Stability of non-deterministic systems

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Abstract A space of non-deterministic dynamical systems of Markov type on compact manifolds is considered. This is a natural space for stochastic perturbations of maps. For such systems, both the combinatorial stability, of the periodic attractors, and the spectral stability, of the invariant measures, are characterized and its genericity established.

1 Introduction

Given a state space X, any function f that associates to each state $x \in X$ a state probability transition f_x on X will be called a (discrete time) stochastic dynamical system or, simply, a Markov system. Deterministic dynamical systems correspond to such functions when each value $f_x = \delta_{f(x)}$ is a Dirac measure sitting on some point $f(x) \in X$.

In the present work we will consider a space $\mathscr{H}(X)$ of stochastic dynamical systems defined on a compact Riemannian manifold X, with volume measure m, which is large and natural to make stochastic perturbations in continuous deterministic dynamical systems. Our main goal is to study and compare, for generic systems $f \in \mathscr{H}(X)$, the combinatorial-topological stability of the limit behaviour of the nondeterministic system $\varphi_f : x \mapsto supp(f_x)$, with the spectral stability of the linear operator $\mathscr{L}_f : \mu \mapsto f_* \mu$, that to each probability distribution μ associates the μ-conditional probability distribution in the next instant, also known as the Perron-Frobenius operator.

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Given a point-set map $\varphi : X \to \mathscr{P}(X)$, any sequence $x_0, x_1, \cdots, x_n$ such that $x_i \in \varphi(x_{i-1})$ for $i = 1, \cdots, n$ will be called an *orbit* of φ, and we shall say that x_n is an *iterate* of the state x_0. If there is $\varepsilon > 0$ such that $d(x_i, \varphi(x_{i-1})) < \varepsilon$ for all $i = 1, \ldots, n$ then $x_0, x_1, \cdots, x_n$ will be called an *ε-pseudo orbit* of φ and we shall say that x_n is an *ε-pseudo iterate* of x_0. We shall say that x_n is a *pseudo iterate* of x_0 if for every $\varepsilon > 0$, x_n is an ε-pseudo iterate of x_0.

The *recurrent set* of φ, denoted by $\Omega(\varphi)$, is the set of all states $x \in X$ such that x is an iterate of x. Analogously, the *chain recurrent set* of φ, denoted by $R(\varphi)$, is the set of all states $x \in X$ such that x is a pseudo iterate of x.

The point-set map φ has two limit sets: the *Ω-final*, or *final recurrent set*, denoted by $\Omega_{\text{final}}(\varphi)$, of all states $x \in \Omega(\varphi)$ such that every iterate of x still has some iterate which comes back to x, and the *R-final*, or *final chain recurrent set*, denoted by $R_{\text{final}}(\varphi)$, of all states $x \in R(\varphi)$ such that every pseudo iterate of x still has some pseudo iterate which comes back to x. These limit sets contain all the asymptotic dynamical behaviour of the point-set map φ.

A point-set map φ is called *open* when its graph

$$\text{graph}\,(\varphi) = \{(x,y) \in X \times X \,:\, y \in \varphi(x), \forall x \in X\,\}$$

is an open set in $X \times X$, and $\varphi(x)$ is connected for all $x \in X$.

Theorem
If φ is an open map then both limit sets $\Omega_{\text{final}}(\varphi)$ and $R_{\text{final}}(\varphi)$ decompose into a finite number of connected pieces which are permuted by φ.

The restriction of φ to each of these pieces is, in some sense, irreducible.

The system φ will be called *combinatorially stable* when this topological decomposition is stable under perturbations. Since any iterate is also a pseudo iterate, there is a natural relation between the connected pieces of $\Omega_{\text{final}}(\varphi)$ and those of $R_{\text{final}}(\varphi)$. We shall see that for a generic system φ, with respect to some natural topology, this relation is bijective, and that whenever this happens the system is combinatorially stable, thus obtaining:

Theorem
There is an open and dense set of combinatorially stable systems.

Given $f \in \mathscr{H}(X)$, both topological decompositions for the limit sets of φ_f, correspond to the unit circle spectral decomposition of the linear operator $\mathscr{L}_f$ acting on the Banach space $L^1(X,m)$. There is an $\mathscr{L}_f$-invariant spectral decomposition $L^1(X,m) = E_0(f) \oplus E_1(f)$ which corresponds to the spectrum partition $\sigma\left(\mathscr{L}_f\right) = \sigma_0\left(\mathscr{L}_f\right) \cup \sigma_1\left(\mathscr{L}_f\right)$, where

$$\sigma_0\left(\mathscr{L}_f\right) = \{\lambda \in \sigma\left(\mathscr{L}_f\right) \;:\; |\lambda| < 1\,\}\,,$$

and

$$\sigma_1\left(\mathscr{L}_f\right) = \{\lambda \in \sigma\left(\mathscr{L}_f\right) \ : \ |\lambda| = 1\ \} \,.$$

This second component of the spectrum consists of a finite number of eigenvalues, all with finite multiplicity. Hence $\dim E_1(f) < +\infty$, and the operator $\mathscr{L}_f$ is quasi-compact.

The system f will be called *spectrally stable* if there is some $0 < k < 1$ such that for every small perturbation g of f:

a) $\mathscr{L}_f\,|_{E_1(f)}$ is conjugated to $\mathscr{L}_g\,|_{E_1(g)}$;
b) $\left\|\mathscr{L}_g\,|_{E_0(g)}\right\| \leq k$.

In our setting the operator $\mathscr{L}_f$ depends continuously on f and, therefore, so does the spectrum $\sigma\left(\mathscr{L}_f\right)$. The spectral stability of f relates with the fact that no eigenvalue can enter, or leave, the unit circle.

In section 5 we establish the spectral stability for a generic system f, with respect to some natural metric in $\mathscr{H}(X)$.

Theorem
There is an open and dense set of systems $f \in \mathscr{H}(X)$ for which φ_f is combinatorially stable and f is spectrally stable.

