gap of order ≤ 1. By induction we define the set of all gaps of order n, $\mathcal{G}_n$, as consisting of those gaps $U \notin \mathcal{G}_0$ such that $\Psi(U)$ is of order $n-1$. Again by induction we can check that for $U \in \mathcal{G}_n$, the restriction of Ψ^{n+1} to an interval intersecting no gaps of order $\leq n$ is an expansive diffeomorphism. As Ψ expands all gaps most have finite order. Thus $\mathcal{G}$ is the disjoint union

$$\mathcal{G} = \bigcup_{n=0}^{\infty} \mathcal{G}_n.$$

Let now (U_1, C, U_2) be triple formed by a pair of gaps $U_1 \in \mathcal{G}_n$, $U_2 \in \mathcal{G}_m$ and the bounded component C of $\mathbb{R} - (U_1 \cup U_2)$. We have to prove that

$$\tau\left(U_1, C, U_2\right) \geq e^{-C_1}\tau(\mathcal{P}).$$

Suppose $n \geq m$. If inside C there are gaps of order $\leq n$ we choose among them U_2' to be the one which is closer to U_1. Otherwise simply define $U_2' = U_2$. Consider the new triple (U_1, C', U_2'), where C' is the bounded component of $\mathbb{R} - (U_1 \cup U_2')$. Now $C' \subseteq C$ and C' contains no gaps of order $\leq n$. Since C' is bounded by gaps of order $\leq n$, $\Psi^n(C')$ is bounded by points in $\partial\mathcal{P}$ and this proves it is an interval of $\mathcal{P}$. Also $\Psi^n(U_1)$ is a gap of $\mathcal{P}$. By the Mean Value Theorem we pick points $\zeta \in C'$ and $\zeta_1 \in U_1$ such that

$$|\Psi^n(C')| = \left|(\Psi^n)'(\zeta)\right| |C'|,$$

$$|\Psi^n(U_1)| = \left|(\Psi^n)'(\zeta_1)\right| |U_1|.$$

Then by the bound on distortion,

$$\begin{aligned}\tau\left(U_1, C, U_2\right) &\geq \frac{|C|}{|U_1|} \geq \frac{|C'|}{|U_1|} = \frac{\left|(\Psi^n)'(\zeta_1)\right|}{\left|(\Psi^n)'(\zeta)\right|} \frac{|\Psi^n(C')|}{|\Psi^n(U_1)|} \\ &\geq e^{-C_1} \frac{|\Psi^n(C')|}{|\Psi^n(U_1)|} \geq e^{-C_1} \tau(\mathcal{P}).\end{aligned}$$

Notice that $|\Psi^n(C')| \leq 1$ and $|\Psi^n(U_1)| \leq 1$, because they are respectively an interval and a gap of $\mathcal{P}$.

■

Remark that it is easy to construct Markov Partitions $\mathcal{P}$ for Ψ, satisfying conditions 1, 2 with arbitrarily large thickness $\tau(\mathcal{P})$. The maximal invariant domain Δ_∞ of Ψ, see (10), is an *infinitely thick* hyperbolic "Cantor set", inside which we can find arbitrarily thick compact invariant Cantor sets.

4. The Basic Set Family

In this section we construct the family of basic sets Λ_k.

4.1. **A Family of Singular Diffeomorphisms.** We start adding to the Standard Map family $f_k(x, y) = (-y + \varphi_k(x), x)$ a *singular perturbation* $(\rho_k(x), 0)$ which transforms it into a family of singular hyperbolic diffeomorphisms $g_k(x, y) = (-y + \psi_k(x), x)$. The new function $\psi_k(x) = \varphi_k(x) + \rho_k(x)$ will satisfy the assumptions made in section 2, and the perturbation $\rho_k(x)$ will vanish outside small $\frac{2}{k^{1/3}}$-neighborhoods of the critical points of φ_k. The size of these neighborhoods is chosen as to the smallest possible provided there exist constants $\lambda \gg 2$ and $0 < \ell \leq \lambda$ satisfying (3) and (4) for all ψ_k. To understand the role of the exponent "1/3" replace ψ by φ_k in the left hand side of (4) and remark that the resulting expression, call it $E_k(x)$, becomes unbounded near $-1/4$ and $1/4$, which up to small errors are the critical points of φ_k. Now suppose that x, in the expression $E_k(x)$, is close to one of these "critical" points, say $\left|x - \frac{1}{4}\right| \leq k^{-\epsilon}$ for some $\epsilon > 0$. An easy computation shows that up to a negligible error $|E_k(x)|$ is bounded from below by

$$\frac{\pi}{k\,|\cos(2\pi x)|^3} - \frac{2\pi}{k\,|\cos(2\pi x)|} \geq k^{3\epsilon - 1}.$$

If we want to choose $\epsilon > 0$ the largest possible so that $\psi_k(x) = \varphi_k(x)$ whenever $\left|x \pm \frac{1}{4}\right| \geq k^{-\epsilon}$ and still have a uniform bound on (4) for all ψ_k, we must have $|E_k(x)| \leq \frac{1}{\ell}$ whenever $\left|x \pm \frac{1}{4}\right| \geq k^{-\epsilon}$. Thus $k^{3\epsilon-1}$

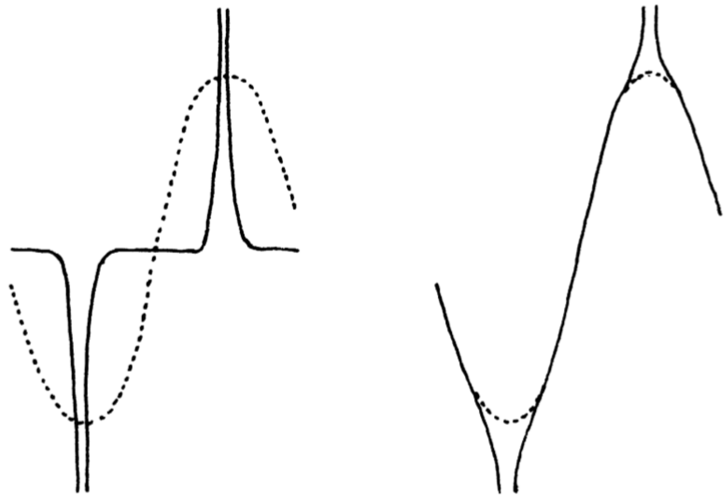

FIGURE 3. Functions ρ_k and ψ_k

must be bounded, implying that $\epsilon \leq 1/3$. So the best choice for our purposes is $\epsilon = 1/3$.

For an explicit definition of ρ_k we take an auxiliary C^∞ function $\beta:\mathbb{R}\to\mathbb{R}$ such that:

$$\beta(x) = \begin{cases} x^{-2} & if \quad |x| \leq 1 \\ 0 & if \quad |x| \geq 2 \end{cases}$$

and all the derivatives of β are monotonous inside $(-\infty, 0)$ and $(0, \infty)$. Define then $\rho_k:\mathbb{R}\to\mathbb{R}\cup\{\infty\}$

$$\rho_k(x) = \sum_{n\in\mathbb{Z}} k^{1/3}\beta\left(k^{1/3}\left(x - \frac{1}{4} + n\right)\right) - k^{1/3}\beta\left(k^{1/3}\left(x + \frac{1}{4} + n\right)\right).$$

The sum is a well defined C^∞ function since it is locally finite, (actually all summands have disjoint supports for $k^{1/3} \geq 8$) and it is obviously periodic,

$$\rho_k(x+1) = \rho_k(x)\,.$$

Setting then $\psi_k:\mathbb{R}\to\mathbb{R}\cup\{\infty\}$, $\psi_k(x) = \varphi_k(x) + \rho_k(x)$, this is a smooth periodic function,

$$\psi_k(x+1) = \psi_k(x) + l,$$

with two poles of second order $-\frac{1}{4}$ and $\frac{1}{4}$ in $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

All estimatives in the following proposition hold for $k^{1/3} \geq 20$. Items 3 and 4 will be needed in the next section to prove that the invariant foliations of g_k depend on k in a differentiable way.

Proposition 10. *For large k,*

(1) $|\psi_k'(x)| \;\geq\; 32\, k^{2/3}$,

(2) $\dfrac{|\psi_k'''(x)|}{\psi_k'(x)^2} + \dfrac{1}{2}\dfrac{|\psi_k''(x)|^2}{|\psi_k'(x)|^3} \;\leq 5$

(3) $\left|\dfrac{\partial \psi_k}{\partial k}\right| \leq \dfrac{1}{k^{2/3}}\,|\psi_k'|$

(4) $\left|\dfrac{\partial \psi_k'}{\partial k}\right| \leq \dfrac{3}{k^{4/3}}\,|\psi_k'|^2$

Two important remarks should be made now. First, inside $\left[-\frac{1}{2},\frac{1}{2}\right]$ the critical points of $\varphi_k(x) = 2x + k\sin(2\pi x)$ are very close to $-\frac{1}{4}$ and $\frac{1}{4}$. Denote them by $0 < \nu_- < \nu_+ < 1$. Then a simple computation shows that

$$i)\quad \left|\nu_+ - \frac{1}{4}\right| \leq \frac{1}{16k} \qquad ii)\quad \left|\frac{1}{4} + \nu_-\right| \leq \frac{1}{16k}.$$

Second, the derivatives $\varphi_k'(x) - 2 = k\cos(2\pi x)$ and $\rho_k'(x)$ always have the same sign. Thus $|\psi_k'(x)| \geq |\varphi_k'(x)|$, except inside $[-\frac{1}{4},\nu_-]\cup[\nu_+,\frac{1}{4}]$. Notice these are very small intervals with length $(16k)^{-1}$. In any case $|\psi_k'(x)| \geq |\rho_k'(x)| - 2$ always holds.

