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# Smoothness of boundaries of regular sets

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**Abstract** We prove that the boundary of an $r$-regular set is a codimension one boundaryless manifold of class $C^1$.

**Keywords** $r$-regularity · $C^1$-boundary · Lipschitz projection · Euclidean distance

**Mathematics Subject Classification (2000)** 41A65 · 65D18

## 1 Introduction

The main task of digital image processing is to infer properties of real objects given their digital images, i.e., discrete data generated by some simple device, like a CCD camera. A fundamental question in digital image processing is: *which properties inferred from discrete representations of real objects, under certain conditions, correspond to properties of their originals?* Most of the known answers to this question are restricted to a certain class of subsets of Euclidean space $\mathbb{R}^2$ ([5,6,8,9,11,12]), or $\mathbb{R}^3$ [10], representing real objects, called *r-regular sets*. As can be read in [11], "*the fact that r-regular sets are widely used in the context of digitalization shows that r-regularity is a fundamental property*". In [6], conditions were derived relating properties of regular sets to the grid size of the sampling device which guarantee that a regular object and its digital image are topologically equivalent. To obtain the topological equivalence it was used that a regular set is always bounded by a codimension one manifold. This property was conjectured in [6, p. 145]. Later, in [11], several properties of $r$-regular sets, which make them attractive for image digitization, were proved. In particular, the author states in [11, Lemma 3.4] that the boundary of an $n$-dimensional $r$-regular set is an $(n-1)$-dimensional manifold, but the sketched idea can not be considered as a mathematical proof.

In this paper we prove, in any dimension, that the boundary of an $r$-regular set is a codimension one boundaryless manifold of class $C^1$. The proof generalizes, to the setting of $r$-regular sets, a classical result on convex sets (see, [4]): *the distance function of a point in an Hilbert space $X$ to a closed convex set $K \subset X$ is always $C^1$, regardless of the boundary behaviour of $K$.* According to Holmes [4], this fact seems to have first been established by Moreau in [7].

The authors' motivation in proving this theorem comes from the study of smooth nondeterministic dynamical systems, that is the dynamics of 'smooth' point-set maps on a compact manifold, where $r$-regular sets can appear as dynamically invariant sets. See [3].

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## 2 Geometry of Convex Projections

Let $K \subset \mathbb{R}^n$ be a compact convex set.

**Proposition 1** *Given $x \in \mathbb{R}^n$, there is a unique point $z \in K$ such that*

(1) $\|x - z\| = d(x, K)$,
(2) $\langle x - z, y - z\rangle \leq 0$, *for all $y \in K$.*

Condition (2) just says that the point $z \in K$ which minimizes the distance to $x$ is contained in the half-space bounded by the hyperplane through $x$ normal to the vector $x - z$,

$$K \subseteq \{ y \in \mathbb{R}^n : \langle x - z, y - z \rangle \leq 0 \} .$$

We define $\pi : \mathbb{R}^n \to K$ to be the map that to each $x \in \mathbb{R}^n$ assigns the unique point $z = \pi(x) \in K$ which minimizes the distance to $x$.

**Proposition 2** *The mapping $\pi : \mathbb{R}^n \to K$ is a Lipschitz projection. More precisely, $\pi \circ \pi = \pi$, and given $x, y \in \mathbb{R}^n$, $\|\pi(x) - \pi(y)\| \leq \|x - y\|$.*

*Proof* It is clear that $\pi \circ \pi = \pi$. Consider the vectors $u = x - \pi(x)$, $v = y - \pi(y)$ and $w = \pi(x) - \pi(y)$, which satisfy $\langle v, w \rangle \leq 0$ and $\langle u, w \rangle \geq 0$. Notice that $u - v + w = x - y$. We have

$$\begin{aligned} &\|u - v + w\|^2 \\ &\quad = \|u - v\|^2 + \|w\|^2 + 2 \langle u - v, w \rangle \\ &\quad = \|u - v\|^2 + \|w\|^2 + \underbrace{2 \langle u, w \rangle - 2 \langle v, w \rangle}_{\geq 0} \geq \|w\|^2 \ , \end{aligned}$$

and hence $\|y - x\|^2 \geq \|\pi(y) - \pi(x)\|^2$. □

Define now $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \|x - \pi(x)\|^2$.