We are naturally led to consider finite state Markov chains when trying to approximate a continuous dynamical system by discretizing the manifold X. Finite state Markov chains are the stochastic, or random dynamical systems on a finite state space. One may think that these dynamical systems are what we actually see when running computer simulations of deterministic dynamical systems. Each such dynamical system is specified by a *stochastic matrix* with the state probability transitions. The stochastic matrix is the Perron operator of this finite state system. The Markov chain also determines an *oriented graph*, encapsulating some qualitative aspects of the system behaviour. The theory of finite state Markov chains establishes a correspondence between spectral properties of the stochastic matrix on one side, and combinatorial properties of the corresponding graph on the other hand. See, e.g., [1].

For measurable spaces and Markov systems satisfying the *Doeblin-condition* similar spectral results were obtained in a more general setting by Doob [2], but the stability problem is never addressed. As far as we know the topological-combinatorial approach of our work is also new (see [3] and [4]).

In the theory of smooth deterministic hyperbolic systems $f : X \to X$ the *spectral decomposition theorem* states that there is a decomposition of the non-wandering set $\Omega(f)$ into a finite number of hyperbolic basic sets which are permuted by f. The dynamics of f partially orders the basic set components of $\Omega(f)$, the minimal, or final, elements being the *attractors* of f. Analogously, in our setting, there are partially ordered decompositions of the recurrent and chain recurrent sets, $\Omega(\varphi)$ and $R(\varphi)$, respectively. The final connected components in $\Omega_{\text{final}}(\varphi)$, and $R_{\text{final}}(\varphi)$, are the *attractors*' equivalent in $\Omega(\varphi)$, and $R(\varphi)$, respectively. As the name indicates this decomposition relates with the spectral decomposition of the linear operator which describes the action of f on the tangent vector fields to X. The ergodic theory of these systems is also well studied. It is well known that in each attractor there is a unique ergodic stochastically stable measure, called the *physical measure* of the attractor. Furthermore, almost every point $x \in X$ lies in the basin of attraction of one of these physical measures. The concept of stochastic stability is usually attributed to Kolmogorov. Roughly, a measure μ is said stochastically stable if it is stable under small stochastic perturbations of the system f. In general it has been conjectured by J. Palis [6] that for a dense set of dynamics, *the system has a finite set of transitive, stochastically stable attractors whose basins of attraction cover almost every point in X*. This conjecture suggested the main motivation for the present study: to understand stochastic stability in the realm of stochastic dynamical systems, at least in a class of Markov systems which is suitable for stochastic perturbations of continuous maps.

2 Topological semigroups of open maps

Several semigroups of point-set maps are defined, namely open, continuous and Lipschitz point-set maps. The key concept of topological semigroup of open maps is introduced.

Throughout this work X will denote a compact Riemannian manifold of dimension n, d will be the geodesic distance on X and m will be the corresponding normalized $(m(X) = 1)$ Riemannian volume. Similar notation will be used on $X \times X$, where d will stand for the metric $d((x_1,y_1),(x_2,y_2)) = \max\{d(x_1,x_2), d(y_1,y_2)\}$.

Let $\mathscr{S}(X)$ be the space of all point-set maps on X, that is, $\varphi \in \mathscr{S}(X)$ if and only if φ is a map from X into the power set of X, i.e. $\varphi : X \to \mathscr{P}(X)$. For a point-set map $\varphi \in \mathscr{S}(X)$ and a subset $A \subseteq X$ the *image* $\varphi(A) \in \mathscr{P}(X)$ is defined by $\varphi(A) = \cup_{x\in A}\varphi(x)$. We say that $A \subseteq X$ is *φ-invariant* when $\varphi(A) \subseteq A$. Analogously, we say that A is *fully φ-invariant* if $\varphi(A) = A$. For two point-set maps $\varphi, \psi \in \mathscr{S}(X)$ the usual composition product $\varphi \circ \psi : X \to \mathscr{P}(X)$ of φ and ψ at x is defined by

$$(\varphi \circ \psi)(x) = \varphi(\psi(x)) = \cup_{y\in\psi(x)}\varphi(y) .$$

Clearly, $\mathscr{S}(X)$ with this composition product is a semigroup.

We define $\mathscr{O}(X)$ to be the space of all open point-set maps φ in $\mathscr{S}(X)$. See the definition of open map in the introduction.

Proposition 1. ([3, Proposition 4.1.]) *Given $\varphi \in \mathscr{O}(X)$ and an open set $C \subseteq X$, if C is connected then $\varphi(C)$ is open and connected.*

From the previous proposition we easily have the following.

Proposition 2. ([3, Proposition 4.2.]) *$\mathscr{O}(X)$ is a subsemigroup of $\mathscr{S}(X)$.*

Given sets $U,V \subseteq X$ we will use the following notation: $B_r(U)$, respectively $\overline{B}_r(U)$, denotes the set of all points whose distance d to U is less than, respectively less than or equal to r, and $\rho_H(U,V) = \inf\{\, r > 0 \, : U \subseteq B_r(V) \text{ and } V \subseteq B_r(U) \,\}$ denotes the Hausdorff distance between U and V. Furthermore, $\overline{U}$ is the closure of U in X and U° is the interior of U in X. Similar notation will be used on $X \times X$.

We say that a point-set map is *Lipschitz* if and only if the map $x \mapsto \varphi(x)$ is Lipschitz with respect to the Hausdorff distance ρ_H, i.e. there is $K > 0$ such that $\rho_H(\varphi(x), \varphi(y)) \le K\, d(x,y)$ for every $y \in X$. We shall denote by $\mathrm{Lip}(\varphi)$ the greatest lower bound of all Lipschitz constants K for this map. We denote by $\mathscr{O}_{\mathrm{Lip}}(X)$ the subset of all Lipschitz point-set maps in $\mathscr{O}(X)$.

We recall the following continuity concepts. A point-set map $\varphi : X \to \mathscr{P}(X)$ with non-empty values is called *lower semi-continuous*, respectively *upper semi-continuous*, if for every $x \in X$ and $\varepsilon > 0$ there is a neighborhood N_x of x in X such that for any $y \in N_x$, $\varphi(x) \subseteq B_\varepsilon(\varphi(y))$, respectively $\varphi(y) \subseteq B_\varepsilon(\varphi(x))$. It is called *continuous* if $\varphi : X \to \mathscr{P}(X)$ is both lower semi-continuous and upper semi-continuous. This means continuity with respect to the Hausdorff distance. We denote by $\mathscr{O}_{\mathrm{Cont}}(X)$ the subset of all continuous point-set maps in $\mathscr{O}(X)$.