Proof:
Let us prove 1. Using the inequality

$$\left|\cos\left(\frac{\pi}{2} + z\right)\right| \geq |z| - \frac{|z|^3}{6} \qquad \text{for} \quad |z| \leq 1$$

we conclude for $\left|x \pm \frac{1}{4}\right| \leq \frac{1}{2\pi}$,

$$\begin{aligned} |\varphi_k'(x)| \;&\geq\; 2\pi k\,|\cos(2\pi x)| - 2 \\ &\geq\; 4\pi^2 k\left|x \pm \frac{1}{4}\right| - \frac{(2\pi)^4 k}{6}\left|x \pm \frac{1}{4}\right|^3 - 2\,. \end{aligned}$$

Consider now two cases $i)$ $\left|x - \frac{1}{4}\right| \;\geq\; \frac{1}{k^{1/3}}$ and $\left|x + \frac{1}{4}\right| \;\geq\; \frac{1}{k^{1/3}}$, $ii)$ $\left|x \pm \frac{1}{4}\right| \;\leq\; \frac{1}{k^{1/3}}$. The minimum value of $|\varphi_k'(x)|$ through case $i)$ is attained when $\left|x \pm \frac{1}{4}\right| = \frac{1}{k^{1/3}}$. Thus if $i)$ is the case

$$|\psi_k'(x)| \geq |\varphi_k'(x)| \geq 4\pi^2 k^{2/3} - \frac{(2\pi)^4}{6} \geq 32k^{2/3}\,.$$

Otherwise in case *ii*)

$$\begin{aligned} |\psi_k'(x)| &\geq 2\pi k\,|\cos(2\pi x)| + \frac{2}{k^{1/3}\left|x\pm\frac{1}{4}\right|^3} - 2 \\ &\geq 4\pi^2 k\left|x\pm\frac{1}{4}\right| + \frac{2}{k^{1/3}\left|x\pm\frac{1}{4}\right|^3} - \frac{(2\pi)^4}{6} - 2 \\ &\geq 32.8\,k^{2/3} - 265 \geq 32\,k^{2/3} \end{aligned}$$

We have used the following inequality

$$4\pi^2 z + \frac{2}{z^3} \geq 32.8 \qquad \text{for} \quad 0\leq z\leq 1$$

In order to prove 2 we decompose its summands as follows:

$$\frac{\psi_k'''(x)}{\psi_k'(x)^2} = \frac{\varphi_k'''(x)}{\psi_k'(x)^2} + \frac{\rho_k'''(x)}{\psi_k'(x)^2}$$

$$\frac{\psi_k''(x)^2}{\psi_k'(x)^3} = \left(\frac{\varphi_k''(x)}{\psi_k'(x)^{3/2}} + \frac{\rho_k''(x)}{\psi_k'(x)^{3/2}}\right)^2$$

Using item 1 and the obvious bounds $8\pi^3 k$ and $4\pi^2 k$ for $|\varphi_k'''(x)|$ and $|\varphi_k''(x)|$ respectively, one can easily see that both summands $\frac{\varphi_k'''(x)}{\psi_k'(x)^2}$ and $\frac{\varphi_k''(x)}{\psi_k'(x)^{3/2}}$ are very small. Actually the first is arbitrarily small, if k is large, while the second can only be forced to be smaller than $\frac{1}{4}$. To estimate the other two summands we consider two cases: *i*) $\left|x-\frac{1}{4}\right| > \frac{1}{2k^{1/3}}$ and $\left|x+\frac{1}{4}\right| > \frac{1}{2k^{1/3}}$ and *ii*) $\left|x\pm\frac{1}{4}\right| \leq \frac{1}{2k^{1/3}}$. In the first case, because the derivatives of β are monotonous, we have $|\rho_k''(x)| \leq k\,|\beta''| \leq 3\cdot 2^5\,k$ and $|\rho_k'''(x)| \leq k^{4/3}\,|\beta'''| \leq 3\cdot 2^8\,k^{4/3}$. In case *ii*) we have explicit formulas for $\rho_k(x)$ and its derivatives so that an estimation is straightforward. Putting together all these estimations we can prove item 2.

Finally to prove 3 and 4 we consider the same two cases *i*) and *ii*) as above. Remark that, because of item 1, the right hand sides of 3 and 4 are bounded from below by 32 and $3\cdot 32^2$ respectively. In case *i*) the proof is trivial because the left hand sides of 3 and 4 have upper bounds which are much lesser than the lower bounds mentioned above:

$$\left|\frac{\partial\psi_k}{\partial k}\right| \leq 1 + \left|\frac{\partial\rho_k}{\partial k}\right| = 1 + O\left(\frac{1}{k^{2/3}}\right)$$

$$\left|\frac{\partial\psi_k'}{\partial k}\right| \leq 2\pi + \left|\frac{\partial\rho_k'}{\partial k}\right| = 2\pi + O\left(\frac{1}{k^{1/3}}\right)$$

In case $ii)$ we have explicit formulas for $\left|\frac{\partial \rho_k}{\partial k}\right|$, $\left|\frac{\partial \rho_k'}{\partial k}\right|$ and $\rho_k'(x)$,

$$\left|\frac{\partial \rho_k}{\partial k}(x)\right| = \frac{1}{3k^{4/3}\left|x \pm \frac{1}{4}\right|^2},$$

$$\left|\frac{\partial \rho_k'}{\partial k}(x)\right| = \frac{2}{3k^{4/3}\left|x \pm \frac{1}{4}\right|^3},$$

$$|\rho_k'(x)| = \frac{2}{k^{1/3}\left|x \pm \frac{1}{4}\right|^3},$$

making it easy to check 3 and 4. ■

We can now estimate constant $\mu = \mu(k)$, see (14),

$$1 \leq \mu(k) \leq 1 + \frac{1}{3k^{1/3}} \tag{17}$$

and the distortion constant $C_1 = C_1(k)$ with $\gamma = \frac{1}{2}$, see (16).

$$0 \leq C_1(k) \leq \frac{9}{k^{1/3}} \tag{18}$$

The distortion $C_1(k)$ converges to 0 as k tends to ∞.

4.2. **Construction of Λ_k.** Using the same notation of section 3, Ψ_k will be the expansive map associated to the singular diffeomorphism g_k. We begin constructing a Cantor set K_k as the maximal invariant set

$$K_k = \bigcap_{n \geq 0} \Psi_k^{-n}\,(\ J_0 \cup J_1 \ mod\ \mathbb{Z}\ )$$

over a Markov Partition $\mathcal{P}_k = \{J_0, J_1\}$ satisfying assumptions 1 and 2 of section 3.3. These intervals are chosen so that

- J_0, J_1 are inside the region $\{\rho_k = 0\}$,
- $\tau(\mathcal{P}_k)$ is large.

Then we set the basic set Λ_k to be the square of K_k relative to the product structure induced on $\mathbb{T}^2$ by the projections $\pi_s, \pi_u : \mathbb{T}^2 \to SS^1$,

$$\Lambda_k = \pi_s^{-1}(K_k) \cap \pi_u^{-1}(K_k)\,.$$

Λ_k will be a compact invariant basic set for both f_k and g_k.

Let B_l^- and B_r^- be small intervals close to $-\frac{1}{4}$, respectively at the left and right of this point, defined by

$$B_l^- = \left\{-\frac{1}{4} - \frac{4}{k^{1/3}} < x < -\frac{1}{4} - \frac{3}{k^{1/3}}\right\}$$

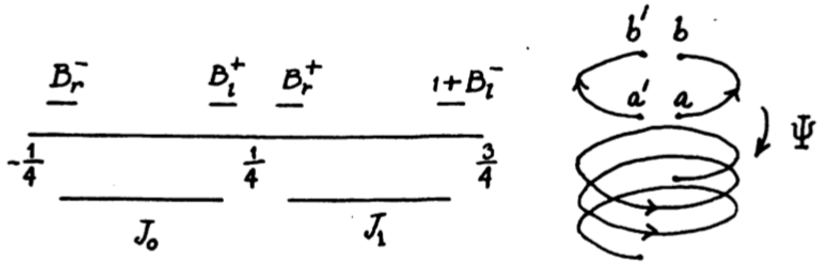

FIGURE 4. The Markov Partition $\mathcal{P}_k$

$$B_r^- = \left\{ -\frac{1}{4} + \frac{3}{k^{1/3}} < x < -\frac{1}{4} + \frac{4}{k^{1/3}} \right\}.$$

Similarly, close to $\frac{1}{4}$, B_l^+ and B_r^+ are the intervals

$$B_l^+ = \left\{ \frac{1}{4} - \frac{4}{k^{1/3}} < x < \frac{1}{4} - \frac{3}{k^{1/3}} \right\}$$

$$B_r^+ = \left\{ \frac{1}{4} + \frac{3}{k^{1/3}} < x < \frac{1}{4} + \frac{4}{k^{1/3}} \right\}.$$

We define $J_0 = [a, b]$ and $J_1 = [b', a' + 1]$ by choosing:

$a \in B_r^-$,	s.t.	$\Psi_k(a) \equiv a \bmod \mathbb{Z}$,
$a' \in B_l^-$	s.t.	$\Psi_k(a') \equiv a \bmod \mathbb{Z}$,
$b \in B_l^+$, $b' \in B_r^+$	s.t.	$\Psi_k(b) \equiv \Psi_k(b') \equiv a' \bmod \mathbb{Z}$.