**Proposition 3** *The mapping $f : \mathbb{R}^n \to \mathbb{R}$ is of class $C^1$ with derivative $Df_x(v) = 2 \langle x - \pi(x), v \rangle$.*

*Proof* Given $x, v \in \mathbb{R}^n$, let $y = x + v$. We have

$$\begin{aligned} \|y - \pi(y)\|^2 &\leq \|y - \pi(x)\|^2 = \|v + x - \pi(x)\|^2 \\ &= \|v\|^2 + \|x - \pi(x)\|^2 + 2 \langle x - \pi(x), v \rangle \end{aligned}$$

and hence

$$f(x + v) - f(x) - 2 \langle x - \pi(x), v \rangle \leq \|v\|^2 \ . \qquad (1)$$

Conversely, interchanging the roles of $x$ and $y$ we get

$$\|x - \pi(x)\|^2 \leq \|v\|^2 + \|y - \pi(y)\|^2 - 2 \langle y - \pi(y), v \rangle \ .$$

Noticing that

$$\begin{aligned} &\langle x - \pi(x), v \rangle - \langle y - \pi(y), v \rangle \\ &\qquad = \langle \pi(y) - \pi(x) - v, v \rangle \\ &\qquad = \langle \pi(y) - \pi(x), v \rangle - \|v\|^2 \\ &\qquad \leq \|\pi(y) - \pi(x)\| \ \|v\| - \|v\|^2 \leq 0 \end{aligned}$$

we get from the previous inequality

$$\|x - \pi(x)\|^2 \leq \|v\|^2 + \|y - \pi(y)\|^2 - 2 \langle x - \pi(x), v \rangle \ ,$$

and therefore

$$f(x + v) - f(x) - 2 \langle x - \pi(x), v \rangle \geq - \|v\|^2 \ . \qquad (2)$$

Combining (1) and (2) we get

$$|f(x + v) - f(x) - 2 \langle x - \pi(x), v \rangle| \leq \|v\|^2 \ ,$$

proving that $f$ is of class $C^1$ with the specified derivative. □

**Remark 1** *Assuming $\pi : \mathbb{R}^n \to \partial U$ is any Lipschitz projection such that $\|x - \pi(x)\| = d(x, \partial U)$, for some open set $U \subset \mathbb{R}^n$ with regular compact boundary, the argument of the previous proposition can be adapted to prove that*

$$|f(x + v) - f(x) - 2 \langle x - \pi(x), v \rangle| \leq C \ \|v\|^2 \ ,$$

*where $C$ is the Lipschitz constant for $\pi$.*

## 3 Geometry of $r$-Convex Projections

The class of $r$-regular sets was independently introduced in [8] and [9]. This class is also referred in [1, 2, 5, 6, 10–12]. Although the details of the definitions in these papers are different, the described class is essentially the same and can be defined as follows. Fix a positive number $r > 0$ and define $\mathcal{U}_r$ as the set of all connected unions of Euclidean open balls of radius $r > 0$. Note that, as any ball of radius greater than $r$ is itself a union of balls of radius $r$, any set in $\mathcal{U}_r$ is a union of balls of radius $r$.

**Definition 1** *An open set $U \subseteq \mathbb{R}^n$ is said to be* **r-regular** *if and only if $U \in \mathcal{U}_r$ and $\overline{U}^c \in \mathcal{U}_r$.*

A set $C \subseteq \mathbb{R}^n$ is called *$r$-convex* if and only if it is an intersection of any number of $r$-ball complements $\mathcal{B}_r(a) = \{x \in \mathbb{R}^n : d(x, a) \geq r\}$ where $a$ runs through some possible infinite set. Notice that complements of open sets in $\mathcal{U}_r$ are $r$-convex sets.

The aim of next results is the proof of

**Theorem 1** *Let $U \subseteq \mathbb{R}^n$ be an $r$-regular set. Then $\partial U$ is a codimension one boundaryless manifold of class $C^1$.*

From now on we assume that $U \subseteq \mathbb{R}^n$ is an $r$-regular set.

**Proposition 4** *Given $x \in \partial U$, there is a unique vector $\eta(x) \in \mathbb{R}^n$ such that:*

(1) $\|\eta(x)\| = r$,
(2) $x + \eta(x) \in U$,
(3) $\{ y \in \mathbb{R}^n : \|x + \eta(x) - y\| < r \} \subseteq U$,
(4) $\{ y \in \mathbb{R}^n : \|x - \eta(x) - y\| < r \} \subseteq \overline{U}^c$.

The mapping $\eta : \partial U \to \mathbb{R}^n$ is a normal vector field along $\partial U$ with constant norm equal to $r$.

Consider the following picture.