Proposition 3. ([3, Proposition 4.6.]) *$\mathscr{O}_{\mathrm{Lip}}(X)$ and $\mathscr{O}_{\mathrm{Cont}}(X)$ are both subsemigroups of $\mathscr{O}(X)$.*

Given $\varphi \in \mathscr{O}(X)$ we define $\overline{\varphi}$, $\widehat{\varphi} : X \to \mathscr{P}(X)$ by setting

$$\mathrm{graph}\,(\overline{\varphi}) = \overline{\mathrm{graph}\,(\varphi)} \quad \text{and}$$

$$\mathrm{graph}\,(\widehat{\varphi}) = \left(\overline{\mathrm{graph}\,(\varphi)}\right)^\circ ,$$

We call the open map $\widehat{\varphi}$, whose graph is the interior of the closure of $\mathrm{graph}\,(\varphi)$, the *regularization* of φ. The following proposition is easily proved.

Proposition 4. ([3, Proposition 4.7.])

(1) *All maps φ in $\mathscr{O}(X)$ are lower semi-continuous.*
(2) *For every map $\varphi \in \mathscr{O}(X)$, $\overline{\varphi}$ is upper semi-continuous.*
(3) *Every map $\varphi \in \mathscr{O}_{\mathrm{Lip}}(X)$ is continuous.*

Because there are several natural non-equivalent ways of endowing $\mathscr{O}(X)$, and its sub-semigroups, with some topology we give the following abstract definition. Consider any sub-semigroup of open maps $\mathscr{O}_1 \subseteq \mathscr{O}(X)$, endowed with some topology.

Definition 1. We say that $\mathscr{O}_1$ is a *topological semigroup of open maps* if

(1) the Hausdorff distance between open map graphs is continuous;
(2) for each $\varphi \in \mathscr{O}_1$, there is a family of open maps $\{\tilde{\varphi}_\varepsilon\}_{\varepsilon>0}$ in $\mathscr{O}_1$ such that

   (a) $\overline{\mathrm{graph}\,(\varphi)} = \bigcap_{\varepsilon>0} \mathrm{graph}\,(\tilde{\varphi}_\varepsilon)$;

(b) for all ε_1, ε_2, if $\varepsilon_1 > \varepsilon_2 > 0$ then $\overline{\mathrm{graph}\,(\tilde{\varphi}_{\varepsilon_2})} \subseteq \mathrm{graph}\,(\tilde{\varphi}_{\varepsilon_1})$; and
(c) $\lim_{\varepsilon \to 0^+} \tilde{\varphi}_\varepsilon = \varphi$ w.r.t. $\mathscr{O}_1$ topology;

(3) given $\varepsilon > 0$, an integer $N \in \mathbb{N}$, and non-empty open subsets $U, V \subseteq X$ such that $\overline{U \times V} \subseteq \mathrm{graph}\left(\varphi^N\right)$, there is a neighborhood $\mathscr{N}$ of φ in $\mathscr{O}_1$ such that for all $\psi \in \mathscr{N}$ and $x \in \overline{U}$, $m(V \setminus \widehat{\psi^N}(x)) < \varepsilon$, where $\widehat{\psi}$ denotes the regularization of ψ.

Condition (2) above is an outer continuity assumption that says every open map φ can be well approximated from above within the topology. Condition (3) expresses a kind of inner, or lower, continuity.

Identifying each $\varphi \in \mathscr{O}(X)$ with its graph we can see $\mathscr{O}(X)$ as a subset of the space of all non-empty connected open subsets of $X \times X$. Therefore, we can consider on $\mathscr{O}(X)$ and its sub-semigroups, topologies induced from general (topological) spaces of open sets. See [5] for an overview on topological spaces of sets. We shall now topologize $\mathscr{O}(X)$ with a topology that is natural to address the subtle concept of *combinatorial stability* for continuous deterministic dynamical systems (see [3, Section 6]). First let $\mathscr{U}(X)$ denote the space of all non-empty connected open subsets of X. We define the following pseudo-metric ρ in $\mathscr{U}(X)$. Given $U, V \in \mathscr{U}(X)$,

$$\rho(U,V) = \max\left\{\rho_H(U,V), \rho_H(U^e,V^e)\right\},$$

where ρ_H stands for the Hausdorff distance and U^e denotes the exterior of U in X. Now consider $\mathscr{O}(X)$ as a subset of $\mathscr{U}(X \times X)$ and let ρ be the induced pseudo-metric, which is given by

$$\rho(\varphi, \psi) = \rho\left(\mathrm{graph}\,(\varphi), \mathrm{graph}\,(\psi)\right).$$

Proposition 5. *With the topology associated to ρ, $\mathscr{O}(X)$ is a topological semigroup of open maps.*

To prove Proposition 5, we first introduce the following two families of open maps. Given $\varphi \in \mathscr{O}(X)$, define φ^*_ε by

$$\mathrm{graph}\,(\varphi^*_\varepsilon) = B_\varepsilon(\mathrm{graph}\,(\varphi)),$$

and define $\varphi^\circ_\varepsilon$ setting $\varphi^\circ_\varepsilon(x)$ to be the largest connected component of the open set

$$\{\, y \in X \;:\; d\left((x,y), \mathrm{graph}\,(\varphi)^c\right) > \varepsilon \,\},$$

where $\mathrm{graph}\,(\varphi)^c$ denotes the complement of $\mathrm{graph}\,(\varphi)$ in $X \times X$. Then $\varphi^*_\varepsilon \in \mathscr{O}(X)$, and $\varphi^\circ_\varepsilon \in \mathscr{O}(X)$ for all small enough $\varepsilon > 0$.

Next we provide the following characterization of the ε-ball for the pseudo-metric ρ. Given open maps $\varphi, \psi : X \to \mathscr{P}(X)$, we will write $\varphi \prec \psi$ to mean that $\overline{\mathrm{graph}\,(\varphi)} \subseteq \mathrm{graph}\,(\psi)$.