Since $|\Psi_k'(x)| \geq (32\, k^{2/3} - 1)/\mu \geq 30\, k^{2/3}$, see (14), all four intervals $B = B_l^-, B_r^-, B_l^+, B_r^+$ are expanded by Ψ_k onto intervals with length $|\Psi_k(B)| \geq 30\, k^{1/3} \gg 1$. Thus it is possible to find a, a', b and b' as above. It is clear that such $\mathcal{P}_k$ is a Markov Partition satisfying assumptions 1, 2 of section 3.3. For some positive integers n, m, n', and m', $[a, b]$ is mapped, orientation preserved, onto $[a - n, a' + m]$ and $[b', a' + 1]$ is mapped, orientation reversed, onto $[a - n', a' + m']$. Furthermore we can choose a, a', b and b' so that $n = n'$, $m = m'$, and so $\Psi_k(J_0) = \Psi_k(J_1)$, and the number of fundamental domains covered by $\Psi_k(J_0) = \Psi_k(J_1)$ is $n_k = n + m$. Then K_k is conjugated to the full one sided shift in $2n_k$ symbols. To estimate n_k observe that inside J_0 we have $|\Psi_k - \varphi_k| \leq \frac{1}{30k^{2/3}}$ since $\psi_k = \varphi_k$. Thus

$$2k \geq n_k \geq |\Psi_k(J_0)| \geq |\varphi_k(J_0)| - \frac{1}{15k^{2/3}}$$

and estimating $|\varphi_k(J_0)|$ we obtain,

$$1 - \frac{32\pi^2}{k^{2/3}} \leq \frac{n_k}{2k} \leq 1.$$

Proposition 11. $\Lambda_k = \pi_s^{-1}(K_k) \cap \pi_u^{-1}(K_k)$ *is a compact invariant basic set for both* f_k *and* g_k*, conjugated to the full Bernoulli shift in* $2n_k$ *symbols.*

Proof:
Λ_k is closed in the complement of the discontinuity circles $C_s \cup C_u$ of the projections π_s, π_u. Also it lies inside the compact set

$$\pi_s^{-1}(J_0 \cup J_1) \cap \pi_u^{-1}(J_0 \cup J_1)$$

which is disjoint from $C_s \cup C_u$. Thus Λ_k is compact. Once we see it is invariant by g_k, Λ_k will obviously be a basic set because it has a global product structure. It will also be a basic set of f_k, because it follows from the definition of J_0, J_1, that $\pi_s^{-1}(J_0 \cup J_1) \cap \pi_u^{-1}(J_0 \cup J_1)$ is inside the region $\{\rho_k(x) = 0\}$. It remains to prove the invariance of Λ_k by g_k. Let $I = \left(-\frac{1}{4}, \frac{3}{4}\right)$ and consider the C^1 diffeomorphism $\Phi_k : \mathbb{T}^2 - (C_s \cup C_u) \to I \times I$, $\Phi_k(x,y) = (\pi_s(x,y), \pi_u(x,y))$, mapping Λ_k onto $K_k \times K_k$. The singular diffeomorphism on $I \times I$ $T_k = \Phi_k \circ g_k \circ \Phi_k^{-1}$ can be explicitly defined by

$$T_k(z,z') = \left(\Psi_k(z), \Psi_\alpha^{-1}(z')\right), \qquad z \in I_\alpha,\ \alpha \in \mathcal{A}$$

where $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ denotes the Markov Partition introduced in section 3.1 for the singular expansive map Ψ_k, and Ψ_α^{-1} stands for the inverse map of the restriction of Ψ_k to I_α. For the sake of rigor we should mention that the components of T_k are to be taken modulus integer translations otherwise they could be outside of I. Assume for the meanwhile that we already know that this map T_k satisfies the conjugacy relation, $T_k \circ \Phi_k = \Phi_k \circ g_k$. Then for Λ_k's invariance it is enough to prove that $K_k \times K_k$ is invariant by T_k. For any $\alpha \in \mathcal{A}$, $\Psi_{I_\alpha} : I_\alpha \cap K_k \to K_k$ is a diffeomorphism and so $\Psi_k(I_\alpha \cap K_k) = K_k$ and also $\Psi_\alpha^{-1}(K_k) = I_\alpha \cap K_k$. Thus $T_k(K_k \times K_k) = K_k \times K_k$. Finally, because $\Psi_k : K_k \to K_k$ is conjugated to a one sided full shift in $2n_k$ symbols it follows that $T_k : K_k \times K_k \to K_k \times K_k$, and therefore $g_k : \Lambda_k \to \Lambda_k$, are conjugated to a full Bernoulli shift in $2n_k$ symbols. Let us now get back to prove the conjugacy relation. Because the projections π_s and π_u respectively semiconjugate g_k and g_k^{-1} with Ψ_k we have

$$\Psi_k(\pi_s(x,y)) = \pi_s(g(x,y)) \ ,$$

$$\Psi_\alpha^{-1}\pi_u(x,y) = \pi_u(g(x,y)) \quad \text{whenever} \quad \pi_u(g_k(x,y)) \in I_\alpha \,.$$

Thus it is enough to prove that $\pi_s(x,y)$ and $\pi_u(g_k(x,y))$ always belong to the same interval I_α, $\alpha \in \mathcal{A}$. Since $\pi_u(g_k(x,y)) = \pi_s(x, -y + \psi(x))$ we have to see that (x,y) and $(x',y') = (x, -y + \psi(x))$ project, along $\mathcal{F}^s$, into the same interval I_α, or which is equivalent, that $g_k(x, y+l(y))$ and $g_k(x', y' + l(y'))$ also project, along $\mathcal{F}^s$, into the same fundamental domain $m+I$, $m \in \mathbb{Z}$. Given $z \in \mathbb{R}$, $l(z) \in \mathbb{Z}$ denotes the only integer such that $-\frac{1}{4} \le z + l(z) < \frac{3}{4}$. Now

$$\begin{aligned} l\left(-y' - l(y') + \psi_k(x')\right) &= l\left(-y' - \psi_k(x')\right) + l(y') \\ &= l(y) + l\left(-y + \psi_k(x)\right) \\ &= l\left(-y - l(y) + \psi_k(x)\right) \end{aligned}$$

shows that $g_k\left(x', y' + l(y')\right)$ and $g_k\left(x, y + l(y)\right)$ have their x coordinates in the same fundamental domain $-l+I$, $l \in \mathbb{Z}$, and so $\pi_s\left(g_k(x', y' + l(y'))\right)$ and $\pi_s\left(g_k(x, y + l(y))\right)$ also belong to the same fundamental domain. ■

For all sufficiently large parameters, say $k \ge \iota_0$, $\Psi_k : SS^1 \to SS^1$ is a singular expansive map and $g_k : \mathbb{T}^2 \to \mathbb{T}^2$ is a singular hyperbolic diffeomorphism. $\iota_0 = 8^3$ is enough for this to be true, but we can take $\iota_0 = 20^3$ so that all estimatives in (10) hold. On their maximal invariant domains,

$$\Delta_\infty(k) = \bigcap_{n \ge 0} \Psi_k^{-n} \left\{ x \,|\, \psi(x) \ne \infty \right\}$$

$$D_\infty(k) = \bigcap_{n \in \mathbb{Z}} g_k^{-n} \left\{ (x, y) \,|\, \psi(x) \ne \infty \right\},$$

these maps are conjugated to full shifts in the infinite alphabet $\mathcal{A} = \mathbb{Z} \times \{0, 1\}$, respectively the one sided full shift $\sigma : \Sigma_+(\mathcal{A}) \to \Sigma_+(\mathcal{A})$, $\Sigma_+(\mathcal{A}) = \mathcal{A}^{\mathbb{N}}$, and the two sided full shift $\sigma : \Sigma(\mathcal{A}) \to \Sigma(\mathcal{A})$, $\Sigma(\mathcal{A}) = \mathcal{A}^{\mathbb{Z}}$. Thus every Cantor set K_{k_0} and every basic set Λ_{k_0} constructed above has a continuation $\tilde{K}_{k_0\,k}$ or $\tilde{\Lambda}_{k_0\,k}$ defined all over $[\iota_0, \infty)$. Moreover, for $k \ge k_0$ the continuation $\tilde{\Lambda}_{k_0\,k}$ of Λ_{k_0} is always a basic set for the Standard Map f_k. To see this let $J_0 = [a_0, b_0]$, $J_1 = [b_0', a_0' + 1]$ be the Markov Partition defining K_{k_0}. Then the C^1 functions $a(k)$, $a'(k)$, $b(k)$ and $b'(k)$ defined by

$$\Psi_k\left(a(k)\right) \equiv \Psi_k\left(a'(k)\right) \equiv a(k) \; mod\; \mathbb{Z} \qquad a(k_0) = a_0, \quad a'(k_0) = a_0'$$

$$\Psi_k\left(b(k)\right) \equiv \Psi_k\left(b'(k)\right) \equiv a'(k) \; mod\; \mathbb{Z} \qquad b(k_0) = b_0, \quad b'(k_0) = b_0'$$

are the boundary points of a family of Markov Partitions $J_0(k) = [a(k),\, b(k)]$, $J_1(k) = [b'(k),\, a'(k) + 1]$, defining the continuation of K_{k_0},

$$\tilde{K}_{k_0\,k} = \bigcap_{n \ge 0} \Psi_k^{-n}\left(J_0(k) \cup J_1(k) \; mod\; \mathbb{Z}\right) .$$

Since $|\Psi_k(x)|$ grows with k, the boundary points $a(k)$, $b(k)$, $a'(k)$ and $b'(k)$ slowly move away from the poles $\pm\frac{1}{4}$. Thus as $k \to \infty$ the intervals $J_0(k)$ and $J_1(k)$ shrink inside the region where ρ_k vanishes, which shows that

$$\tilde{\Lambda}_{k_0\,k} = \pi_s^{-1}(\tilde{K}_{k_0\,k}) \cap \pi_u^{-1}(\tilde{K}_{k_0\,k}),$$

the continuation of Λ_{k_0}, for $k \geq k_0$ lies inside $\{\rho_k(x) = 0\}$ and so it is a basic set for the Standard Map f_k. Finally remark that for each k the definitions of the Cantor set K_k and the basic set Λ_k depend on an arbitrary choice of a Markov Partition $\mathcal{P}_k = \{J_0, J_1\}$. This selection can easily be made explicit so that these families become dynamically increasing in the sense of item 1, Theorem A, and continuous with respect to the Hausdorff metric except on a discreet set $\{k_0,\, k_1,\, k_2, \cdots\}$, formed by an increasing sequence of parameters $k_n \to \infty$, where it is only right continuous, meaning $\Lambda_{k_i} = \lim_{k \to k_i^+} \Lambda_k$.