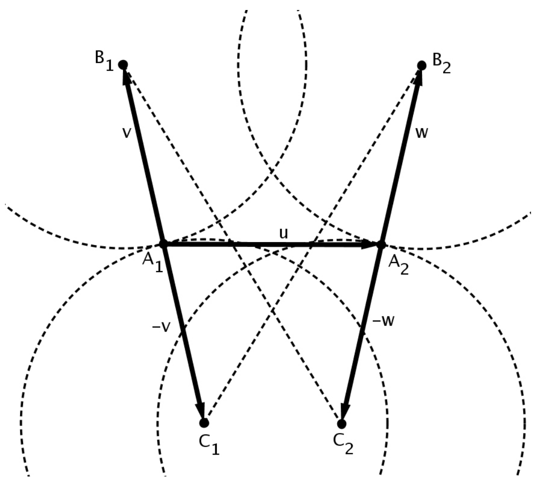


**Fig. 1** Assumptions of Lemma 1

The assumptions of the following lemma say that $v$ and $w$ are vectors of some fixed length $r$, $u$ is a small vector, and the two specified diagonals have length greater than $2\,r$. This means the radius $r$ circles centered at the diagonal endpoints $B_1$ and $B_2$ do not intersect the interiors of the radius $r$ circles centered at the diagonal endpoints $C_1$ and $C_2$.

**Lemma 1** *Assume we have*

(a) $\|v\| = \|w\| = r$,
(b) $\|u + (v + w)\| > 2r$, *and*
(c) $\|u - (v + w)\| > 2r$.

*Then*

(1) $\|v - w\| \leq \|u\|$,
(2) $|\langle v + w, u\rangle| \leq \frac{1}{2}\ \|u\|^2$,
(3) $|\langle v, u\rangle| \leq \frac{3}{4}\ \|u\|^2$,
(4) $|\langle v - w, u\rangle| \leq \|u\|^2$, *and*
(5) $|\langle v, v - w\rangle| \leq \frac{1}{2}\ \|u\|^2$.

*Proof* Using the parallelogram identity, we derive that

$$8\,r^2 < \|u + (v + w)\|^2 + \|u - (v + w)\|^2 = 2\ \|u\|^2 + 2\ \|v + w\|^2\ , \tag{3}$$

and

$$\|v + w\|^2 + \|v - w\|^2 = 2\ \|v\|^2 + 2\ \|w\|^2 = 4\,r^2\ .$$

Hence plugging

$$\|v + w\|^2 = 4\,r^2 - \|v - w\|^2 \tag{4}$$

in (3) we get

$$8\,r^2 < 2\ \|u\|^2 + 2\ \|v + w\|^2 = 8\,r^2 + 2\ \|u\|^2 - 2\ \|v - w\|^2$$

$$\Leftrightarrow\ \|v - w\|^2 \leq \|u\|^2\ \Leftrightarrow\ \|v - w\| \leq \|u\|\ .$$

This proves item (1).

From (b) and (c), plugging (4) in, we have

$$4\,r^2 < \|u\|^2 + \|v + w\|^2 \pm 2\,\langle u, v + w\rangle = \|u\|^2 + 4\,r^2 - \|v - w\|^2 \pm 2\,\langle u, v + w\rangle$$

which implies

$$\mp 2\,\langle u, v + w\rangle < \|u\|^2 - \|v - w\|^2 \leq \|u\|^2\ .$$

Hence

$$|\langle u, v + w\rangle| \leq \frac{1}{2}\ \|u\|^2\ ,$$

which proves item (2).

Items (3) and (4) follow because

$$\begin{aligned} |\langle u, v\rangle| &\leq \frac{1}{2}\ (|\langle u, v - w\rangle| + |\langle u, v + w\rangle|) \\ &\leq \frac{1}{2}\left(\|u\|\ \|v - w\| + \frac{1}{2}\ \|u\|^2\right) \\ &\leq \frac{1}{2}\left(\|u\|^2 + \frac{1}{2}\ \|u\|^2\right) = \frac{3}{4}\ \|u\|^2\ , \end{aligned}$$

and

$$\begin{aligned} |\langle u, v - w\rangle| &\leq \|u\|\ \|v - w\| \\ &\leq \|u\|^2\ . \end{aligned}$$

Finally,

$$\begin{aligned}\|v-w\|^2 &= \langle v-w, v-w\rangle \\ &= 2\,\langle v, v-w\rangle - \langle v+w, v-w\rangle \\ &= 2\,\langle v, v-w\rangle - \underbrace{(\|v\|^2 - \|w\|^2)}_{=0}\end{aligned}$$

Hence

$$\langle v, v-w\rangle = \frac{1}{2}\,\|v-w\|^2 \le \frac{1}{2}\,\|u\|^2 \ ,$$

and this proves item (5). □

**Lemma 2** *Under the same assumptions of the previous lemma consider vectors $v'$ and $w'$ colinear with $v$ and $w$ respectively such that $\|v'\| < \delta$ and $\|w'\| < \delta$. Then $\|u\| \le \sqrt{\frac{2\,r}{2\,r-3\delta}}\ \|u+w'-v'\|$.*

Notice that the coefficient $\sqrt{\frac{2\,r}{2\,r-3\delta}}$ gets close to 1 as $\delta \to 0$.