Proposition 6. *Given $\varepsilon > 0$, for every $\varphi, \psi \in \mathscr{O}(X)$,*

$$\rho(\varphi,\psi) < \varepsilon \quad \Longrightarrow \quad \psi_\varepsilon^\circ \prec \widehat{\varphi} \prec \psi_\varepsilon^* \,.$$

Proof. First $\rho_H(\varphi,\psi) < \varepsilon$ implies that

$$\overline{\text{graph}\,(\widehat{\varphi})} \subseteq \text{graph}\,(\overline{\varphi}) \subseteq B_\varepsilon(\text{graph}\,(\psi)) = \text{graph}\,(\psi_\varepsilon^*)\,.$$

On the other hand $\rho_H(\text{graph}\,(\varphi)^e, \text{graph}\,(\psi)^e) < \varepsilon$ implies that

$$\text{graph}\,(\varphi)^e \subseteq B_\varepsilon(\text{graph}\,(\psi)^e) \subseteq B_\varepsilon(\text{graph}\,(\psi)^c),$$

which in turn implies that $\psi_\varepsilon^\circ \prec \widehat{\varphi}$. □

We prove now Proposition 5.

Proof. It is clear that Definition 1(1) holds. To prove Definition 1(2) we just need to take $\tilde{\varphi}_\varepsilon \equiv \varphi_\varepsilon^*$. To prove Definition 1(3), let $U, V \subseteq X$ be non-empty open sets such that $\overline{U \times V} \subseteq \text{graph}\left(\varphi^N\right)$. Taking $\delta > 0$ small enough we have $\overline{U \times V} \subseteq \text{graph}\left((\varphi_\delta^\circ)^N\right)$. Consider the $\delta-$neighbourhood $\mathscr{N} = B_\delta(\varphi)$ with respect to the pseudo-metric ρ. If $\psi \in \mathscr{N}$ then, by Proposition 6, $\varphi_\delta^\circ \prec \widehat{\psi}$, implying that

$$\overline{U \times V} \subseteq \text{graph}\left((\varphi_\delta^\circ)^N\right) \subseteq \text{graph}\left((\widehat{\psi})^N\right) \subseteq \text{graph}\left(\widehat{\psi^N}\right)\,.$$

Therefore, $m(V \setminus \widehat{\psi^N}(x)) = 0$ for all $x \in \overline{U}$. □

3 Combinatorial stability of open maps

Combinatorial stability of open maps is defined and characterized. Its genericity is proved.

Let us briefly recall the main dynamical concepts for open maps (see [3]). Given $\varphi \in \mathscr{O}(X)$, a sequence $x_0, x_1, \cdots, x_n$ such that $x_i \in \varphi(x_{i-1})$ for all $i = 1, \cdots, n$ is called a $\varphi-$*orbit*, and we say that x_n is a $\varphi-$*iterate* of x_0. If for every $\varepsilon > 0$, y is a φ_ε^*-iterate of x, where φ_ε^* is the open map whose graph is an $\varepsilon-$radius ball of graph(φ), we say that y is a $\varphi-$*pseudo-iterate* of x. The *recurrent* and *chain-recurrent sets* of φ are defined respectively by $\Omega(\varphi) = \{x \in X : x \text{ is a } \varphi-\text{iterate of } x\}$ and $R(\varphi) = \{x \in X : x \text{ is a } \varphi-\text{ pseudo-iterate of } x\}$. Both these sets split into equivalence classes, each class being formed by states which are accessible from each other. The set of all these classes is then partially ordered by the dynamics of φ. At the bottom of this hierarchy are two special limit sets: the *final recurrent* and the *final chain-recurrent sets*, denoted respectively by $\Omega_{\text{final}}(\varphi)$ and $R_{\text{final}}(\varphi)$, of all states $x \in \Omega(\varphi)$ ($x \in R(\varphi)$) such that every iterate (pseudo-iterate) of x still has some iterate (pseudo-iterate) which comes back to x. These limit sets contain all the asymptotic dynamical behaviour of φ. They both decompose into a finite number of equivalence classes, called respectively $\Omega-$final and $R-$final classes. We denote by $\Lambda^{\Omega}_{\text{final}}(\varphi)$ respectively $\Lambda^{R}_{\text{final}}(\varphi)$ the set of all equivalence classes of the limit sets $\Omega_{\text{final}}(\varphi)$ and $R_{\text{final}}(\varphi)$. Each $\Omega-$final and $R-$final class decomposes into a finite number of connected pieces, called respectively *$\Omega-$final* and *$R-$final components*, which are permuted by φ. See Theorems 5.1 and 5.2 of [3]. The restriction of φ to each of these pieces is in some sense irreducible. We call period of a final class to the number of its connected components. The period of a connected component is the period of its class. We denote by $\Sigma^{\Omega}_{\text{final}}(\varphi)$ respectively $\Sigma^{R}_{\text{final}}(\varphi)$ the set of connected pieces of the limit sets $\Omega_{\text{final}}(\varphi)$ and $R_{\text{final}}(\varphi)$. Thus, each open map $\varphi \in \mathscr{O}(X)$ induces a permutation π_φ on the set $\Sigma^{\Omega}_{\text{final}}(\varphi)$ of $\Omega-$final components.

Definition 2. Let φ, $\psi \in \mathscr{O}(X)$. We say that φ is *combinatorially equivalent* to ψ, and write $\varphi \bowtie \psi$, if and only if the permutations π_φ and π_ψ are conjugated, that is, there is a bijective map $h : \Sigma^{\Omega}_{\text{final}}(\varphi) \to \Sigma^{\Omega}_{\text{final}}(\psi)$ such that the following diagram comutes:

$$\begin{array}{ccc} \Sigma^{\Omega}_{\text{final}}(\varphi) & \xrightarrow{h} & \Sigma^{\Omega}_{\text{final}}(\psi) \\ \pi_\varphi \big\downarrow & & \big\downarrow \pi_\psi \\ \Sigma^{\Omega}_{\text{final}}(\varphi) & \xrightarrow{h} & \Sigma^{\Omega}_{\text{final}}(\psi) \end{array} .$$

Definition 3. Given a topological subsemigroup $\mathscr{O}_1 \subseteq \mathscr{O}(X)$, we say that $\varphi \in \mathscr{O}(X)$ is *combinatorially stable in* $\mathscr{O}_1$ if and only if there is a neighbourhood $\mathscr{U}$ of φ in $\mathscr{O}_1$ such that all $\psi \in \mathscr{U}$ are combinatorially equivalent to φ.