4.3. **Measuring Λ_k.**

Proposition 12. *For all sufficiently large k, $\tau(K_k) \geq \frac{k^{1/3}}{9}$.*

Proof:
By the localization of the extreme points a, a', b, b' of the Markov Partition it is clear that both gaps (a', a) and (b, b') have length $\leq \frac{4}{k^{1/3}}$, and both intervals $J_0 = [a, b]$ and $J_1 = [b', a'+1]$ have length $\geq \frac{1}{2} - \frac{8}{k^{1/3}}$. Thus, using the distortion estimative (18), it follows that

$$\tau(K_k) \geq e^{-C_1(k)} \tau(\mathcal{P}_k) \geq \left(1 - \frac{10}{k^{1/3}}\right)\left(\frac{k^{1/3}}{8} - 2\right) \geq \frac{k^{1/3}}{9} \qquad \blacksquare$$

Lemma 5. *The map $\Phi_k : \mathbb{T}^2 - (C_s \cup C_u) \to I \times I$, defined by $(x, y) \mapsto (\pi_s(x, y), \pi_u(x, y))$, is a C^1 diffeomorphism close to the identity*

(1)
$$|\Phi_k(x, y) - (x, y)| \leq \frac{1}{30\, k^{2/3}}$$

(2)
$$\|D\Phi_k(x, y) - Id\,\| \leq \frac{1}{k^{1/3}}$$

Proof:
To prove that Φ_k is C^1 close to the identity, we only have to see that π_s is C^1 close to the vertical projection $(x, y) \mapsto x$, because by symmetry

π_u will then be C^1 close to the horizontal projection $(x,y) \mapsto y$. By definition 3) of section 3.1, for $0 \le y < 1$, $g_s\left(\pi_s(x,y), y\right) = x$. Thus

$$\begin{aligned} |\pi_s(x,y) - x| &= |g_s\left(\pi_s(x,y),0\right) - g_s\left(\pi_s(x,y),y\right)| \\ &\le \int_0^y |\alpha^s\left(g_s(\pi_s(x,t),t),t\right)|\,dt \le \frac{1}{30\,k^{2/3}}. \end{aligned}$$

Differentiating the relation above we get

$$1 = \frac{\partial g_s}{\partial x}\,\frac{\partial \pi_s}{\partial x}$$

$$0 = \frac{\partial g_s}{\partial x}\,\frac{\partial \pi_s}{\partial y} + \frac{\partial g_s}{\partial y} = \frac{\partial g_s}{\partial x}\,\frac{\partial \pi_s}{\partial y} + \alpha^s$$

Because α^s is small and $\frac{\partial g_s}{\partial x}$ is close to 1 we get $\frac{\partial \pi_s}{\partial x}$ and $\frac{\partial \pi_s}{\partial y}$ respectively close to 1 and 0. The calculations are left to the reader. ■

Proposition 13. *For all sufficiently large k,*

(1) $B_{\delta_k}\left(\Lambda_k\right) = \mathbb{T}^2, \qquad \delta_k = \dfrac{4}{k^{1/3}}$

(2) $\tau^s_{loc}(\Lambda_k) = \tau^u_{loc}(\Lambda_k) \ge \dfrac{k^{1/3}}{9}$,

(3) $HD(\Lambda_k) \ge 2\dfrac{\log 2}{\log\left(2 + \frac{9}{k^{1/3}}\right)}$,

Proof:
1) The idea is to remark that, by construction, all gaps of K_k have length $< \frac{4}{k^{1/3}}$. Thus $B_{\frac{2}{k^{1/3}}}(K_k) = SS^1$ and also $B_{\frac{2}{k^{1/3}}}(K_k \times K_k) = \mathbb{T}^2$, where the second ball is associated to metric defined on $\mathbb{T}^2$ by the *max* norm $|(x,y)| = \max\{|x|, |y|\}$. Now because Φ_k is C^1 close to the identity it has a Lipschitz constant close to one. This is enough to conclude that $B_{\frac{3}{k^{1/3}}}(\Lambda_k) = \mathbb{T}^2$.
2) The local thickness of a Cantor set K at a point $x \in K$ is defined as

$$\tau_{loc}(K,x) = \lim_{\delta \to 0}\, \sup\{\tau(A) :\ A \subseteq B_\delta(x) \cap K\ \}$$

where the supremum is taken over by all compact subsets A of K. From the definition it is clear that always

$$\tau_{loc}(K,x) \ge \tau(K).$$

Another important remark is that *local thickness* is invariant by diffeomorphisms. For surface diffeomorphisms *local stable* and *unstable thickness* of a basic set Λ are defined as follows. See [PT, N 3]. Take sections Σ^s and Σ^u through the point $x \in \Lambda$ respectively transversal to the stable and unstable foliations. Then

$$\tau^s_{loc}(\Lambda, x) = \tau_{loc}\left(\Sigma^s \cap W^s(\Lambda), x\right)$$
$$\tau^u_{loc}(\Lambda, x) = \tau_{loc}\left(\Sigma^u \cap W^u(\Lambda), x\right)$$

The invariance by diffeomorphisms enables one to prove this definition is independent of the transversal section. It can also be proved that the definition is independent of point $x \in \Lambda$. Thus $\tau^s_{loc}(\Lambda)$ and $\tau^u_{loc}(\Lambda)$ are two well defined numbers. Because of all remarks above it is obvious, in our setting that

$$\tau^s_{loc}(\Lambda_k) = \tau^u_{loc}(\Lambda_k) = \tau_{loc}(K_k) \geq \tau(K_k) \geq \frac{k^{1/3}}{9}.$$

3) For Dynamically defined Cantor sets the following relation holds between *thickness* and *Hausdorff Dimension*.

$$HD(K) \geq \frac{\log 2}{\log\left(2 + 1/\tau\right)} \qquad \tau = \tau(K)$$

See [PT, N 3] . Thus because Λ_k is diffeomorphic to $K_k \times K_k$,

$$HD(\Lambda_k) = 2HD(K_k) \geq 2\frac{\log 2}{\log\left(2 + \frac{9}{k^{1/3}}\right)}$$

■

5. Persistent Tangencies

We prove Theorem C in this section. Push the g-invariant foliation $\mathcal{F}^u$ by the Standard Map f into a new foliation $\mathcal{G}^u = f_*\mathcal{F}^u$. Then $\mathcal{G}^u$ and $\mathcal{F}^s$ have two circles of mutual tangencies. We project the basic set Λ_k along the foliations $\mathcal{F}^s$ and $\mathcal{G}^u$ to one of these circles and obtain two Cantor sets K^s and K^u, respectively. Then applying the gap lemma to these Cantor sets we conclude that for all sufficiently large k there is a tangency between stable and unstable leaves of Λ_k. Finally we show that all these tangencies unfold generically.

5.1. **Circles of tangencies.** We begin defining a pair of new foliations $\mathcal{G}^u$ and $\mathcal{G}^s$, respectively the forward and backward images of $\mathcal{F}^u$ and $\mathcal{F}^s$ by the Standard Map f. These foliations are defined by the vector fields $(\beta^u, 1)$ and $(1, \beta^s)$, where

$$(\beta^u(x, y), 1) = Df_{f^{-1}(x,y)}\left(1, \alpha^u f^{-1}(x, y)\right),$$
$$(1, \beta^s(x, y)) = Df^{-1}_{f(x,y)}\left(\alpha^s f(x, y), 1\right).$$

A simple computation shows then

$$\beta^u(x,y) = \varphi'(y) - \alpha^u f^{-1}(x,y),$$
$$\beta^s(x,y) = \varphi'(x) - \alpha^s f(x,y).$$

The set of tangencies between $\mathcal{G}^u$ and $\mathcal{F}^s$ is

$$\{\beta^u = \alpha^s\} = \{\ (x,y) :\ \varphi'(y) = \alpha^s(x,y) + \alpha^u(y, -x+\varphi(y))\ \},$$

and similarly the set of tangencies between $\mathcal{F}^u$ and $\mathcal{G}^s$ is

$$\{\beta^s = \alpha^u\} = \{\ (x,y) :\ \varphi'(x) = \alpha^s(-y+\varphi(x), x) + \alpha^u(x,y)\ \}.$$

Both these tangency sets consist of two circles. Denote by ν_- and ν_+ the critical points of φ. A straightforward application of the Implicit Function Theorem gives

Proposition 14. *The set $\{\beta^u = \alpha^s\}$ is the union of two horizontal circles, $\{(x, \sigma_+(x)) \mid x \in SS^1\}$ and $\{(x, \sigma_-(x)) \mid x \in SS^1\}$, which are graphs of C^1 functions σ_+, $\sigma_+ : SS^1 \to SS^1$ satisfying*

(1)
$$|\sigma_\pm(x) - \nu_\pm| \le \frac{1}{270\, k^{5/3}}$$

(2)
$$\left|\sigma'_\pm(x)\right| \le \frac{1}{12\, k^{4/3}}$$

Symmetrically, $\{\beta^s = \alpha^u\}$ consists of two vertical circles which are graphs, $\{(\varrho_+(x), x) \mid x \in SS^1\}$ and $\{(\varrho_-(x), x) \mid x \in SS^1\}$, of C^1 functions $\varrho_\pm : SS^1 \to SS^1$ satisfying the same conditions 1 and 2 above.

Fix the critical point ν_+ of φ_k near $\frac{1}{4}$ and denote by $S_h \subseteq \{\beta^u = \alpha^s\}$, respectively by $S_v \subseteq \{\beta^s = \alpha^u\}$, the horizontal circle of tangencies near $\{(x, \nu_+) : x \in SS^1\}$, respectively the vertical circle near $\{(\nu_+, x) : x \in SS^1\}$.