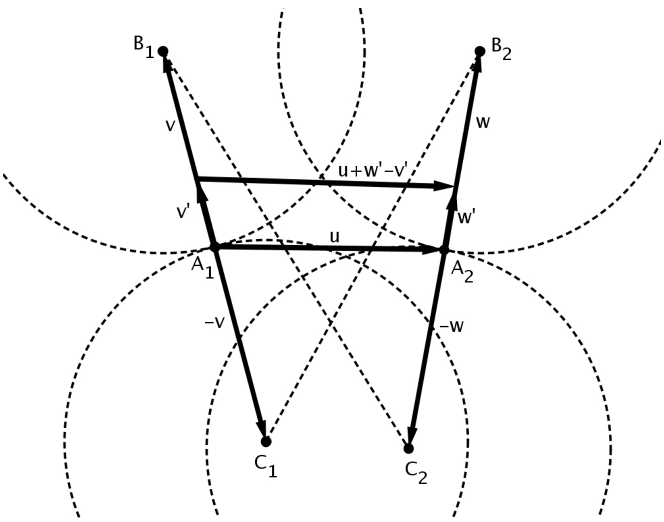


**Fig. 2** Assumptions of Lemma 2

*Proof* By item (3) of previous lemma we have

$$|\langle u, v'\rangle| = \frac{\|v'\|}{r}\ |\langle u, v\rangle| < \frac{3\,\delta}{4\,r}\ \|u\|^2 \ .$$

Because the vectors $v$ and $w$ play the same role we also have

$$|\langle u, w'\rangle| = \frac{\|w'\|}{r}\ |\langle u, w\rangle| < \frac{3\,\delta}{4\,r}\ \|u\|^2 \ ,$$

and, therefore,

$$|\langle u, v'\rangle| + |\langle u, w'\rangle| < \frac{3\,\delta}{2\,r}\ \|u\|^2 \ .$$

Thus

$$\begin{aligned}\|u+w'-v'\|^2 &= \|w'-v'\|^2 + \|u\|^2 + 2\,\langle u, w'-v'\rangle \\ &\ge \|u\|^2 - |\langle u, w'\rangle| - |\langle u, v'\rangle| \\ &\ge \|u\|^2 - \frac{3\,\delta}{2\,r}\ \|u\|^2 \\ &= \left(1 - \frac{3\,\delta}{2\,r}\right)\ \|u\|^2\end{aligned}$$

which implies that

$$\begin{aligned}\|u\|^2 &\le \left(1 - \frac{3\,\delta}{2\,r}\right)^{-1}\ \|u+w'-v'\|^2 \\ &= \frac{2\,r}{2\,r-3\,\delta}\ \|u+w'-v'\|^2 \ ,\end{aligned}$$

and proves the lemma. □

**Proposition 5** *The normal vector field $\eta : \partial U \to \mathbb{R}^n$ is Lipschitz continuous, with* $\mathrm{Lip}(\eta) \le 1$,

$$\|\eta(x) - \eta(y)\| \le \|x-y\|\,.$$

*Proof* Given $x, y \in \partial U$ we have

$$B(x+\eta(x), r) \cap B(y-\eta(y), r) = \emptyset$$

which is equivalent to $\|x-y+\eta(x)+\eta(y)\| > 2r$. Analogously,

$$B(x-\eta(x), r) \cap B(y+\eta(y), r) = \emptyset$$

which is equivalent to $\|x-y-\eta(x)-\eta(y)\| > 2r$. The Lipchitz inequality follows by applying Lemma 1 (1) to the vectors $u = x-y$, $v = \eta(x)$ and $w = \eta(y)$. □

Define $\pi : \mathbb{R}^n \to \partial U$ as the minimizing projection $\|x - \pi(x)\| = d(x, \partial U)$.