Theorem 1 (Combinatorial stability characterization). *For any $\varphi \in \mathscr{O}(X)$, φ is combinatorially stable in $(\mathscr{O}(X), \rho)$ if and only if φ satisfies the following combinatorial stability condition: φ induces the same permutation on $\Sigma^{\Omega}_{\text{final}}(\varphi)$ and $\Sigma^{R}_{\text{final}}(\varphi)$, or, equivalently, $\left|\Lambda^{\Omega}_{\text{final}}(\varphi)\right| = \left|\Lambda^{R}_{\text{final}}(\varphi)\right|$ and $\left|\Sigma^{\Omega}_{\text{final}}(\varphi)\right| = \left|\Sigma^{R}_{\text{final}}(\varphi)\right|$.*

Proof. In [3, Theorem 5.3.]) we have proved that given any topological semigroup of open maps $\mathscr{O}_1$, φ is combinatorially stable in $\mathscr{O}_1$ if and only if φ satisfies the combinatorial stability condition. Thus the proof follows immediately in view of Proposition 5. □

Theorem 2 (Genericity of combinatorial stability). *The set of combinatorially stable systems is open and dense in $(\mathscr{O}(X), \rho)$.*

Proof. In [3, Theorem 5.4.]) we have proved that for any topological semigroup of open maps $\mathscr{O}_1$, the set of $\mathscr{O}_1$-combinatorially stable maps is open and dense in the semigroup $\mathscr{O}_1$. Thus the proof follows immediately in view of Proposition 5. □

4 Topological semigroups of Markov systems

A semigroup $\mathscr{H}(X)$ of Markov systems is defined. The key concept of topological semigroup of Markov systems is introduced. For each $f \in \mathscr{H}(X)$, the Perron-Frobenius operator $\mathscr{L}_f$ is recalled. Invariant measures are defined.

We denote by $\mathscr{M}_{\text{prob}}(X)$ the space of all Borel probability measures on the compact manifold X. This is a subset of the Banach space $\mathscr{M}(X)$ of all finite Borel real measures on X, with the usual total variation norm $\|\mu\|$. $\mathscr{M}(X)$ is the dual of the Banach space of continuous real-valued functions on X, denoted here by $C^0(X)$, endowed with the uniform proximity norm $\|\cdot\|_\infty$. The space $\mathscr{M}_{\text{prob}}(X)$ is a compact and convex subset of $\mathscr{M}(X)$ with respect to the *weak-$*$* topology, which is the weak topology of $\mathscr{M}(X)$ as dual of $C^0(X)$. We will call here *Markov system* to any weak-$*$ continuous mapping $p : X \to \mathscr{M}_{\text{prob}}(X)$. The probability measure $p(x) = p_x$ is referred as the transition probability at state $x \in X$. We denote by $\mathscr{M}\mathscr{S}(X)$ the set of all Markov systems. A Markov system $p : X \to \mathscr{M}_{\text{prob}}(X)$ will also be referred as a stochastic dynamical system. A Markov system is called deterministic if for some continuous mapping $f : X \to X$, we have $p(x) = \delta_{f(x)}$ for every $x \in X$, where $\delta_{f(x)}$ denotes the Dirac measure sitting at the point $f(x)$. The Perron-Frobenius operator of a Markov system $p : X \to \mathscr{M}_{\text{prob}}(X)$ is the linear operator $\mathscr{L}_p : \mathscr{M}(X) \to \mathscr{M}(X)$, defined by

$$\mathscr{L}_p(\mu) = \int_X p(x)\, d\mu(x)\,, \qquad \text{for every } \mu \in \mathscr{M}(X)\,.$$

The integral of the measure-valued function p is well defined, in a sense that can be found, for instance, in [7]. The adjoint operator $\mathscr{L}_p^* : C^0(X) \to C^0(X)$, is given by

$$\mathscr{L}_p^*(\psi)(x) = \int_X \psi(y)\, d\, p_x(y)\,, \qquad \text{for every } \psi \in C^0(X)\,.$$

Both $\mathscr{L}_p$ and $\mathscr{L}_p^*$ are bounded linear operators with norms less or equal than 1. The convolution of two Markov systems $p, q \in \mathscr{M}\mathscr{S}(X)$ is $p * q : X \to \mathscr{M}_{\text{prob}}(X)$, where

$$(p * q)(x) = \mathscr{L}_p(q_x) = \mathscr{L}_p(\mathscr{L}_q(\delta_x)) \qquad \text{for every } x \in X\,.$$

The space $(\mathscr{M}\mathscr{S}(X), *)$ is a semigroup with identity, where the identity is the deterministic Markov system $x \mapsto \delta_x$. The map $p \mapsto \mathscr{L}_p$ is a semigroup homomorphism taking $\mathscr{M}\mathscr{S}(X)$ into the algebra of bounded linear operators on the Banach space $\mathscr{M}(X)$. We will say that a measure $\mu \in \mathscr{M}(X)$ is $p-$invariant when $\mathscr{L}_p\mu = \mu$, and that a measurable set $A \subseteq X$ is $p-$invariant when $\mathscr{L}_p^*\chi_A = \chi_A$, where χ_A denotes the characteristic function $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in X - A \end{cases}$.

We denote by $L^1(X,m)$ the Banach space of m-integrable functions on X with the usual L^1-norm, $\|h\|_1 = \int_X |h(x)|\, dm(x)$. This space is isometrically embedded in $\mathscr{M}(X)$ through the inclusion map $L^1(X,m) \hookrightarrow \mathscr{M}(X)$, $h \mapsto hm$. We say that a Markov system $p : X \to \mathscr{M}(X)$ is *absolutely continuous with respect to m* if $p_x =$

$f_x m$, with $f_x \in L^1(X,m)$, for every $x \in X$. Absolutely continuous Markov systems are defined by *stochastic transition functions* $f : X \times X \to \mathbb{R}$ such that:

(a) $f(x,y)$ is measurable on $X \times X$,
(b) $f(x,y) \geq 0$, for every $(x,y) \in X \times X$,
(c) $\int_X f(x,y)\, dm(y) = 1$, for every $x \in X$,
(d) the real valued function $X \to \mathbb{R}$, $x \mapsto \int f(x,y)\, \psi(y)\, dm(y)$, is continuous for every test function $\psi \in C^0(X)$.