Lemma 6.
$$f(S_v) = S_h\ .$$

Proof:
It is geometrically obvious that f maps $\{\beta^s = \alpha^u\}$, the set of tangencies between $(f^{-1})_* \mathcal{F}^s$ and $\mathcal{F}^u$, onto the set of tangencies between $\mathcal{F}^s$ and $f_* \mathcal{F}^u$, $\{\beta^u = \alpha^s\}$. Also f maps $\{(\nu, x) : x \in SS^1\}$ onto $\{(x, \nu) : x \in SS^1\}$. Thus by continuity $f(S_v) = S_h$. ■

We define the *projection of* Λ_k *along* $\mathcal{F}^s$ *into* S_h as

$$K_h^s = S_h \cap \pi_s^{-1}(K),$$

and the *projection of* Λ_k *along* $\mathcal{G}^u$ *into* S_h as

$$K_h^u = S_h \cap f\pi_u^{-1}(K).$$

Remark that an intersection point $x \in K_h^s \cap K_h^u$ is a point of tangency between one stable and one unstable leaf of Λ_k. Both K_h^s and K_h^u are compact sets because $\pi_s^{-1}(K)$ and $f\pi_u^{-1}(K)$ are closed in the complement of C_s. To get the *persistent tangency* phenomenon, we estimate the thickness of the Cantor sets K_h^s and K_h^u.

Proposition 15. *For all sufficiently large* k*,*

$$\tau(K_h^s) \geq \frac{k^{1/3}}{10} \qquad \tau(K_h^u) \geq \frac{k^{1/3}}{10}$$

Proof:
We need the following easy fact. Let $h : SS^1 \to SS^1$ be a Lipschitz homeomorphism with $Lip(h) \leq \mu$, $Lip(h^{-1}) \leq \mu$. Then for any compact set $K \subseteq SS^1$,

$$\frac{1}{\mu^2} \;\leq\; \frac{\tau\,(h(K))}{\tau(K)} \;\leq\; \mu^2.$$

Now if $h : SS^1 \to SS^1$ is a diffeomorphism C^1 close to the identity we can choose μ close to 1 such that $Lip(h) \leq \mu$ and $Lip(h^{-1}) \leq \mu$ to conclude that $\tau(K)$ is close to $\tau\,(h(K))$. More generally if $h : SS^1 \to SS^1$ is C^1 close to an isometric rotation $\theta : SS^1 \to SS^1$ then also $\tau(K)$ is close to $\tau\,(h(K))$. We just have to remark that θ preserves thickness and apply the same argument to $h \circ \theta^{-1}$.

Consider on S_h the metric induced by its natural parametrization $SS^1 \ni x \mapsto (x, \sigma(x)) \in S_h$, via which we make the identification $S_h \equiv SS^1$. The projection $\pi_s : S_h \to SS^1$ restricted to S_h is a diffeomorphism C^1 close to the identity which maps K_h^s onto K. The order of the C^1 closeness is $\frac{1}{k^{1/3}}$. See proposition (5). Thus if k is large we can find a Lipschitz constant less then $\sqrt{\frac{10}{9}}$ for both the projection and its inverse which gives us

$$\tau(K_h^s) \geq \frac{9}{10}\tau(K) \geq \frac{k^{1/3}}{10}.$$

In order to estimate $\tau(K_h^u)$ we remark that by symmetry the same argument above proves that (eventually for larger k),

$$\tau(K_v^u) \geq \sqrt{\frac{9}{10}}\tau(K) \geq \frac{k^{1/3}}{3\sqrt{10}}.$$

where $K_v^u = S_v \cap \pi_u^{-1}(K)$ is the projection of Λ_k along $\mathcal{F}^u$ into S_v. Again on S_v we consider the metric induced by its parametrization $SS^1 \ni x \mapsto (\varrho(x), x) \in S_v$, and make the identification $S_v \equiv SS^1$. The Standard Map f takes S_v onto S_h, mapping K_v^u onto K_h^u,

$$f(K_v^u) = f(S_v) \cap f\left(\pi_u^{-1}(K)\right) = S_h \cap f\left(\pi_u^{-1}(K)\right) = K_h^u.$$

By the previous remarks it is enough to prove now that the restriction diffeomorphism $f : S_v \to S_h$ is C^1 close to the isometric rotation $\theta : SS^1 \to SS^1$, $\theta(x) = \varphi(\nu) - x$. We prove below that so it is with

$$\|f - \theta\|_{C^1} \leq \frac{1}{81\,k^2}.$$

Then if k is large $f \circ \theta^{-1}$ and $\theta \circ f^{-1}$ have Lipschitz constants $\leq \sqrt[4]{\frac{10}{9}}$ which gives us,

$$\tau(K_h^u) = \tau(f(K_v^u)) \geq \sqrt{\frac{9}{10}}\tau(K_v^u) \geq \frac{k^{1/3}}{10}.$$

To estimate $\|f - \theta\|_{C^1}$ notice that f maps $(\varrho(x), x)$ to $(-x + \varphi(\varrho(x)), \varrho(x))$. Thus, modulus the above identifications $S_h \equiv SS^1$ and $S_v \equiv SS^1$, we have $f(x) = \varphi(\varrho(x)) - x$, and by proposition (14)

$$\begin{aligned}
|f(x) - \theta(x)| &= |(\varphi(\varrho(x)) - x) - (\varphi(\nu) - x)| = |\varphi(\varrho(x)) - \varphi(\nu)| \\
&\leq |\varphi'(z)|\,|\varrho(x) - \nu| \leq \frac{4\pi^2}{270\,k^{2/3}}\,\frac{1}{270\,k^{5/3}} \leq \frac{1}{1500\,k^{7/3}}, \\
|f'(x) - \theta'(x)| &= |\varphi'(\varrho(x))|\,|\varrho'(x)| \leq \frac{4\pi^2}{270\,k^{2/3}}\,\frac{1}{12\,k^{4/3}} \leq \frac{1}{81\,k}.
\end{aligned}$$

■

5.2. **Gap lemma.** We now use the following circle version of Newhouse's *Gap Lemma* to get the persistent tangency phenomenon.

Proposition 16. *If* $K_1\, K_2 \subseteq SS^1$ *are compact sets such that* $\tau(K_1)\,\tau(K_2) > 1$ *then* $K_1 \cap K_2 \neq \emptyset$.

Proof:
It follows easily from the usual Gap Lemma for Cantor sets in the real line. See [PT]. Lift K_1 and K_2 to periodic closed Cantor sets $\tilde{K}_1$, $\tilde{K}_2 \subseteq \mathbb{R}$. It is obvious that $\tau(K_1) = \tau(\tilde{K}_1)$, $\tau(K_2) = \tau(\tilde{K}_2)$ and that none of the Cantor sets $\tilde{K}_1$, $\tilde{K}_2$ is contained in a gap of the other because they are both unbounded. Thus we can apply the usual Gap Lemma to conclude $\tilde{K}_1 \cap \tilde{K}_2 \neq \emptyset$ and so $K_1 \cap K_2 \neq \emptyset$. ■

From proposition (15) we get

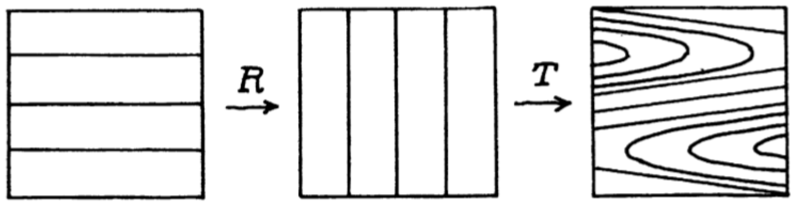

FIGURE 5. $f = T \circ R, \quad R(x, y) = (-y, x) \quad T(x, y) = (x + \varphi(y), y)$.

Corollary 17. *For all sufficiently large parameters k there is a tangency in S_h between one stable leaf $W^s(f_k, x)$ and another unstable one $W^u(f_k, y)$ of two points $x, y \in \Lambda_k$.*

Proof:
Given k large, since $\tau(K_h^s)\,\tau(K_h^u) \geq \frac{k^{2/3}}{100} \gg 1$, there is some point $Q \in K_h^s \cap K_h^u$. Q is a tangency point between $\pi_s^{-1}(z)$ and $f\pi_u^{-1}(z')$ for some pair $(z, z') \in K_k \times K_k$. Now, $\pi_s^{-1}(z)$ is a piece of stable leaf of Λ_k for both f_k and g_k, because it lies with all its forward iterates inside the region $\{f_k = g_k\}$. Similarly, $f\pi_u^{-1}(z')$ is a piece of unstable leaf of Λ_k as a basic set of the Standard Map f_k, because all backward iterates of $\pi_u^{-1}(z')$ are inside $\left\{f_k^{-1} = g_k^{-1}\right\}$. ■

5.3. **Generic Unfolding.** All tangencies in $K_h^s \cap K_h^u$ between stable leaves in $\pi_s^{-1}(K)$ and unstable ones in $f\pi_u^{-1}(K)$ are quadratic and unfold generically. We will give complete analytic proofs of these facts.

Even so the following heuristic description should be enough to convince ourselves. We have seen that the leaves in $\pi_s^{-1}(K)$ are almost vertical and symmetrical so that those in $\pi_u^{-1}(K)$ are almost horizontal. Now, the same factorization of section 2.1 holds for the Standard Map, see fig (5), so when we push $\pi_u^{-1}(K)$ by f we first rotate 90 degrees counterclockwise to an almost vertical foliation and then slide along horizontal circles in a way that verticals are folded to a foliation $\mathcal{G}$ of curves parallel to the graph of φ, $G = \{(\varphi(x), x) : x \in SS^1\}$. Thus the tangency circles between $\mathcal{F}^s$ and $\mathcal{G}^u$ are very close to the circles of tangencies between vertical lines and the foliation $\mathcal{G}$ of horizontal displacements of G, which are the critical circles $\{(x, \nu_-) : x \in SS^1\}$ and $\{(x, \nu_+) : x \in SS^1\}$. Thus the difference of curvatures at a tangency point is close to the second derivative $\varphi''(\nu_\pm) \approx 4\pi^2 k$. The tangencies are quadratic!