**Proposition 6** *The mapping $\pi : \mathbb{R}^n \to \partial U$ is a Lipchitz projection. More precisely, $\pi \circ \pi = \pi$, and given $x, y \in \mathbb{R}^n$ and $\delta > 0$ such that $d(x, \partial U) < \delta$ and $d(y, \partial U) < \delta$, $\quad \|\pi(x) - \pi(y)\| \le \sqrt{\frac{2\,r}{2\,r-3\,\delta}}\ \|x-y\|$.*

*Proof* Just apply Lemma 2 to the vectors $u = \pi(y) - \pi(x)$, $v = \eta(\pi(x))$, $w = \eta(\pi(y))$, $v' = x - \pi(x)$ and $w' = y - \pi(y)$. Notice that $u + w' - v' = y - x$. □

Combining the two Lipschitz maps $\eta$ and $\pi$ we define a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \langle\, x - \pi(x),\, \eta(\pi(x))\,\rangle\ .$$

Although we did not prove $\eta$ and $\pi$ to be of class $C^1$, it happens that

**Proposition 7** *Function $f$ is of class $C^1$ with differential given by*

$$Df_x(v) = \langle\, v,\, \eta(\pi(x))\,\rangle\ .$$

*Proof* It's enough to prove that for some constant $C > 0$,

$$|f(y) - f(x) - \langle y - x, \eta(\pi x)\rangle| \le C\ \|y - x\|^2\ .$$

We shall apply the inequalities in Lemma 1 (3), (4), and (5) with $u = \pi(y) - \pi(x)$, $v = \eta(\pi x)$ and $w = \eta(\pi y)$.

$$\begin{aligned}
&|f(y) - f(x) - \langle y - x, \eta(\pi x)\rangle| = \\
&= |\langle y - \pi y, \eta(\pi y)\rangle - \langle x - \pi x, \eta(\pi x)\rangle - \langle y - x, \eta(\pi x)\rangle| \\
&= |\langle y - \pi y, \eta(\pi y)\rangle - \langle y - \pi x, \eta(\pi x)\rangle| \\
&\le |\langle y - \pi y, \eta(\pi y)\rangle - \langle y - \pi x, \eta(\pi y)\rangle| \\
&\quad + |\langle y - \pi x, \eta(\pi y)\rangle - \langle y - \pi x, \eta(\pi x)\rangle| \\
&= |\langle \pi x - \pi y, \eta(\pi y)\rangle| + |\langle y - \pi x, \eta(\pi y) - \eta(\pi x)\rangle| \\
&= |\langle \pi x - \pi y, \eta(\pi y)\rangle| + |\langle y - x, \eta(\pi y) - \eta(\pi x)\rangle| \\
&\quad + |\langle x - \pi x, \eta(\pi y) - \eta(\pi x)\rangle| \\
&= |\langle u, w\rangle| + |\langle y - x, w - v\rangle| + \frac{\|x - \pi x\|}{r}\ |\langle v, w - v\rangle| \\
&\le \frac{3}{4}\ \|u\|^2 + \|y - x\|\ \|v - w\| + \frac{\|x - \pi x\|}{2\,r}\ \|u\|^2 \\
&\le \frac{3}{4}\ \|\pi y - \pi x\|^2 + \|y - x\|\ \|\pi y - \pi x\| + \frac{\delta}{2\,r}\ \|\pi y - \pi x\|^2 \\
&\le C\ \|y - x\|^2\ ,
\end{aligned}$$

where in the last step we use that $\pi$ is Lipschitz in any neighbourhood $N_\delta = \{\, x \in \mathbb{R}^n\ :\ d(x, \partial U) < \delta\,\}$, and the constant $C = C_\delta$ is given explicitely by

$$C_\delta = \frac{3}{4}\,\frac{2\,r}{2\,r - 3\,\delta} + \sqrt{\frac{2\,r}{2\,r - 3\,\delta}} + \frac{\delta}{2\,r}\,\frac{2\,r}{2\,r - 3\,\delta}\ .$$

□

Because function $f$ has gradient $\eta \not\equiv 0$ along $\partial U$ it follows that $\partial U = f^{-1}(0)$ is a codimension one boundaryless manifold of class $C^1$, which proves Theorem 1.

We note that Theorem 1 can not be improved to higher smoothness classes, as simple examples like the union of a family of radius $r$ balls with centre in some closed interval show (see Fig. 3).

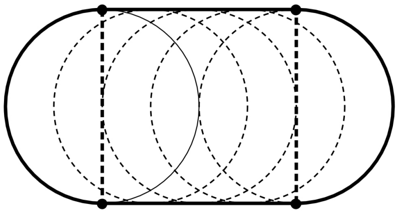

**Fig. 3** An $r$-regular set with $C^1$ but not $C^2$ boundary

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