A function $f : X \times X \to \mathbb{R}$ satisfying (a), (b), (d) and

(c') $\int_X f(x,y)\, dm(y) \leq 1$, for every $x \in X$,

is called a *sub-stochastic transition function*.

The subset of all absolutely continuous Markov systems forms a sub-semigroup, without identity, of $\mathscr{M}\mathscr{S}(X)$. Given two transition functions $f,g : X \times X \to \mathbb{R}$, the convoluted Markov system is defined by the usual function convolution

$$(f * g)(x,z) = \int_X f(x,y) g(y,z)\, dm(y) \; .$$

From now on we shall identify each absolutely continuous Markov system with its probability transition function $f : X \times X \to \mathbb{R}$. Given any such absolutely continuous Markov system f, the operator $\mathscr{L}_f$ takes $\mathscr{M}(X)$ into $L^1(X,m)$ and its restriction to $L^1(X,m)$ is given by

$$\mathscr{L}_f(q)(y) = \int_X q(x)\, f(x,y)\, dm(x) \qquad q \in Ł^1(X,m) \; .$$

The adjoint action on $L^\infty(X,m)$ is given by

$$\mathscr{L}_f^*(g)(x) = \int_X f(x,y)\, g(y)\, dm(y) \qquad g \in Ł^\infty(X,m) \; .$$

Given a Markov system $p : X \to \mathscr{M}_{\text{prob}}(X)$, $\sigma(\mathscr{L}_p)$ will denote the spectrum of the Perron-Frobenius operator $\mathscr{L}_p$. The *spectral radius* of $\mathscr{L}_p$, i.e. the lowest upper bound for absolute values of elements in $\sigma(\mathscr{L}_f)$, will be denoted by $r(\mathscr{L}_p)$. Of course $r(\mathscr{L}_p) = 1$. The *discrete spectrum* of $\mathscr{L}_p$, i.e. the set of all eigenvalues in $\sigma(\mathscr{L}_p)$ that are isolated and have finite multiplicity, will be denoted by $\sigma_{\text{disc}}(\mathscr{L}_p)$. The complement of $\sigma_{\text{disc}}(\mathscr{L}_p)$ in $\sigma(\mathscr{L}_p)$ is called the *essential spectrum* of $\mathscr{L}_p$, and denoted by $\sigma_{\text{ess}}(\mathscr{L}_p)$. The *essential spectral radius* of $\mathscr{L}_p$, i.e. the lowest upper bound for absolute values of elements in $\sigma_{\text{ess}}\left(\mathscr{L}_f\right)$, is denoted by $r_{\text{ess}}(\mathscr{L}_p)$. It is well known, see for instance [8], that the Perron operator $\mathscr{L}_f$ of any absolutely continuous Markov system f is a weakly compact operator. In particular, $r_{\text{ess}}\left(\mathscr{L}_f\right) = 0$ and, therefore, the spectrum $\sigma(\mathscr{L}_f)$ is at most countable. All spectrum points in $\sigma(\mathscr{L}_f) - \{0\}$ are isolated eigenvalues with finite multiplicity. Given an absolutely continuous Markov system f, we can decompose the spectrum of $\mathscr{L}_f$ as:

$$\sigma(\mathscr{L}_f) = \sigma_0(\mathscr{L}_f) \cup \sigma_1(\mathscr{L}_f),$$

where $\sigma_0(\mathscr{L}_f) = \{\lambda \in \sigma(\mathscr{L}_f) : |\lambda| < 1\}$, and $\sigma_1(\mathscr{L}_f) = \sigma(\mathscr{L}_f) - \sigma_0(\mathscr{L}_f)$. Of course $\sigma_1(\mathscr{L}_f)$ is finite while $\sigma_0(\mathscr{L}_f)$ is at most countable but closed for the complex plane topology. Consequently, $\sigma_0(\mathscr{L}_f)$ and $\sigma_1(\mathscr{L}_f)$ are disjoint compact sets and, therefore, there is an associated decomposition of $L^1(X,m)$ into two $\mathscr{L}_f$-invariant subspaces:

$$L^1(X,m) = E_0(f) \oplus E_1(f) ,$$

where $E_1(f)$ has finite dimension. We shall denote by $r_{\text{int}}(\mathscr{L}_f)$ the *interior spectral radius* of $\mathscr{L}_f$, i.e. the lowest upper bound of all absolute values of elements in $\sigma_0(\mathscr{L}_f)$.

Given any absolutely continuous Markov system f a sequence $x_0, x_1, \cdots, x_n$ such that $f(x_{i-1}, x_i) > 0$ for all $i = 1, \cdots, n$ is called an $f-$*orbit*, and we say that x_n is an $f-$*iterate* of x_0. An absolutely continuous Markov system is called *irreducible* if for almost all points $x, y \in X$ there is some $n \in \mathbb{N}$ such that the probability transition density from x to y in n iterates is positive, i.e. $f^n(x,y) > 0$. A *recurrence time* is any integer $n \in \mathbb{N}$ such that the set $E_n = \{x \in X : f^n(x,x) > 0\}$ has positive measure. Given an absolutely continuous irreducible Markov system f the greatest common divisor d of all recurrence times $n \in \mathbb{N}$ is called the period of f. An irreducible Markov system f is called *acyclic* if it has period one. The state space X of an irreducible Markov system f of period d can be decomposed into a finite union of f^d-invariant subsets $X = X_0 \cup \cdots \cup X_{d-1}$ such that each restriction $(f^d)_{X_i} : X_i \times X_i \to \mathbb{R}$, is an irreducible acyclic Markov system on X_i.

We shall denote by f_R the restriction to $R \times R$ of a given function $f : X \times X \to \mathbb{R}$, for any subset $R \subseteq X$. If f is stochastic transition function then:

1. f_R is a sub-stochastic transition function.
2. f_R is a stochastic transition function $\Leftrightarrow$ R is f-invariant.