As the parameter k grows the stable leaves in $\pi_s^{-1}(K)$ become more and more vertical with very small displacements along S_h and the

FIGURE 6. Moving with k

same is true about $\pi_u^{-1}(K)$ becoming horizontal without moving much in the vertical direction. When k increases the critical values of φ_k are pushed apart with velocity one and in the same way f_k pushes the leaves in $\pi_u^{-1}(K)$ along the circle S_h. Thus as we move the parameter k, while the leaves of $\pi_s^{-1}(K)$ are almost still, those in $f\pi_u^{-1}(K)$ move comparatively fast along S_h with velocity close to one. All tangencies unfold generically!

Fix a point $(x_0, y_0) \in S_h$ and denote by $\gamma^s \subseteq \mathcal{F}^s$, respectively $\gamma^u \subseteq \mathcal{G}^u$ the stable and unstable leaves of these foliations through (x_0, y_0). The following proposition shows that the tangency between γ^s and γ^u is *quadratic.*

Proposition 18. *γ^s and γ^u are graphs of C^2 functions $\phi_s, \phi_u : \mathbb{R} \to \mathbb{R}$.*

$$\gamma^s = \{\, (\phi_s(y), y) : y \in \mathbb{R} \,\}$$
$$\gamma^u = \{\, (\phi_u(y), y) : y \in \mathbb{R} \,\} \quad \textit{and}$$
$$|\phi_u''(y_0) - \phi_s''(y_0)| \geq 4\pi^2 k - \frac{3}{k^{1/3}}.$$

Proof:
As $g_s\left(\pi_s(x_0, y_0), y_0\right) = x_0$, see definition 3) of section 3.1,

$$\gamma^s = \{(g_s(\pi_s(x_0, y_0), y), y) : y \in \mathbb{R}\}$$

is the stable leaf of $\mathcal{F}^s$ through (x_0, y_0). Defining $\phi_s : \mathbb{R} \to \mathbb{R}$, $\phi_s(y) = g_s(x_0', y)$ where $x_0' = \pi_s(x_0, y_0)$, γ^s is the graph of ϕ_s and it is of class C^2 since it solves the C^1 differential equation $\frac{\partial x}{\partial y} = \alpha^s(x, y)$. In particular $\phi_s''(y) = \frac{\partial \alpha^s}{\partial x}\alpha^s + \frac{\partial \alpha^s}{\partial y}$, and

$$|\phi_s''(y)| \leq |\alpha^s| \left|\frac{\partial \alpha^s}{\partial x}\right| + \left|\frac{\partial \alpha^s}{\partial y}\right| \leq \frac{1}{k^{1/3}}.$$

Analogously, as $g_u\left(y_0, \pi_u f^{-1}(x_0, y_0)\right) = -x_0 + \varphi(y_0)$,

$$\tilde{\gamma}^u = \left\{ \left(y, g_u(y, \pi_u f^{-1}(x_0, y_0))\right) : y \in \mathbb{R} \right\}$$

is the leaf of $\mathcal{F}^u$ through $f^{-1}(x_0,y_0)=(y_0,-x_0+\varphi(y_0))$. Thus

$$\gamma^u = f(\tilde{\gamma}^u) = \{(-g_u(y,y_0')+\varphi(y),y) : y\in\mathbb{R}\}$$

where $y_0' = \pi_u f^{-1}(x_0,y_0)$, is the graph of $\phi_u : \mathbb{R}\to\mathbb{R}$ $\phi_u(y) = -g_u(y,y_0')+\varphi(y)$. In the same way we see that $\tilde{\phi}_u(y) = g_u(y,y_0')$ is a function of class C^2 with second derivative smaller than $k^{-1/3}$. An elementary calculation, using proposition (14), shows that

$$|\varphi''(y_0)| = |\varphi''(\sigma_+(x_0))| \geq 4\pi^2 k\left(1-\frac{1}{10k^2}\right),$$

and so

$$\begin{aligned} |\phi_u''(y_0)-\phi_s''(y_0)| &\geq |\varphi''(y_0)|-|\phi_s''(y_0)|-\left|\tilde{\phi}_u''(y_0)\right| \\ &\geq 4\pi^2 k\left(1-\frac{1}{10k^2}\right)-\frac{2}{k^{1/3}} \geq 4\pi^2 k-\frac{3}{k^{1/3}}. \end{aligned}$$ ■

Now in order to prove that these tangencies are generically unfold as k varies, we analyse the dependence of the invariant foliations on the parameter k. Consider the Markov Partition $\{I_\alpha\}_{\alpha\in\mathcal{A}}$, $\mathcal{A}=\mathbb{Z}\times\{0,1\}$, for $\Psi_k:\Delta_\infty(k)\to\Delta_\infty(k)$ defined in section 3.1. The full shift $\sigma:\Sigma(\mathcal{A})\to\Sigma(\mathcal{A})$, $\Sigma(\mathcal{A})=\mathcal{A}^{\mathbb{Z}}$, is conjugated to $\Psi_k:\Delta_\infty(k)\to\Delta_\infty(k)$ by $\Phi_k:\Sigma(\mathcal{A})\to\Delta_\infty(k)$,

$\Phi_k(\underline{a}) =$ *the unique point x of $\Delta_\infty(k)$ with itinerary $\underline{a}=(a_n)_{n\geq 0}$ meaning that $\forall n\geq 0$ $\Psi_k^n(x)\in I_{a_n}(k)$,*

Let $\iota_0>0$ be as given in section 4.2. Then $\Phi:\Sigma(\mathcal{A})\times[\iota_0,\infty)\to SS^1$ defined by $\Phi(\underline{a},k)=\Phi_k(\underline{a})$ is a continuous function and we have

Proposition 19. *For each $\underline{a}\in\Sigma(\mathcal{A})$, $k\mapsto\Phi_k(\underline{a})$ is differentiable and $\frac{\partial\Phi}{\partial k}:\Sigma(\mathcal{A})\times[\iota_0,\infty)\to SS^1$ is continuous satisfying*

$$\left|\frac{\partial\Phi}{\partial k}(\underline{a},k)\right| \leq \frac{2}{k^{2/3}}\ . \tag{19}$$

Proof:

Let $\mathcal{S}$ be the space of all sequences $\underline{x}=(x_n)_{n\geq 0}$ of real numbers, with the usual pointwise convergence, and define the open subset $\mathcal{U}\subseteq\mathbb{R}\times\mathcal{S}$,

$$\mathcal{U} = \left\{(k,\underline{x})\ :\ k\geq\iota_0,\ \ \forall n\geq 0,\ x_n\neq -\frac{1}{4}\ mod\ \mathbb{Z}\right\}.$$

For each $\alpha\in\mathcal{A}$, consider the map $G_\alpha:[\iota_0,\infty)\times I\to I_\alpha$, where $I=\left(-\frac{1}{4},\frac{3}{4}\right)$, defined by $\Psi_k(G_\alpha(k,x)) = x\ \ mod\ \mathbb{Z}$. Up to an integer translation $x\mapsto G_\alpha(k,x)$ is the inverse of Ψ_{I_α}. Then we define the continuous map,

$$F : \Sigma(\mathcal{A}) \times \mathcal{U} \to \mathcal{S}$$
$$F(\underline{a}, k, \underline{x}) = (x_n - G_{a_n}(k, x_{n+1}))_{n\geq 0} \ .$$

Now remark that $F(\underline{a}, k, \underline{x}) = 0$ means that all the following three equivalent statements are true

$$1) \ \ x_n = G_{a_n}(k, x_{n+1}), \ \ \forall n \geq 0 \ ,$$

$$2) \ \ \Psi_k(x_n) = x_{n+1} \quad \text{and} \quad x_n \in I_{a_n}(k) \ , \ \ \forall n \geq 0 \ ,$$

$$3) \ \ x_n = \Psi_k^n \left(\Phi(\underline{a}, k)\right) \ , \ \ \forall n \geq 0 \ .$$

Thus if we define, $\underline{\Phi} : \Sigma(\mathcal{A}) \times [\iota_0, \infty) \to \mathcal{S}$ by

$$\underline{\Phi}(\underline{a}, k) = (\Psi_k^n(\Phi(\underline{a}, k)))_{n\geq 0}$$

$\underline{\Phi}$ is a continuous map such that

$$(k, \underline{\Phi}(\underline{a}, k)) \in \mathcal{X} \quad F\left(\underline{a}, k, \underline{\Phi}(\underline{a}, k)\right) = 0.$$

We now want to conclude by an implicit function theorem argument that $\underline{\Phi}$ is differentiable in k and $\frac{\partial \underline{\Phi}}{\partial k}$ is continuous in $(\underline{a}, k)$, which will imply the same about Φ. For this to be true we need to know that for each $\underline{a} \in \Sigma(\mathcal{A})$, $(k, \underline{x}) \mapsto F(\underline{a}, k, \underline{x})$ is a C^1 function with derivatives depending continuously on $(\underline{a}, k, \underline{x})$. Now the maps $G_\alpha : D \to \mathbb{R}$ are of class C^1 because of lemma 7 below, proving $\Psi(k, x) = \Psi_k(x)$ is a C^1 function of (k, x). It is easy to prove, after lemma (7), that $\frac{\partial}{\partial k} F\left(\underline{a}, k, \underline{x}\right)$ and $D_3 F\left(\underline{a}, k, \underline{x}\right)$ are continuous functions of $(\underline{a}, k, \underline{x})$. Remark now that $(k, \underline{x}) \mapsto F(\underline{a}, k, \underline{x})$ is the linear projection $(k, \underline{x}) \mapsto \underline{x}$ minus a perturbation $\underline{G}(\underline{a}, k, \underline{x}) = (G_{a_n}(k, x_{n+1}))_{n\geq 0}$ with very small derivatives,

$$D_3 \underline{G}(\underline{a}, k, \underline{x}) \, \underline{u} \ = \ \left(\frac{\partial G_{a_n}}{\partial x}(k, x_{n+1}) u_{n+1} \right)_{n\geq 0} .$$

$$\left| \frac{\partial G_\alpha}{\partial x} \right| \ = \ \frac{1}{|\Psi_k'|} \ \leq \ \frac{1}{30 \, k^{2/3}}.$$

Thus $D_3 F(\underline{a}, k, \underline{x}) = I - D_3 \underline{G}(\underline{a}, k, \underline{x})$ is invertible, which shows that an implicit function theorem argument applies to prove continuity of $\frac{\partial \underline{\Phi}}{\partial k}(\underline{a}, k)$.