Let $\mathscr{H}(X)$ be the set of all absolutely continuous Markov systems (i.e. probability transition functions) $f : X \times X \to \mathbb{R}$ satisfying the following extra conditions:

(1) f is bounded on $X \times X$;
(2) f is lower semi-continuous on $X \times X$;
(3) for each $x \in X$, the open set $\varphi_f(x) = \{y \in X : f(x,y) > 0\}$ is connected.

The space $\mathscr{H}(X)$ is a convolution sub-semigroup of $\mathscr{M}\mathscr{S}(X)$. Item (2) in the definition of $\mathscr{H}(X)$ ensures that $\varphi_f \in \mathscr{O}(X)$. Thus, this semigroup carries a natural homomorphism $\varphi : \mathscr{H}(X) \to \mathscr{O}(X)$.

Given $f \in \mathscr{H}(X)$ and an open φ_f-invariant set $R \subseteq X$, let

$$\tau_f(R) = \frac{1}{2} \sup_{x,z\in R} \int_R |f(x,y) - f(z,y)|\, dm(y) \tag{1}$$

$$= 1 - \min_{x,z\in R} \int_R f(x,y) \wedge f(z,y)\, dm(y)\ .$$

and

$$\tau_f^*(R) = \inf_{n\geq 1} \left[\, \tau_{f^n}(R) \right]^{1/n}\ .$$

The quantity $-\ln\left(\tau_f^*(R)\right)$ is a kind of *mixing rate* for the action of $\mathscr{L}_f$ on $\mathscr{M}_{\mathrm{prob}}(R)$, which measures how fast the $\mathscr{L}_f$-iterates of any probability distribution on R will converge to the unique $\mathscr{L}_f$-invariant measure supported in R. Next, we make some trivial remarks on this concept:

1. $\tau_{f^n}(R) = 0 \Leftrightarrow$ the transition probabilities $f_x^n(\cdot) = f^n(x,\cdot)$ do not depend on x, for x over R.
2. If for some pair of points $x, y \in R$, the transition probabilities f_x^n and f_y^n have disjoint supports, then $\tau_{f^n}(R) = 1$.
3. If $\tau_f^*(R) < 1$ then the restriction Markov system f_R on R is irreducible and acyclic.

Under the same invariance assumption on $R \subseteq X$, $\varphi_f(R) \subseteq R$, we define

$$\beta_f(R) = 1 - \min_{x\in X} \int_R f(x,y)\, dm(y) \tag{2}$$

$$= \sup_{x\in X} \int_{R^c} f(x,y)\, dm(y)$$

and

$$\beta_f^*(R) = \inf_{n\geq 1} \left[\, \beta_{f^n}(R) \right]^{1/n}\ .$$

The quantity $-\ln(\beta_f^*(R))$ is a kind of *escape rate*, which measures how fast the restriction to R^c of the $\mathscr{L}_f$-iterates of any probability distribution on X will tend to zero. We also make some obvious remarks on this concept:

1. $\beta_{f^n}(R) = 0 \Leftrightarrow \varphi_{f^n}(X) = (\varphi_f)^n(X) \subseteq R$.

2. If for some point $x \in X$, the transition probability f_x^n has support disjoint from R, then $\beta_{f^n}(R) = 1$.

3. If $\beta_f^*(R) < 1$ then for every $x \in X$ the probability density $(f_x^n)_{R^c}$ converges to zero in $L^1_{R^c}$, as $n \to +\infty$.

We shall say that an open φ_f-invariant set $R \subseteq X$ is an *acyclic spectral attractor* for $f \in \mathscr{H}(X)$ if and only if R is connected and $\tau_f^*(R) < 1$. When the set R splits as a disjoint union of d connected sets,

$$R = R_0 \cup \varphi_f(R_0) \cup \cdots \cup \varphi_f^d(R_0),$$

such that R_0 is an acyclic spectral attractor for f^d we say that R is a periodic spectral attractor of period d. We shall say that an open φ_f-invariant set $R \subseteq X$ is *spectrally attractive* for $f \in \mathscr{H}(X)$ if and only if $\beta_f^*(R) < 1$.

We can extract some spectral information on $\mathscr{L}_f$ from the combinatorics of φ_f.

Proposition 7. ([4, Proposition 5.9.]) *Given $f \in \mathscr{H}(X)$, each $\Omega-$final class of period d is a periodic spectral attractor of period d for f.*

Proposition 8. ([4, Proposition 5.10.]) *Given $f \in \mathscr{H}(X)$, $\Omega_{\text{final}}(\varphi_f)$ is spectrally attractive for f.*

Corollary 1. ([4, Corollary 5.11.]) *Given $f \in \mathscr{H}(X)$, let $\Sigma^{\Omega}_{\text{final}}\left(\varphi_f\right) = \{R_1, R_2, \cdots, R_s\}$. Let κ_f be the maximum between $\beta_f^*(R)$ and $\tau_f^*(R_i)$, for $i = 1, \cdots, s$. Then*

$$r_{\text{int}}\left(\mathscr{L}_f\right) \leq \kappa_f.$$

Corollary 2. ([4, Corollary 5.12.]) *Given $f \in \mathscr{H}(X)$, let $\Sigma^{\Omega}_{\text{final}}\left(\varphi_f\right) = \{R_1, R_2, \cdots, R_s\}$, where each component R_i is $\varphi_f^{d_i}-$invariant for some power $d_i \geq 1$. Then there is a f^{d_i}-invariant measure supported on R_i, $\mu_i = \mathscr{L}_{f^{d_i}}\, \mu_i$, for each $i = 1, \cdots, s$, such that:*

1. *The sum $E_1(f)$ of all generalized eigenspaces associated with eigenvalues in the unit circle is the $s-$dimensional space spanned by the measures $\mu_1, \cdots, \mu_s$.*
2. *The action of $\mathscr{L}_f$ on the invariant subspace $E_1(f)$ w.r.t. the basis $\{\mu_1, \cdots, \mu_s\}$ is represented by the permutation matrix associated with the permutation π_{φ_f}.*
3. *The eigenvalues of $\mathscr{L}_f$ in the unit circle are, with multiplicity, the $d-$unity roots $\mathbb{U}^d = \{\lambda \in \mathbb{C} : \lambda^d = 1\}$, counted for every cycle of length d in permutation π_{φ_f}.*
4. *The operator induced by $\mathscr{L}_f$ on the quotient $\mathscr{M}_{\text{prob}}(X)/E_1(f)$ is contractive, i.e., it has norm less than one.*

Consider now any sub-semigroup of Markov systems $\mathscr{H}_1 \subseteq \mathscr{H}(X)$, endowed with some topology.