Let us now estimate $\frac{\partial \Phi}{\partial k}$. Assume $\underline{a} \in \Sigma(\mathcal{A})$ is a fixed point of σ. For some $m \in \mathbb{Z}$ we have $\Psi_k\left(\Phi(\underline{a}, k)\right) = \Phi(\underline{a}, k) + m$. Differentiating this relation with respect to k we have,

$$\frac{\partial \Psi_k}{\partial k} \left(\Phi(\underline{a}, k)\right) + \Psi_k' \left(\Phi(\underline{a}, k)\right) \, \frac{\partial \Phi}{\partial k}(\underline{a}, k) \ = \ \frac{\partial \Phi}{\partial k}(\underline{a}, k) \, .$$

Thus, using lemma (7) below,

$$\left|\frac{\partial\Phi}{\partial k}(\underline{a},k)\right| \le \frac{\left|\frac{\partial\Psi_k}{\partial k}\right|}{|\Psi_k'|-1} \le \frac{3}{2\,k^{2/3}}\frac{|\Psi_k'|}{|\Psi_k'|-1} \le \frac{2}{k^{2/3}}.$$

Consider now the case $\underline{a} \in \Sigma(\mathcal{A})$ is a prefixed point, meaning that for some $m \in \mathbb{Z}$, $\sigma^m(\underline{a}) = \sigma^{m+1}(\underline{a})$. Write $x_n(k) = \Psi_k^n(\Phi(\underline{a},k)) = \Phi(\sigma^n(\underline{a}),k)$. Differentiating $x_{n+1}(k) = \Psi_k(x_n(k))$ we get

$$\frac{\partial x_{n+1}}{\partial k} = \frac{\partial \Psi_k}{\partial k}(x_n) + \Psi_k'(x_n)\frac{\partial x_n}{\partial k} \qquad (^*)$$

By regressive induction in n we can prove that

$$\forall\, 0 \le n \le m \qquad \left|\frac{\partial x_n}{\partial k}\right| \le \frac{2}{k^{2/3}}.$$

In fact this relation holds for $n = m$, since $\sigma^m(\underline{a})$ is a fixed point and $\frac{\partial x_n}{\partial k} = \frac{\partial\Phi}{\partial k}(\sigma^n(\underline{a}),k)$. Assuming it holds for some $0 < n \le m$, then by $(^*)$ and lemma 7,

$$\left|\frac{\partial x_{n-1}}{\partial k}\right| \le \frac{\left|\frac{\partial\Psi_k}{\partial k}(x_{n-1})\right| + \left|\frac{\partial x_n}{\partial k}\right|}{|\Psi_k'(x_{n-1})|} \le \frac{3}{2\,k^{2/3}} + \frac{2}{k^{2/3}}\,\frac{1}{30\,k^{2/3}} \le \frac{2}{k^{2/3}}$$

and it holds for $n-1$ too. Thus it is true for $n = 0$ which proves

$$\left|\frac{\partial\Phi}{\partial k}(\underline{a},k)\right| = \left|\frac{\partial x_0}{\partial k}\right| \le \frac{2}{k^{2/3}}\,.$$

Then by continuity of $\frac{\partial\Phi}{\partial k}$, since the prefixed points are dense in $\Sigma(\mathcal{A})$, relation $\left|\frac{\partial\Phi}{\partial k}(\underline{a},k)\right| \le \frac{2}{k^{2/3}}$ is always true. ∎

Lemma 7. *The family of expanding maps,*

$$\Psi : [\iota_0,\infty) \times I_0 \cup I_1 \to \mathbb{R}, \qquad \Psi(k,x) = \Psi_k(x),$$

where $I_0 = \left(-\frac{1}{4},\frac{1}{4}\right)$, $I_1 = \left(\frac{1}{4},\frac{3}{4}\right)$, *is a* C^1 *function in both variables and satisfies*

$$\left|\frac{\partial\Psi}{\partial k}(k,x)\right| \le \frac{3}{2k^{2/3}}\,|\Psi_k'(x)|\,.$$

To prove this lemma we need another one.

Lemma 8. *The stable and unstable functions* $\alpha^s(k,x,y)$ *and* $\alpha^u(k,x,y)$ *are of class* C^1 *in* $[\iota_0,\infty)\times\mathbb{T}^2$.
Furthermore,

$$\left|\frac{\partial\alpha^s}{\partial k}(k,x,y)\right| \quad \left|\frac{\partial\alpha^u}{\partial k}(k,x,y)\right| \le \frac{4}{k^{4/3}}. \tag{20}$$

Proof:
The operator T_0 of section 2.1 acts as Lipschitz contraction on the space $\mathcal{X}$ of all continuous functions $\alpha\colon[\iota_0,\infty)\times\mathbb{T}^2\to[-1,1]$. Thus $\alpha^s(k,x,y)$ and $\alpha^u(k,x,y)$ are continuous. To prove they are C^1 functions we apply the Fiber Contraction Theorem, lemma (1), as in section 2.1, making essential use of items 3 and 4 of proposition (10). We omit the proof of this fact, assuming we already know α^s is of class C^1 and proceed to estimate $\frac{\partial\alpha^s}{\partial k}$. Differentiating

$$\alpha^s(k,x,y) = \frac{1}{\psi_k'(x) - \alpha^s(k,-y+\psi_k(x),x)}$$

with respect to k, we obtain

$$\frac{\partial\alpha^s}{\partial k} = \frac{\dfrac{\partial\alpha^s}{\partial k} + \dfrac{\partial\alpha^s}{\partial x}\dfrac{\partial\psi_k}{\partial k} - \dfrac{\partial\psi_k'}{\partial k}}{(\psi_k' - \alpha^s)^2}.$$

By items 3 and 4 of proposition (10),

$$\left|\frac{\partial\alpha^s}{\partial k}\right| \;\le\; \left|1 - \frac{1}{(\psi_k'-\alpha^s)^2}\right|^{-1} \frac{\frac{1}{k^{2/3}}\left|\frac{\partial\alpha^s}{\partial x}\right||\psi_k'| + \frac{3}{k^{4/3}}|\psi_k'|^2}{(\psi_k'-\alpha^s)^2}$$

Finally, because $\left(1 - \dfrac{1}{(\psi_k'-\alpha^s)^2}\right)^{-1}$ and $\left(\dfrac{\psi_k'}{\psi_k'-\alpha^s}\right)^2$ are very close to 1, and also because $\left|\dfrac{\partial\alpha^s}{\partial x}\right|\dfrac{|\psi_k'|}{(\psi_k'-\alpha^s)^2} = O\left(\dfrac{1}{k}\right)$ is very small, we obtain $\left|\dfrac{\partial\alpha^s}{\partial k}\right| \;\le\; \dfrac{4}{k^{4/3}}\,.$ ■

Proof of lemma (7)
Ψ is implicitly defined by $g_s\left(k,\Psi(k,x),x\right) = \psi_k(x)$, for $0\le x\le 1\,.$ So by the Parametric Implicit Function Theorem Ψ is C^1. Of course $g_s(k,x,y)$ is C^1 since it is the flow of a C^1 parametric o.d.e. :

$$\begin{cases} g_s(k,x,0) = x \\ \frac{\partial g_s}{\partial y}(k,x,y) = \alpha^s\left(k, g_s(k,x,y), y\right). \end{cases}$$

We have

$$\frac{\partial}{\partial y}\left(\frac{\partial g_s}{\partial k}\right) = \frac{\partial}{\partial k}\left(\frac{\partial g_s}{\partial y}\right) = \frac{\partial}{\partial k}\left(\alpha^s(k,g_s(k,x,y),y)\right) = \frac{\partial\alpha^s}{\partial k} + \frac{\partial\alpha^s}{\partial x}\frac{\partial g_s}{\partial k}$$

so $\frac{\partial g_s}{\partial k}$ is solution of a linear equation and by the Gronwall lemma,

$$\left|\frac{\partial g_s}{\partial k}(k,x,y)\right| \le \frac{4}{k^{4/3}}\exp\left\{\int_0^y \left|\frac{\partial\alpha^s}{\partial x}\right|\,dt\right\} \;\le\; \frac{4\mu}{k^{4/3}}.$$

Thus, using (17) it follows that $\left|\dfrac{\partial g_s}{\partial k}\right| \leq \dfrac{5}{k^{4/3}}$. Now differentiating with respect to k the above relation we get $\dfrac{\partial g_s}{\partial k} + \dfrac{\partial g_s}{\partial x}\dfrac{\partial \Psi_k}{\partial k} = \dfrac{\partial \psi_k}{\partial k}$. So using item 2 of proposition (7),

$$\begin{aligned} \left|\frac{\partial \Psi}{\partial k}\right| &\leq \frac{\left|\frac{\partial \psi_k}{\partial k}\right| + \left|\frac{\partial g_s}{\partial k}\right|}{\left|\frac{\partial g_s}{\partial x}\right|} \leq \mu\left(\frac{1}{k^{2/3}}\left|\psi_k'\right| + \frac{5}{k^{4/3}}\right) \\ &\leq \frac{\mu^2}{k^{2/3}}\left|\Psi_k'\right| + O\left(\frac{1}{k^{4/3}}\right) \leq \frac{3}{2\,k^{2/3}}\left|\Psi_k'\right| \qquad \blacksquare \end{aligned}$$

We now want to study how the leaves of $\mathcal{F}^s$ and $\mathcal{G}^u$ move along the tangency circle S_h. Take a stable leaf of $\mathcal{F}^s$ in $\mathbb{T}^2 - C_s$ with itinerary $\underline{a} \in \Sigma(\mathcal{A})$. The continuation of this leaf is given by

$$k \mapsto \pi_s^{-1}\left(\Phi(\underline{a}, k)\right).$$

Call $\Phi_s(\underline{a}, k)$ to the intersection of $\pi_s^{-1}\left(\Phi(\underline{a}, k)\right)$ with S_h. Similarly the continuation of an unstable leaf of $\mathcal{G}^u$ in $f\left(\mathbb{T}^2 - C_u\right)$ with itinerary $\underline{a} \in \Sigma(\mathcal{A})$ is given by

$$k \mapsto f\pi_u^{-1}\left(\Phi(\underline{a}, k)\right),$$

and we call $\Phi_u(\underline{a}, k)$ to the intersection of this leaf with S_h. The genericity of the unfolding of a tangency between two leaves

$$k \mapsto \pi_s^{-1}\left(\Phi(\underline{a}, k)\right) \qquad k \mapsto f\pi_u^{-1}\left(\Phi(\underline{b}, k)\right),$$

where $\underline{a}, \underline{b} \in \Sigma(\mathcal{A})$ is established by the following proposition:

Proposition 20. *For all* $\underline{a} \in \Sigma(\mathcal{A})$,

(1) $\left|\dfrac{\partial \Phi_s}{\partial k}(\underline{a}, k)\right| \leq \dfrac{3}{k^{2/3}}$

(2) $\left|\dfrac{\partial \Phi_u}{\partial k}(\underline{a}, k)\right| \geq 1 - \dfrac{3}{k^{2/3}}$

Proof:
The projection π_s induces a diffeomorphism $\pi_s : S_h \to SS^1$ C^1 close to the "identity". Denote its inverse by $h : SS^1 \to S_h$. Of course both h and π_s depend on k. In the proof of lemma (7) we established $\left|\frac{\partial g_s}{\partial k}\right| \leq \frac{5}{k^{4/3}}$. Thus differentiating the relation $g_s\left(\pi_s(x, y), y\right) = x$ with respect to k we obtain,
$\dfrac{\partial g_s}{\partial k} + \dfrac{\partial g_s}{\partial x}\dfrac{\partial \pi_s}{\partial k} = 0$, and so $\left|\dfrac{\partial \pi_s}{\partial k}\right| \leq \left|\dfrac{\partial g_s}{\partial k}\right| \left|\dfrac{\partial g_s}{\partial x}\right|^{-1} \leq \dfrac{5\mu}{k^{4/3}}$.

Differentiating $\pi_s \circ h = id_{SS^1}$ we have $\dfrac{\partial \pi_s}{\partial k} + D\pi_s \dfrac{\partial h}{\partial k} = 0$, or equivalently $\dfrac{\partial h}{\partial k} = -h' \dfrac{\partial \pi_s}{\partial k}$, and so we get $\left|\dfrac{\partial h}{\partial k}\right| \leq \dfrac{6}{k^{4/3}}$. Finally, since $\Phi_s(\underline{a}, k) = h\left(\Phi(\underline{a}, k)\right)$, we have

$$\left|\frac{\partial \Phi_s}{\partial k}\right| \leq \left|\frac{\partial h}{\partial k}\right| + |h'| \left|\frac{\partial \Phi}{\partial k}\right| \leq \frac{6}{k^{4/3}} + \frac{5}{4}\frac{2}{k^{2/3}} \leq \frac{3}{k^{2/3}}.$$

Similarly, if $h : SS^1 \to S_v$ is the inverse of the projection diffeomorphism $\pi_u : S_v \to SS^1$, we can prove

$$\left|\frac{\partial h \circ \Phi}{\partial k}\right| \leq \frac{2.5}{k^{2/3}}\,.$$

To finish the proof notice that $\Phi_u(\underline{a}, k) = f_k h\left(\Phi(\underline{a}, k)\right)$. Using proposition 14 we can show that over the vertical circle S_v,

$$\left|\frac{\partial f_k}{\partial k}\right| \geq 1 - \frac{1}{10\,k^2},$$

$$|Df_k| \leq 1 + \frac{1}{6\,k^{2/3}}.$$

Thus,

$$\begin{aligned}\left|\frac{\partial \Phi_u}{\partial k}\right| &= \left|\frac{\partial f_k}{\partial k} + Df_k \frac{\partial h \circ \Phi}{\partial k}\right| \geq \left|\frac{\partial f_k}{\partial k}\right| - |Df_k| \left|\frac{\partial h \circ \Phi}{\partial k}\right| \\ &\geq 1 - \frac{1}{10\,k^2} - \left(1 + \frac{1}{6\,k^{2/3}}\right)\frac{2.5}{k^{2/3}} \geq 1 - \frac{3}{k^{2/3}}\,. \qquad \blacksquare\end{aligned}$$

Now Theorem C is an immediate consequence of corollary 17 and the fact that in a basic set Λ the stable and the unstable manifolds of every point in Λ are dense in Λ.

6. Many Elliptic Points

In this last section we conclude our work proving Theorem B. The basic technique is a *renormalization* procedure which permits us to conclude the existence of elliptic periodic points arbitrarily close to a homoclinic tangency in phase-parameter space.

6.1. **Renormalization.** Consider a 1-parameter family of surface diffeomorphisms $\varphi_\mu : M^2 \to M^2$ of class C^k, generically unfolding a quadratic homoclinic tangency at point Q and at parameter $\mu = 0$. Renormalization near the homoclinic tangency $(Q, 0)$ means the following:
For every large $n \geq 0$ one finds a small box near $(Q, 0) \in M \times \mathbb{R}$, shrinking to this point as $n \to \infty$, which is mapped by $(x, \mu) \mapsto \left(\varphi_\mu^n(x), \mu\right)$ near itself. Then in this tiny box one computes adequate rescaling changes in phase and parameter coordinates,

$$\mathbb{R}^3 \ni (x, y, a) \mapsto (\Psi_{n,a}(x, y), \mu_n(a)) \in M \times \mathbb{R}$$

such that in this new coordinates the map φ_μ^n,

$$i.e. \quad \Psi_{n,a}^{-1} \circ \varphi_{\mu_n(a)}^n \circ \Psi_{n,a},$$

converges to a normal form $\varphi_a(x, y)$ in the C^k topology.
Thus any feature or property of the dynamics of normal form φ_a, which is stable under small perturbations, will also be present in the dynamics of φ_μ for parameter values very close to parameter $\mu = 0$. For dissipative systems, in fact it is enough to assume the saddle P associated to the tangency is dissipative $|\det D\varphi_\mu(P)| < 1$, the above scheme works having as limit the *Quadratic Family of Endomorphisms*,

$$\varphi_a(x, y) = (a - x^2, x).$$

Of course area expansive saddles $|\det D\varphi_\mu(P)| > 1$, reduce to dissipative ones considering φ_μ^{-1}. In the conservative case, that is if all φ_μ preserve the same area form, it turns out that the same scheme works having as limit the *Henón Conservative Family*

$$\varphi_a(x, y) = (-y + a - x^2, x).$$

This was recently established by N. Romero [MR]. For the Henón family we can easily compute that an elliptic fixed point Q is created through the unfolding of a saddle node bifurcation at parameter $a = -1$. Then as a runs between -1 and 3 the eigenvalues of Q go through the unit circle from 1 to -1 and at parameter $a = 3$ Q goes through a period doubling bifurcation becoming thereafter hyperbolic. As elliptic points are persistent under conservative perturbations we arrive at the following conclusion.

Proposition 21. *Let $\varphi_\mu : M^2 \to M^2$ be a family of area preserving C^k diffeomorphisms, P be a hyperbolic saddle of φ_0, and assume $W^s(P)$ and $W^u(P)$ generically unfold a quadratic homoclinic tangency at $\mu =$*

0 . *Then there is a sequence* $(Q_n, \mu_n)_{n \geq n_0}$ *in phase-parameter space such that:*

- $(Q_n, \mu_n) \in M \times \mathbb{R}$ *converges to* $(P, 0)$*,*
- Q_n *is a generic elliptic periodic point of* φ_{μ_n} *with period* n *.*

A periodic point P of a conservative diffeomorphism $f : M^2 \to M^2$ is said to be a *generic elliptic* point if both eigenvalues of Df^n_P, where $f^n(P) = P$, are in the unit circle without resonances of order ≤ 3, that is $\lambda, \overline{\lambda} \in SS^1$, with $\lambda^2 \neq 1$, $\lambda^3 \neq 1$ and the first coefficient of f's Birkhoff normal form at point P is nonzero. This implies KAM Theory applies and P is a full density point of "Cantor set" of invariant curves around P. See [A, Mo].

6.2. **Conclusion.** Let us prove Theorem B. The shift $\sigma : \Sigma(\mathcal{A}) \to \Sigma(\mathcal{A})$ has a countable number of periodic points. Enumerate them P_1, P_2, P_3, $\cdots$. For each k we will denote by $P_n(k)$ the corresponding periodic point of g_k in D_∞, $P_n(k) = \Phi(\underline{a}, k)$. Consider k_0 as in Theorem C. Then for each $n \geq 0$ and $m \geq 0$ define U_{nm} as the set of all parameters $k > k_0$ such that $P_n(k) \notin \Lambda_k$ or there is a generic elliptic periodic point Q of f_k with $|P_n(k) - Q| < \frac{1}{m}$. We prove that U_{nm} is an open dense subset of $[k_0, \infty)$. The density follows from Theorem C and proposition (21). Let $k \in U_{nm}$. If $P_n(k) \notin \Lambda_k$ then by the right continuity of the family Λ_k there is a neighborhood of k in which $P_n(k') \notin \Lambda_{k'}$, thus a neighborhood contained in U_{nm}. If $P_n(k) \in \Lambda_k$, because generic elliptic points are persistent under conservative perturbations, the existence of an elliptic point near $P_n(k)$ holds in a neighborhood of k, thus a neighborhood contained in U_{nm}. Defining $R = \bigcap_{n,m \geq 0} U_{nm}$, R is a residual set of parameters k for which Λ_k is accumulated by generic elliptic periodic points. The proof is finished! ■

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