Definition 4. We say that $\mathscr{H}_1$ is a *topological semigroup of Markov systems* over a topological semigroup of open maps $\mathscr{O}_1$ if and only if for any $f \in \mathscr{H}_1$:

(1) $\varphi_f \in \mathscr{O}_1$;
(2) The map $f \mapsto \varphi_f$ is continuous for the topology of $\mathscr{O}_1$;
(3) $\mathscr{H}_1$ admits outer approximations in the sense that given $f \in \mathscr{H}_1$, for every neighborhood $\mathscr{N}$ of f in $\mathscr{H}_1$ there is $g \in \mathscr{N}$ such that $\varphi_f \prec \varphi_g$;
(4) $\lim_{g \to f} \left\| \mathscr{L}_f^*\, \varphi - \mathscr{L}_g^*\, \varphi \right\|_\infty = 0$ for all $\varphi \in C^0(X)$;
(5) The quantities $\tau_f(R)$ and $\beta_f(R)$, defined in (1) and (2), vary upper semicontinuously with f, for any set $R \subseteq X$.

We now topologize the semigroup $\mathscr{H}(X)$ turning it into a topological semigroup of Markov systems. Consider

$$d_\infty(f,g) = \max_{(x,y)\in X\times X} |f(x,y) - g(x,y)|$$

and

$$d_1(f,g) = \max_{x\in X} \int_X |f(x,y) - g(x,y)|\, dm(y).$$

Define

$$\rho_\infty(f,g) = \max\{d_\infty(f,g), \rho(\varphi_f, \varphi_g)\}$$

and

$$\rho_1(f,g) = \max\{d_1(f,g), \rho(\varphi_f, \varphi_g)\}.$$

Proposition 9. *With the topology associated to any of the metrics ρ_∞ and ρ_1, $\mathscr{H}(X)$ is a topological semigroup of Markov systems.*

Proof. In [4, Proposition 6.2.], we have proved the following result. Consider any sub-semigroup $\mathscr{O}_1 \subseteq \mathscr{O}(X)$ with a topology defined by some metric ρ which makes it a topological semigroup of open maps. Consider the metrics ρ_∞ and ρ_1 as defined above. Then $\mathscr{H}_1 = \{ f \in \mathscr{H}(X) : \varphi_f \in \mathscr{O}_1 \}$ with the topology associated with any of the metrics ρ_∞ and ρ_1 is a topological semigroup of Markov systems over $\mathscr{O}_1$. Thus the proof follows immediately in view of Proposition 5. □

5 Spectral stability of Markov systems

Spectral stability of Markov systems is defined and characterized. Its genericity is proved. Continuity of the invariant measures.

The main goal of this section is to relate, for generic systems $f \in \mathscr{H}(X)$, the combinatorial stability of φ_f with the spectral stability of f, defined below. The novelty here with respect to finite state Markov system theory is that in this context, because we are dealing with continuous systems, it makes sense to define stability. Assume $\mathscr{H}_1 \subseteq \mathscr{H}(X)$ is a sub-semigroup endowed with some topology.

Definition 5. We say that $f \in \mathscr{H}(X)$ is *spectrally stable in* $\mathscr{H}_1$ if and only if there is a neighborhood $\mathscr{U}$ of f in $\mathscr{H}_1$ and there is $0 < k < 1$ such that for all $g \in \mathscr{U}$:

(1) there is a linear map $h_g : E_1(f) \to E_1(g)$ that conjugates $\mathscr{L}_f\,|_{E_1(f)}$ to $\mathscr{L}_g\,|_{E_1(g)}$;
(2) the map h_g depends continuously on f w.r.t. the topology in $\mathscr{H}_1$, in the sense that for any $\varphi \in C^0(X)$, $\lambda_\varphi \circ h_g$ converges to λ_φ as g tends to f in $\mathscr{H}_1$, where $\lambda_\varphi : L^1(X,m) \to \mathbb{R}$ is defined by $\lambda_\varphi(\mu) = \int \varphi\, d\mu$;
(3) $\sigma_0(\mathscr{L}_g) \cap \{\lambda \in \mathbb{C} : k < |\lambda| < 1\} = \emptyset$.

We note that item (2) above is equivalent to say that the invariant measures of $\mathscr{L}_f$ vary continuously with f w.r.t. the weak-$*$ topology. The fixed points of this linear operator are precisely the system invariant measures. The spectral stability of f relates with the fact that no eigenvalues can enter, or leave, the unit circle.

Theorem 3 (Spectral stability characterization). *For any $f \in \mathscr{H}(X)$, f is spectrally stable in any of the spaces $(\mathscr{H}(X), \rho_\infty)$ and $(\mathscr{H}(X), \rho_1)$ if and only if φ_f satisfies the combinatorial stability condition.*

Proof. In [4, Theorem A]) we have proved that given any topological semigroup of Markov systems $\mathscr{H}_1$, f is spectrally stable in $\mathscr{H}_1$ if and only if φ_f satisfies the combinatorial stability condition. Thus the proof follows immediately in view of Proposition 9. □

Theorem 4 (Genericity of spectral stability). *The set of spectrally stable systems is open and dense in any of the spaces $(\mathscr{H}(X), \rho_\infty)$ and $(\mathscr{H}(X), \rho_1)$.*

Proof. In [4, Theorem B]) we have proved that for any topological semigroup of Markov systems $\mathscr{H}_1$, the set of $\mathscr{H}_1$-spectrally stable Markov systems is open and dense in the semigroup $\mathscr{H}_1$. Thus the proof follows immediately in view of Proposition 9. □

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